Music 445: second order filters

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1 Two pole filters

The difference equation of a two pole filter is given by:

$$y(n) = b_0 x(n) - a_1 y(n-1) - a_2 y(n-2) \tag{1}$$

The Z-transform is given by:

$$Y(z) = b_0 X(z) - a_1 z^{-1} Y(z) - a_2 z^{-2} Y(z) \tag{2}$$

and the transfer function is given by:

$$H(z) = \frac{b_0}{1 + a_1 z^{-1} + a_2 z^{-2}} \tag{3}$$

finally the frequency response:

$$H(e^{j\omega}) = \frac{b_0}{1 + a_1 e^{-j\omega} + a_2 e^{-2j\omega}} \tag{4}$$

Using the quadratic formula, it is easy to find the location of the two poles:

$$z = -\frac{a_1}{2} \pm \sqrt{\left(\frac{a_1}{2}\right)^2 - a_2} \tag{5}$$

Let's express the two poles p_1 and p_2 in polar form, i.e.

$$p_1 = Re^{j\theta}$$

$$p_2 = Re^{-j\theta}$$

(since they must form a complex conjugate pair).

in this case, we obtain:

$$H(z) = \frac{b_0}{(z^{-1} - p_1)(z^{-1} - p_2)} =$$

$$\frac{b_0}{(z^{-1} - Re^{j\theta})(z^{-1} - Re^{-j\theta})} =$$

$$\frac{b_0}{(z^{-2} - Re^{j\theta} z^{-1} - Re^{-j\theta} z^{-1} + R^2} =$$

$$= \frac{b_0}{(z^{-2} - 2Rcos(\theta) + R^2)}$$

where $\theta = 2\pi f_c$, where f_c is the cutoff frequency in Hz.

R is the pole radius that determines damping.

For filter stability, we must have $R < 1$.

1.1 Quality factor and bandwidth

The magnitude R if a pole determines the damping or bandwidth Bw of the resonator. The quality factor Q is given by:

$$Q = \frac{f_c}{Bw}$$

where f_c is the cutoff frequency as before.

So increasing the quality factor decreases the bandwidth.