FAST MUSIC – AN EFFICIENT IMPLEMENTATION OF THE MUSIC ALGORITHM FOR FREQUENCY ESTIMATION OF APPROXIMATELY PERIODIC SIGNALS

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ABSTRACT

Noise subspace methods are popular for estimating the parameters of complex sinusoids in the presence of uncorrelated noise and have applications in musical instrument modeling and microphone array processing. One such algorithm, MUSIC (Multiple Signal Classification) has been popular for its ability to resolve closely spaced sinusoids. However, the computational efficiency of MUSIC is relatively low, since it requires an explicit eigenvalue decomposition of an autocorrelation matrix, followed by a linear search over a large space. In this paper, we discuss methods for and the benefits of converting the Toeplitz structure of the autocorrelation matrix to *circulant* form, so that eigenvalue decomposition can be replaced by a Fast Fourier Transform (FFT) of one row of the matrix. This transformation requires modeling the signal as at least approximately periodic over some duration. For these periodic signals, the pseudospectrum calculation becomes trivial and the accuracy of the frequency estimates only depends on how well periodicity detection works. We derive a closed-form expression for the pseudospectrum, yielding large savings in computation time. We test our algorithm to resolve closely spaced piano partials.

1. INTRODUCTION

Sinusoidal parameter estimation is a classical problem with applications in radar, sonar, music, and speech, among others. When the frequencies of sinusoids are well resolved, looking for spectral peaks is adequate. It is shown in [1] that the maximum likelihood (ML) frequency estimate for a single sinusoid in Gaussian white noise is given by the frequency of the magnitude peak in the periodogram. The ML approach is extended and Cramer-Rao bounds are derived for multiple sinusoids in noise in a follow-on paper by the same authors [2]. Some other estimators are covered in [3].

For closely spaced sinusoidal frequencies, however, other approaches have been developed. Noise subspace methods are a class of sinusoidal parameter estimators that utilize the fact that the noise subspace of the measured signal is orthogonal to the signal subspace. Pisarenko Harmonic Decomposition [4] makes use of the eigenvector associated with the minimum eigenvalue of the estimated autocorrelation matrix to find frequencies. However, it has been found to exhibit relatively poor accuracy [3]. Schmidt [5] improved over Pisarenko with the MUSIC (MUltiple SIgnal Classification) algorithm which could estimate the frequencies of multiple closely spaced signals more accurately in the presence of noise. In MUSIC, a pseudospectrum is generated by projecting a complex sinusoid onto all of the noise subspace eigenvectors, defining peaks where this projection magnitude is minimum. This method is shown to be asymptotically unbiased. An enhancement to MUSIC, root-MUSIC, was proposed in [6]. It uses the properties of the signal-space eigenvectors to define a rational spectrum with poles and zeros. It is said to have better resolution than MUSIC at low SNRs. Similarly, another popular algorithm, ESPRIT [7], was invented by Roy et al. which makes use of the underlying rotational invariance of the signal subspace. The generalized eigenvalues of the matrix pencil formed by an auto-covariance matrix and a cross-covariance matrix gives the unknown frequencies. ESPRIT performs better than MUSIC, especially when the signal is sampled nonuniformly. More recently, another enhancement to MUSIC, gold-MUSIC [8] has been proposed which uses two stages for coarse and fine search, respectively.

One of the disadvantages of the MUSIC algorithm is its computational complexity. Typical eigenvalue decomposition algorithms are of the order $O(N^3)$ [9]. In this paper, we use the fact that, for periodic signals, the autocorrelation matrix is *circulant* when it spans an integer multiple of the signal's period. In this case, looking for the eigenvalues with largest magnitude is equivalent to looking for peaks in the power spectrum. We know in the circulant case that all noise eigenvectors are DFT sinusoids [10], and hence we can derive a closed-form solution when we project our search space onto the noise subspace, thereby reducing further the calculations required to find the pseudospectrum. Replacing eigenvalue decomposition in MUSIC with efficient Fourier transform based methods has been previously studied in [11] where the eigenvectors are derived to be some linear combinations of the data vectors, while maintaining the orthonormality constraint. In this paper, we take a different approach and show that for periodic signals, the MUSIC pseudospectrum can be exactly calculated using a sum of *aliased sinc* functions and its accuracy only depends on the accuracy with which the periodicity of the autocorrelation function is detected. We also propose speeding up MUSIC for non-periodic signals by initializing QR factorization for eigenvalue decomposition with the DFT matrix.

We test our algorithm to resolve closely spaced partials of the A3 note played on a piano. It is a well known fact that the strings corresponding to a particular piano key are slightly mistuned. Coupled motion of piano strings has been studied in detail by Weinreich in [12, 13]. There is a slight difference in frequency of the individual strings, giving rise to closely spaced peaks in the spectra. We show that FAST MUSIC can resolve two closely spaced peaks much faster than MUSIC.

The rest of this paper is organized as follows : Section 2 gives an outline of the MUSIC algorithm, Section 3 derives the FAST MUSIC algorithm, Section 4 describes the experimental results on a) an artificially synthesized signal containing two sinusoids with additive white noise and b) a partial of the A3 note played on the piano which contains beating frequencies. We conclude the paper in Section 5 and delineate the scope for future work. Estimation of real-valued sine wave frequencies with MUSIC has been studied in [14]. For all derivations in this paper, we work with real signals, because we are interested in audio applications. This saves some time in computing the pseudospectrum since we know it will be symmetric. Our derivations can be easily extended to complex signals.

2. MUSIC

2.1. Model

We wish to estimate the parameters of a signal composed of additive sinusoids from noisy observations. Let $y(n)$ be the noisy signal, composed of a deterministic part, $x(n)$, made of r real sinusoids and random noise, $w(n)$. We assume that $w(n) \sim N(0, \sigma^2)$, and that $w(n)$ and $x(n)$ are uncorrelated. The sinusoidal phases ϕ_i's are assumed to be i.i.d. and uniformly distributed $\phi_i \sim U(-\pi, \pi)$.

$$\begin{aligned} y(n) &= \sum_{i=1}^{r} A_i \cos(\omega_i n + \phi_i) + w(n) \\ y(n) &= s(n) + w(n) \end{aligned}$$

In vector notation, the signal $\mathbf{y} \in \mathbb{R}^M$ can be characterized by the $M \times M$ autocorrelation matrix $K_y = \mathbb{E}(\mathbf{y}\mathbf{y}^T)$. For a zero-mean signal, the autocorrelation matrix coincides with the covariance matrix. Since this matrix is Toeplitz and symmetric positive-definite, its eigenvalues are real and nonnegative (and positive when $\sigma > 0$). We can perform an eigenvalue decomposition on this matrix to get a diagonal matrix Λ consisting of the eigenvalues, and an eigenvector matrix Q. The $2r$ eigenvectors corresponding to the $2r$ largest eigenvalues, Q_s, contain signal plus noise information, whereas the remaining *M-2r* eigenvectors, Q_w, only represent the noise subspace. Thus, we have the following relationships:

$$\begin{aligned} K_y &= K_s + K_w = K_s + \sigma^2 I \\ K_y &= Q \Lambda Q^H \\ K_y &= \begin{bmatrix} Q_s & Q_w \end{bmatrix} \begin{bmatrix} \Lambda' & 0 \\ 0 & \sigma^2 I_{M-2r} \end{bmatrix} \begin{bmatrix} Q_s^H \\ Q_w^H \end{bmatrix} \end{aligned} \tag{1}$$

2.2. Pseudospectrum Estimation

Let a vector of M harmonic frequencies be denoted as $\mathbf{b}(\omega) = [1, e^{j\omega}, e^{2j\omega} \cdots e^{(M-1)j\omega}]^T$. We project this vector onto Q_w, i.e., the subspace occupied by the noise (where there is no signal component). MUSIC defines the following pseudospectrum as a function of a set of ω's:

$$\begin{aligned} P(\omega) &= \frac{1}{\mathbf{b}(\omega)^H Q_w Q_w^H \mathbf{b}(\omega)} \\ P(\omega) &= \frac{1}{||Q_w^H \mathbf{b}(\omega)||^2} \end{aligned} \tag{2}$$

For a particular value of ω that is actually present in the signal, the sum of projections of $\mathbf{b}$ onto the eigenvectors spanning the noise subspace will be zero. This is because the subspace occupied by the signal is orthogonal to that occupied by noise since they are uncorrelated. Thus, we see that $P(\omega)$ will take on a very high value in such cases (theoretically infinite). In conclusion, we can find peaks in the function $P(\omega)$ and those will correspond to our estimated frequencies. Since the search space can consist of any number of densely packed frequencies, very closely spaced peaks can show up in the pseudospectrum. However, as the search-space grows, so does computational complexity.

3. FAST MUSIC

3.1. Deriving the autocorrelation matrix

In vector form, $\mathbf{y} \in \mathbb{R}^M$ can be written as:

$$\begin{aligned} \mathbf{y} &= S\mathbf{a} + \mathbf{w} \\ \mathbf{w} &\sim N(0, \sigma^2 I) \end{aligned} \tag{3}$$

$$\begin{bmatrix} y(n) \\ y(n-1) \\ \vdots \\ y(n-M+1) \end{bmatrix} = \begin{bmatrix} 1 & 0 & \cdots \\ \cos(\omega_1) & \sin(\omega_1) & \cdots \\ \vdots & \vdots & \cdots \\ \cos[(M-1)\omega_1] & \sin[(M-1)\omega_1] & \cdots \end{bmatrix} \times \begin{bmatrix} A_1 \cos(\omega_1 n + \phi_1) \\ A_1 \sin(\omega_1 n + \phi_2) \\ \vdots \\ A_r \sin(\omega_r n + \phi_r) \end{bmatrix} + \begin{bmatrix} w(n) \\ w(n-1) \\ \vdots \\ w(n-M+1) \end{bmatrix} \tag{4}$$

Since $\mathbf{y}$ is zero-mean, its covariance matrix is

$$K_y = \mathbb{E}(\mathbf{y}\mathbf{y}^T) = SK_aS^T + \sigma^2 I. \tag{5}$$

We now want to get K_y in terms of K_a. We have assumed $\phi_i \sim U(-\pi, \pi)$ (uniformly identically distributed random phase). We observe that every term of K_a is of the form $K_a(i,j) = \mathbb{E}[A_i \cos(\omega_i n + \phi_i)\, A_j \cos(\omega_j n + \phi_j)]$, or $\mathbb{E}[A_i \sin(\omega_i n + \phi_i) A_j \sin(\omega_j n + \phi_j)]$, or $\mathbb{E}[A_i \sin(\omega_i n + \phi_i) A_j \cos(\omega_j n + \phi_j)]$, or $\mathbb{E}[A_i \cos(\omega_i n + \phi_i) A_j \sin(\omega_j n + \phi_j)]$. All of these terms are zero, except the first two when $i = j$, i.e. $\mathbb{E}[A_i^2 \cos(\omega_i n + \phi_i)^2] = \mathbb{E}[A_i^2 \sin(\omega_i n + \phi_i)^2] = \frac{A_i^2}{2}$, which makes it a diagonal matrix:

$$K_a = \begin{bmatrix} \frac{A_1^2}{2} & 0 & \cdots & 0 & 0 \\ 0 & \frac{A_1^2}{2} & \cdots & 0 & 0 \\ \vdots & \vdots & \cdots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & \frac{A_r^2}{2} \end{bmatrix} \tag{6}$$

The autocorrelation matrix of the observed signal is given in (7). This is an $M \times M$ real, symmetric Toeplitz matrix.

3.2. For periodic signals

Under conditions to be specified, it is possible to replace the eigenvalue decomposition required in MUSIC by a Fast Fourier Transform (FFT). In this subsection, we derive the order of the autocorrelation matrix for which it is circulant instead of only Toeplitz. We also derive a closed-form expression for finding the pseudospectrum.

$$SK_aS^T + \sigma^2 I = \begin{bmatrix} \sum_i \frac{A_i^2}{2} + \sigma^2 & \sum_i \frac{A_i^2}{2}\cos\omega_i & \sum_i \frac{A_i^2}{2}\cos 2\omega_i & \cdots & \sum_i \frac{A_i^2}{2}\cos(M-1)\omega_i \\ \sum_i \frac{A_i^2}{2}\cos\omega_i & \sum_i \frac{A_i^2}{2} + \sigma^2 & \sum_i \frac{A_i^2}{2}\cos\omega_i & \cdots & \sum_i \frac{A_i^2}{2}\cos(M-2)\omega_i \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \sum_i \frac{A_i^2}{2}\cos(M-1)\omega_i & \sum_i \frac{A_i^2}{2}\cos(M-2)\omega_i & \cdots & \cdots & \sum_i \frac{A_i^2}{2} + \sigma^2 \end{bmatrix} \tag{7}$$

3.2.1. Circulancy of the autocorrelation matrix

We have seen that the autocorrelation matrix K_y is symmetric Toeplitz. However it is to be noted that for $k = 1, 2, ...$, if M is an integer such that $M = 2\pi n/\omega_i, n \in \mathbb{Z}^+$, then

$$\sum_{i=1}^{r} \frac{A_i^2}{2}\cos(M-k)\omega_i = \sum_{i=1}^{r} \frac{A_i^2}{2}\cos k\omega_i \tag{8}$$

If we choose M carefully, then the autocorrelation matrix may be written as :

$$\begin{bmatrix} \sum_i \frac{A_i^2}{2} + \sigma^2 & \sum_i \frac{A_i^2}{2}\cos\omega_i & \cdots & \sum_i \frac{A_i^2}{2}\cos\omega_i \\ \sum_i \frac{A_i^2}{2}\cos\omega_i & \sum_i \frac{A_i^2}{2} + \sigma^2 & \cdots & \sum_i \frac{A_i^2}{2}\cos 2\omega_i \\ \sum_i \frac{A_i^2}{2}\cos 2\omega_i & \sum_i \frac{A_i^2}{2}\cos\omega_i & \vdots & \sum_i \frac{A_i^2}{2}\cos 3\omega_i \\ \vdots & \vdots & \ddots & \vdots \\ \sum_i \frac{A_i^2}{2}\cos\omega_i & \sum_i \frac{A_i^2}{2}\cos 2\omega_i & \cdots & \sum_i \frac{A_i^2}{2} + \sigma^2 \end{bmatrix} \tag{9}$$

This matrix is circulant! Hence, its eigenvectors are given by the DFT sinusoids and its eigenvalues are the DFT coefficients of the first row [10]. The eigenvalues can be computed using an FFT algorithm when M is a power of 2 or highly composite. The relationship between eigenvalues of Toeplitz matrices and those of asymptotically equivalent circulant matrices have been studied in [15].

For example, if the signal consists of 3 sinusoids with frequencies $\frac{\pi}{2}$, $\frac{\pi}{4}$ and $\frac{\pi}{5}$ then the minimum order of M which will make the autocorrelation matrix circulant is given by $2 \times LCM(2, 4, 5) = 40$. However, if any of the denominators is irrational, then the LCM does not exist, and hence no value of M will make the autocorrelation matrix circulant. Of course, in reality we do not know the frequencies and cannot determine M this way. However, we can instead detect when the autocorrelation corresponds to a signal that is periodic. If we can find the periodicity of the estimated autocorrelation function and set M to be that period, then the resulting autocorrelation matrix will be circulant. Since no signal is truly precisely periodic, this procedure can be viewed as introducing an approximation based on assuming the signal is periodic. Such a periodic/harmonic approximation is common when the underlying signal source is known to be a quasi periodic oscillator such in voiced speech, bowed strings, woodwinds, flutes, brasses, organs, and so on.

In this paper, we use the Average Magnitude Difference Function [16] to detect the period. We find all local minima in the AMDF and pick the period as the lowest minimum index which is smaller than its adjacent neighbors. We set M to be an integer multiple of the detected period. This gives us more data points for the FFT, thus increasing accuracy. It also comes at a higher cost, but the FFT is still orders of magnitude faster than eigenvalue decomposition, hence the trade-off is justified.

3.2.2. Searching over a large range of frequencies

Suppose we want to calculate the pseudospectrum for $N \geq M$ distinct frequencies $\omega_k = 2\pi\frac{k}{N}$ for $k = -\frac{N}{2}, \ldots, \frac{N}{2} - 1$ covering the range $[-\pi, \pi)$, i.e, the search space has N points. Each search space vector is

$$\mathbf{b}(k) = [1, e^{\frac{2\pi jk}{N}}, e^{\frac{4\pi jk}{N}} \ldots, e^{\frac{2\pi(M-1)jk}{N}}]^T. \tag{10}$$

The noise subspace consists of $M - 2r$ vectors. Instead of projecting on to the noise subspace, Q_w, we can make the computation easier by using the signal subspace, Q_s instead, which only has $2r$ vectors. This is because the noise subspace and the signal subspace are orthogonal complements and hence the following holds

$$Q_sQ_s^H + Q_wQ_w^H = I \tag{11}$$

The projection onto the noise subspace can be simplified as

$$\begin{aligned} ||Q_w^H\mathbf{b}||^2 &= \mathbf{b}^H Q_w Q_w^H \mathbf{b} \\ &= \mathbf{b}^H(I - Q_sQ_s^H)\mathbf{b} \\ &= \mathbf{b}^H\mathbf{b} - \mathbf{b}^H Q_s Q_s^H \mathbf{b} \\ &= ||\mathbf{b}||^2 - ||Q_s^H\mathbf{b}||^2 \end{aligned} \tag{12}$$

Since $\mathbf{b}(k)$ is a vector of length M consisting of complex exponentials of unit magnitude, $||\mathbf{b}(k)||^2 = M$. The matrix $Q_s \in \mathbb{C}^{M \times 2r}$ is composed of columns of signal eigenvectors, such that each column is denoted as

$$\mathbf{q}_s = \frac{1}{\sqrt{M}}[1, e^{\frac{2\pi jm_i}{M}}, e^{\frac{4\pi jm_i}{M}}, \cdots, e^{\frac{2\pi(M-1)jm_i}{M}}]^T \tag{13}$$

where m_i are the complex frequencies associated with the signal eigenvectors [10], i.e, the indices of the top $2r$ FFT magnitudes. The projection of $\mathbf{b}(k)$ onto the signal subspace can be written as :

$$\begin{aligned} Q_s^H\mathbf{b}(k) &= \frac{1}{\sqrt{M}} \begin{bmatrix} \sum_{p=0}^{M-1} \exp[2\pi jp(\frac{k}{N} - \frac{m_1}{M})] \\ \vdots \\ \sum_{p=0}^{M-1} \exp[2\pi jp(\frac{k}{N} - \frac{m_{2r}}{M})] \end{bmatrix} \\ &= \frac{1}{\sqrt{M}} \begin{bmatrix} e^{-\pi j(\frac{k}{N} - \frac{m_1}{M})(M-1)} \frac{\sin[\pi(\frac{k}{N} - \frac{m_1}{M})M]}{\sin[\pi(\frac{k}{N} - \frac{m_1}{M})]} \\ \vdots \\ e^{-\pi j(\frac{k}{N} - \frac{m_{2r}}{M})(M-1)} \frac{\sin[\pi(\frac{k}{N} - \frac{m_{2r}}{M})M]}{\sin[\pi(\frac{k}{N} - \frac{m_{2r}}{M})]} \end{bmatrix} \end{aligned} \tag{14}$$

The pseudospectrum can be approximated as:

$$\begin{aligned} P(k) &= \frac{1}{||\mathbf{b}(k)||^2 - ||Q_s^H \mathbf{b}(k)||^2} \\ &= \frac{1}{M - \frac{1}{M}\sum_{i=1}^{2r}\left[\frac{\sin[\pi(\frac{k}{N}-\frac{m_i}{M})M]}{\sin[\pi(\frac{k}{N}-\frac{m_i}{M})]}\right]^2} \\ &= \frac{1}{M - \sum_{i=1}^{2r}\left[asinc_M(\frac{k}{N}-\frac{m_i}{M})\right]^2} \end{aligned} \tag{15}$$

where $asinc$ stands for the *aliased sinc* function[1]. The pseudospectrum is independent of the data and only depends on the calculated period, M. At signal frequencies, when $\frac{k}{N} = \frac{m_i}{M}$, one *aliased sinc* term in the summation dominates and we can evaluate it using L'Hospital's rule.

$$\lim_{x \to 0} asinc_M(x) = M \tag{16}$$

Therefore, at signal frequencies, the pseudospectrum is theoretically infinite.

$$P(k) \approx \infty \quad if \quad \frac{k}{N} = \frac{m_i}{M} \tag{17}$$

For the special case of periodic signals, MUSIC is essentially equivalent to looking for the top $2r$ peaks in the power spectrum and using the positions of those peaks to form the signal space.

3.2.3. Algorithm Summary

1. Estimate the autocorrelation function (ACF) of the given signal.
2. Find the periodicity M of the ACF and take the FFT of its first M samples. This is equivalent to computing the power spectrum.
3. Sort the FFT magnitudes in descending order. The indices corresponding to the largest $2r$ magnitudes are the signal eigenvector frequencies.
4. Form search space vectors according to (10) with $k = -\frac{N}{2}, \ldots, \frac{N}{2} - 1$.
5. Calculate the pseudospectrum according to (15) and find $2r$ peaks in it.
6. Do parabolic interpolation on the peaks to get more accurate frequency estimates [17].

3.3. For non-periodic signals

Most signals in practical applications are non-periodic. In that case, these derivations do not hold exactly. However, we can still speed up the eigenvalue decomposition process. From (1), we can write the diagonal eigenvalue matrix as

$$\Lambda = Q^H K_y Q \tag{18}$$

The eigenvectors Q are usually estimated with QR factorization [9]. We can use the DFT matrix W as an initial value for QR factorization, which will ensure its convergence in fewer steps.

$$\Lambda + \epsilon = W^H K_y W \tag{19}$$

For exactly periodic signals ϵ is a null matrix. For approximately periodic signals, ϵ is a non-diagonal matrix with small entries. We can see that within some iterations W will converge to Q. The speed of convergence will depend on how close to being periodic the signal is.

4. EXPERIMENTS AND RESULTS

4.1. Synthesized signal

To compare FAST MUSIC with MUSIC, we tested a signal composed of cosines at frequencies 0.004 Hz and 0.005 Hz at $f_s = 1$ Hz and added normally distributed noise $w(n)$ at an SNR of 10dB.

$$y(n) = \cos(0.01\pi n) + 0.5\cos(0.008\pi n + \phi) + w(n) \tag{20}$$

To detect periodicity of the autocorrelation function, we need at least two periods of the signal. This signal has a periodicity of $M = 1000$ samples. Thus, we made the signal 2500 samples long. It is to be noted that the closer the frequencies in the signal, the larger will be its periodicity, and hence we will need more samples of data to accurately determine it.

To measure computation time, we compared various eigenvalue decomposition algorithms with Fast Fourier Transform algorithms for increasing orders of the autocorrelation matrix. The results can be seen in Figure 1. QR factorization is used to find eigenvalues and eigenvectors for MUSIC. QR factorization with Gram Schmidt orthogonalization is slow, symmetric tridiagonal QR with implicit Wilkinson shift is slightly faster whereas reduction to the Hessenberg form is fastest. More details about these algorithms can be found in [9]. The Fourier transform algorithms are orders of magnitude faster, with the DFT dominating at lower orders and self-sorting mixed radix FFT [18] and resampled split radix FFT [19] giving faster speeds at higher orders. It is to be noted that the order of the autocorrelation matrix for which circulancy is achieved is not likely to be a power of 2, and hence we cannot use the well-known radix-2 FFT algorithm. However, we can first resample the data to a power of 2 [20] and then apply a radix-2/split radix FFT algorithm or use a mixed-radix FFT algorithm on any composite order. All of these functions have been implemented in MATLAB. More efficient implementations can be done in C, where the FFT functions should overtake the DFT at much lower orders.

We also conducted 1000 Monte-Carlo simulations on the above example, with uniformly distributed random phase. We plotted the mean-squared errors vs SNR for MUSIC and FAST MUSIC, along with the Cramer-Rao bounds (CRB) as given in [21] in Figure 2. The order of the autocorrelation matrix for MUSIC is set to 200. For FAST MUSIC, the period is calculated for each simulation and found to be 1000 samples. The number of points in the search space is 2000. The poor performance of FAST MUSIC at low SNRs is due to the inaccuracy in periodicity detection. In Figure 2a, FAST MUSIC overtakes the CRB at high SNRs, where periodicity is detected accurately, hence MSE $= 0$. At high SNRs, FAST MUSIC also outperforms MUSIC. Increasing the order of the autocorrelation matrix would have improved the accuracy of MUSIC at a cost of high computational time, so we decided to work with a reasonable order of 200, while FAST MUSIC used 2000 samples in the autocorrelation function (an integer multiple of the period). As seen in Figure 2b, both methods have significant bias.

[1] https://ccrma.stanford.edu/~jos/sasp/Rectangular_Window.html

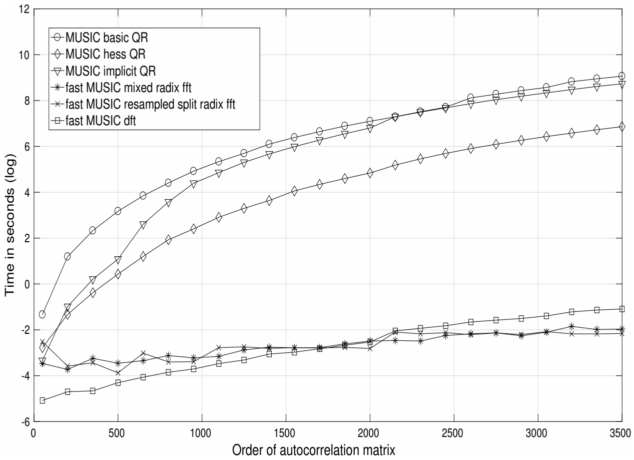
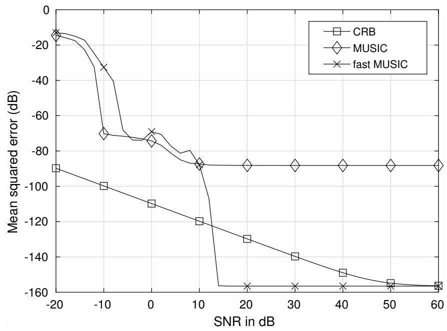
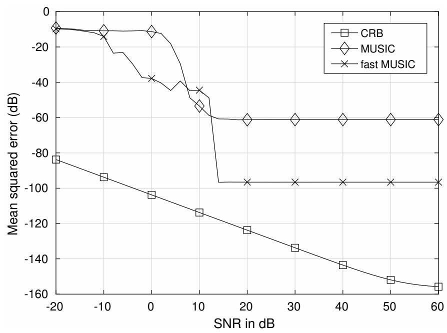


Figure 1: Computation time in log seconds versus matrix order

Bias can be reduced arbitrarily in FFT based peak finding methods [17] by increasing the amount of zero-padding, as well as by other methods [22]. We expect it to reduce similarly in FAST-MUSIC and MUSIC when the number of points in the search space is increased.



(a) Frequency = 0.004 Hz



(b) Frequency = 0.005 Hz

Figure 2: MSE vs SNR plots

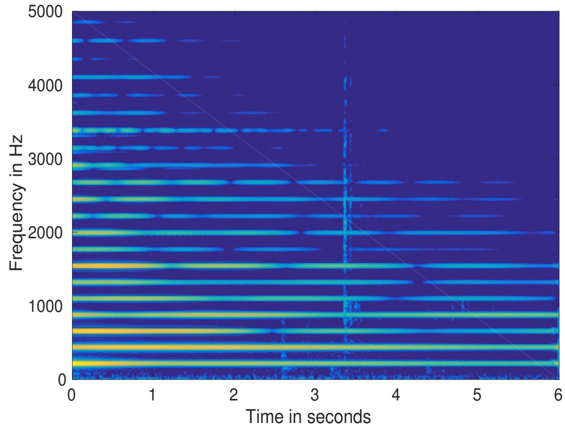


Figure 3: Spectrogram of piano note

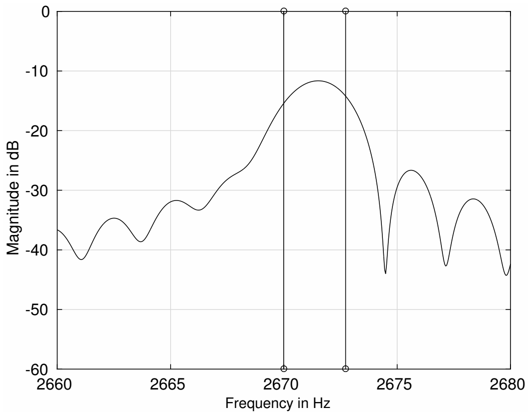
4.2. Piano data

We tested our algorithm on the A3 note played on the piano. The spectrogram of the steady state portion of the note is given in Figure 3. We can observe beating in some of the partials. We decided to work with the 11th partial, located close to 2600 Hz, where a beating of roughly 1 Hz is observed. We bandpass-filtered the signal using a 4th-order Butterworth filter with cut-off frequencies at 2400 Hz and 2900 Hz. We ran FFTs with the rectangular window and FAST MUSIC on different data lengths, as shown in Figure 4, where the vertical lines indicate the frequencies detected by FAST MUSIC. One disadvantage of using the rectangular window is high side lobe height as seen in Figure 4, but we compromise side lobe height for the narrowest main lobe width for the sake of best resolution. We see that for window size of 2^{14}, the FFT magnitude does not exhibit two separately discernible peaks at all, whereas FAST MUSIC provides two peak frequencies with some error. This is because we specify the number of sinusoids to be 2 in FAST MUSIC, whereas the FFT has no prior information about the number of peaks expected in the magnitude spectrum. For longer window sizes, both FFT and FAST MUSIC are able to resolve the two peaks with greater accuracy. One potential application is in piano tuning, where FAST MUSIC could be used to quickly resolve closely spaced peaks caused by the coupled motion of the piano strings.

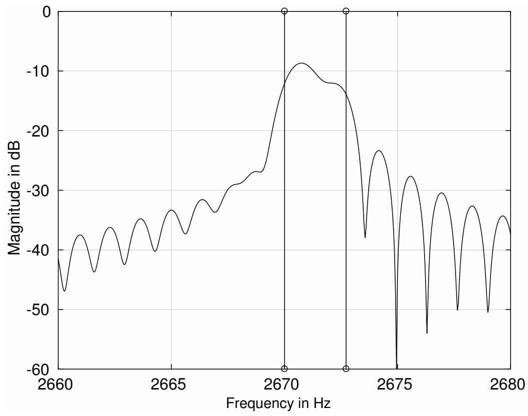
5. DISCUSSION AND FUTURE WORK

In this paper, we have proposed a computationally efficient interpretation of the MUSIC algorithm for periodic signals that makes use of the peaks in the power spectrum. The autocorrelation matrix has been derived and approximated by a circulant matrix. This approximation has allowed us to replace computationally intensive eigenvalue decomposition algorithms with an FFT. We have subsequently derived a closed-form expression for searching over a range of frequencies. These modifications have yielded a significant improvement in computational speed. For non-periodic signals, we have proposed initialization of QR factorization with the DFT matrix to speed up eigenvalue decomposition.[2]

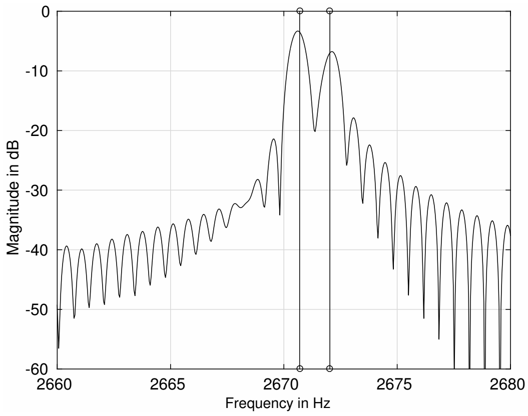
[2]The code and the simulations can be found at `https://github.com/orchidas/fast_MUSIC`



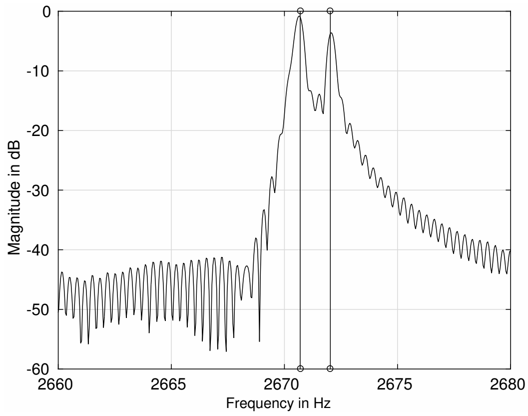
(a) Data length= 2^{14}



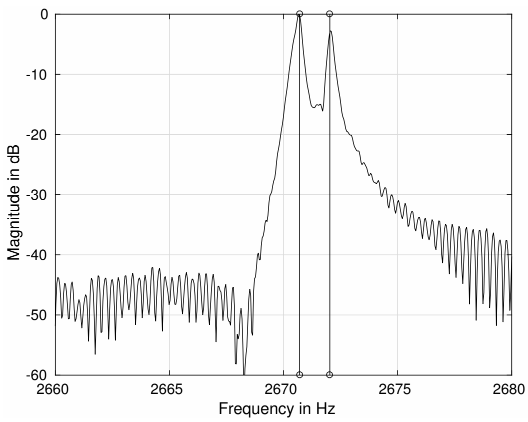
(b) Data length= 2^{15}



(c) Data length= 2^{16}



(d) Data length= 2^{17}



(e) Data length= 2^{18}

Figure 4: FFT magnitude plots and FAST MUSIC frequency estimates (vertical lines)

A key factor in the accuracy of FAST MUSIC is the precision in periodicity detection. If the period is off by a significant number of samples, the autocorrelation matrix is no longer circulant and FAST MUSIC falls apart. AMDF based periodicity detector is simple but time consuming, not foolproof and often yields wrong results if the number of lags in the autocorrelation function is very high. Ideally, a better method for periodicity detection should be used.

Another issue is finding the number of sinusoids present in a given signal, when not known a priori. To do so, one can look at the relative magnitude of the eigenvalues (power spectrum peak values in our case). This works well if the signal to noise ratio is sufficiently high and the peak separation sufficient. More robust partitioning schemes have been used in [8]. Once the signal frequencies are known, the estimation of amplitudes is simple and can be done using linear least squares.

We found that the estimator mean squared errors for both FAST-MUSIC and MUSIC were dominated by bias at high SNRs. Future work should reduce or eliminate the bias so that the relative performance can be observed at high SNRs. FAST MUSIC also needs to be better evaluated with non-periodic signals. We have not tested its performance with non-periodic signals in this paper.

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