# Discretizing the Casio SK-1 Percussion Filter (and a "circuit-bent" extension)

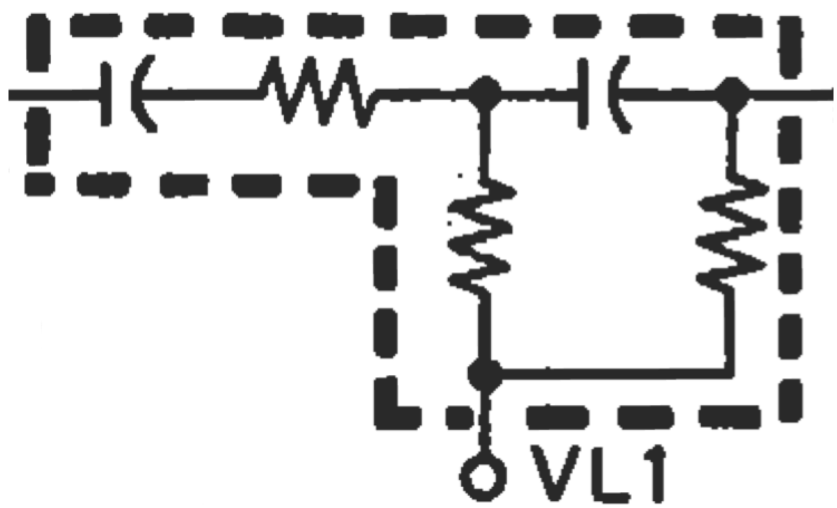
Kurt James Werner

May 13, 2012

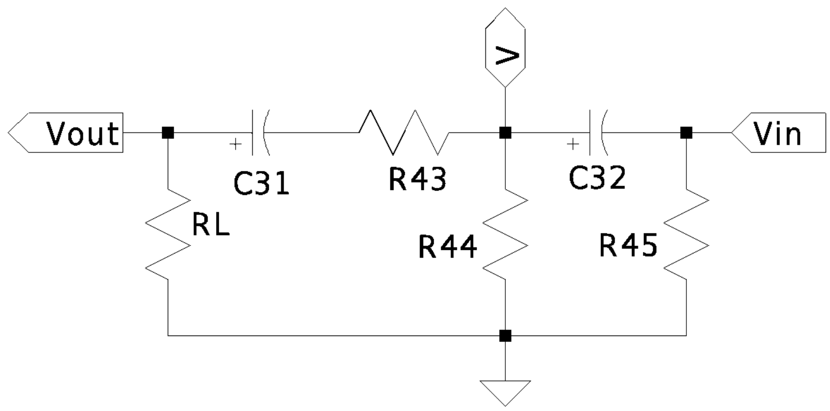
## 1 Overview

It is desirable to find the digital transfer function $H(z)$ approximating the analog circuitry of the Casio SK-1. One of these is the Percussion Filter. $H(z)$ can be obtained by performing the bilinear transform on the continuous-time transfer function $H(s)$. $H(s)$ can be obtained by manipulating the continuous-time expressions that describe the circuit's behavior. This analysis assumes linear / ideal circuit behavior.

A schematic diagram of the Percussion Filter circuit, as found in the Realistic Concertmate-500 service manual (the alternate name of the Casio SK-1, when it was sold through RadioShack). $VL1$ is ground, $V_{in}$ enters on the right, and $V_{out}$ leaves on the left.




To analyze the circuit, a load resistor $R_L$ will be added from $V_{out}$ to ground. This represents the load that the rest of the circuit will contribute to the Percussion Filter. For now, a "reasonable" value of $1M\,\Omega$ will be used. This can be replaced by a more accurate value once it is obtained. A node $V$ is defined to make algebraic manipulations simpler (it will cancel out early on). Components are labelled as in the Realistic Concertmate-500 service manual:




Component values as given in the Realistic Concertmate-500 service manual:

$$
\begin{aligned}
R_{43} &= 100\,k\Omega \\
R_{44} &= 100\,k\Omega \\
R_{45} &= 6.7\,k\Omega \\
C_{31} &= 0.1\,\mu F \\
C_{32} &= 0.1\,\mu F
\end{aligned}
$$

# 2 The Continuous-Time Tranfer Function

## 2.1 Voltage Divider

Once expressions for $V_{in}$ and $V_{out}$ are obtained, their ratio will give the continuous-time transfer function $H(s)$. $V_{in}$ and $V_{out}$ can be found using voltage dividers. In a voltage divider, the input voltage is proportional to the series impedance of $Z_1 + Z_2$, and the output voltage is proportional to the impedance $Z_2$.

$$V_i = i * (Z_1 + Z_2) \rightarrow i = \frac{V_i}{Z_1 + Z_2}$$

$$V_o = i * Z_2 \rightarrow i = \frac{V_o}{Z_2}$$

$$\frac{V_i}{Z_1 + Z_2} = \frac{V_o}{Z_2} \rightarrow V_o = V_i \frac{Z_2}{Z_1 + Z_2}$$

## 2.2 Finding the Transfer Function

Expressions for $V_{in}$ and $V_{out}$ are obtained by application of the voltage divider equation. Keep in mind that the impedance of a resistor is the same as its resistance ($Z_R = R$) and that the impedance of a capacitor is inversely proportional to its capacitance multiplied by $s$ ($Z_C = \frac{1}{sC}$), where $s$ is the differentiation operator on the s-plane, $R$ is the value of the resistor in Ohms ($\Omega$), and $C$ is the value of the capacitor in Farads($F$).

$$V_{out} = V \frac{R_L}{R_{43} + R_L + \frac{1}{sC_{31}}}$$

$$V = V_{in} \frac{Z}{Z + \frac{1}{sC_{32}}} \rightarrow V_{in} = V \frac{Z + \frac{1}{sC_{32}}}{Z}$$

Where $Z$ is the impedance of $R_{44}$ in parallel with the series impedance of $R_{43}$, $C_{31}$, and $R_L$.

$$\begin{aligned} Z &= R_{44} \| (R_{43} + C_{31} + R_L) \\ &= \frac{R_{44} \left( R_{43} + R_L + \frac{1}{sC_{31}} \right)}{R_{43} + R_{44} + R_L + \frac{1}{sC_{31}}} \\ &= \frac{R_{44} \left( R_{43} + R_L \right) sC_{31} + R_{44}}{\left( R_{43} + R_{44} + R_L \right) sC_{31} + 1} \\ &= \frac{Z_{num}}{Z_{denom}} \end{aligned}$$

With expressions for $V_{in}$ and $V_{out}$ in terms of impedences, $H(s)$ is easily obtained:

$$\begin{aligned} H(s) &= \frac{V_{out}}{V_{in}} \\ &= \frac{V \frac{R_L}{R_{43} + R_L + \frac{1}{sC_{31}}}}{V \frac{Z + \frac{1}{sC_{32}}}{Z}} \end{aligned}$$

Node voltage $V$ drops out of the expression:

$$\begin{aligned} H(s) &= \frac{\frac{R_L}{R_{43} + R_L + \frac{1}{sC_{31}}}}{\frac{Z + \frac{1}{sC_{32}}}{Z}} \\ &= \left( \frac{R_L}{R_{43} + R_L + \frac{1}{sC_{32}}} \right) \left( \frac{Z}{Z + \frac{1}{sC_{32}}} \right) \end{aligned}$$

Avoiding compound fractions, simplifying, and collecting terms:

$$
\begin{aligned}
H(s) &= \left(\frac{R_L s C_{32}}{(R_{43}+R_L) s C_{32}+1}\right)\left(\frac{Z s C_{32}}{Z s C_{32}+1}\right) \\
&= \left(\frac{R_L s C_{32}}{(R_{43}+R_L) s C_{32}+1}\right)\left(\frac{\frac{Z_{num}}{Z_{denom}} s C_{32}}{\frac{Z_{num}}{Z_{denom}} s C_{32}+1}\right) \\
&= \left(\frac{R_L s C_{32}}{(R_{43}+R_L) s C_{32}+1}\right)\left(\frac{Z_{num} s C_{32}}{Z_{num} s C_{32}+Z_{denom}}\right) \\
&= \left(\frac{R_L s C_{32}}{(R_{43}+R_L) s C_{32}+1}\right)\left(\frac{R_{44}(R_{43}+R_L) s^2 C_{31} C_{32}+R_{44} s C_{32}}{R_{44}(R_{43}+R_L) s^2 C_{31} C_{32}+R_{44} s C_{32}+(R_{43}+R_{44}+R_L) s C_{31}+1}\right) \\
&= \left(\frac{R_L s C_{32}}{(R_{43}+R_L) s C_{32}+1}\right)\left(\frac{((R_{43}+R_L) s C_{32}+1)(R_{44} s C_{32})}{R_{44}(R_{43}+R_L) s^2 C_{31} C_{32}+R_{44} s C_{32}+(R_{43}+R_{44}+R_L) s C_{31}+1}\right) \\
&= R_L s C_{32} \frac{R_{44} s C_{31}}{R_{44}(R_{43}+R_L) s^2 C_{31} C_{32}+R_{44} s C_{32}+(R_{43}+R_{44}+R_L) s C_{31}+1} \\
&= \frac{R_{44} R_L C_{31} C_{32} s^2}{R_{44}(R_{43}+R_L) C_{31} C_{32} s^2+((R_{43}+R_{44}+R_L) C_{31}+R_{44} C_{32}) s+1}
\end{aligned}
$$

### 2.3 Coefficients

This transfer function is second-order in the numerator and second-order in the denominator (as would be expected for a circuit with two capacitors). The transfer function of a general second-order filter is:

$$H(s) = \frac{\beta_2 s^2 + \beta_1 s + \beta_0}{\alpha_2 s^2 + \alpha_1 s + \alpha_0}$$

The Casio SK-1 Percussion Filter can be described as a second-order filter with analog filter coefficients:

$$
\begin{aligned}
\beta_2 &= R_{44} R_L C_{31} C_{32} \\
\beta_1 &= 0 \\
\beta_0 &= 0 \\
\alpha_2 &= R_{44}(R_{43}+R_L) C_{31} C_{32} \\
\alpha_1 &= (R_{43}+R_{44}+R_L) C_{31} + R_{44} C_{32} \\
\alpha_0 &= 1
\end{aligned}
$$

## 3 The Bilinear Transform

### 3.1 Definition

Much like a transfer function in continuous time, a transfer function in discrete time can be written in terms of digital filter coefficients.

$$H(z) = \frac{b_0 + b_1 z^{-1} + b_2 z^{-2}}{a_0 + a_1 z^{-1} + a_2 z^{-2}}$$

The bilinear transform is used to transform from a continuous-time representation of a system to a discrete-time representation by substituting in an approximation of the trapezoidal rule for integration (where $T$ is the sampling period):

$$s = \frac{2}{T}\frac{1 - z^{-1}}{1 + z^{-1}}$$

Substituting this expression for each $s$ in the continuous-time representation of the transfer function $H(s)$ will give the discrete-time representation of the transfer function $H(z)$. This can be done manually, or with a symbolic manipulator. However, in the case of commonly used filters (such as the biquad), expressions for the digital filter coeffcients in terms of the analog filter coefficients are already available.

$$\begin{aligned}
b_0 &= 4\beta_2 + 2T\beta_1 + T^2\beta_0 \\
b_1 &= -8\beta_2 + 2\beta_0 T^2 \\
b_2 &= 4\beta_2 - 2T\beta_1 + T\beta_0 \\
a_0 &= 4\alpha_2 + 2T\alpha_1 + T^2\alpha_0 \\
a_1 &= -8\alpha_2 + 2T^2\alpha_0 \\
a_2 &= 4\alpha_2 - 2T\alpha_1 + T^2\alpha_0
\end{aligned}$$

For most implementations, $\alpha_0$ should equal 1, so the whole filter can be normalized by dividing by $\alpha_0$:

$$\begin{aligned}
b_0 &= \frac{4\beta_2 + 2T\beta_1 + T^2\beta_0}{4\alpha_2 + 2T\alpha_1 + T^2\alpha_0} \\
b_1 &= \frac{-8\beta_2 + 2T^2\beta_0}{4\alpha_2 + 2T\alpha_1 + T^2\alpha_0} \\
b_2 &= \frac{4\beta_2 - 2T\beta_1 + T\beta_0}{4\alpha_2 + 2T\alpha_1 + T^2\alpha_0} \\
a_0 &= 1 \\
a_1 &= \frac{-8\alpha_2 + 2T^2\alpha_0}{4\alpha_2 + 2T\alpha_1 + T^2\alpha_0} \\
a_2 &= \frac{4\alpha_2 - 2T\alpha_1 + T^2\alpha_0}{4\alpha_2 + 2T\alpha_1 + T^2\alpha_0}
\end{aligned}$$

## 3.2 Alternatives

The bilinear transform is only a first-order approximation of the exact transformation form the $s$-plane to the $z$-plane. Higher-order approximations may also be employed, or other methods such as impulse invariance.

# 4 The Discrete-Time Transfer Function

## 4.1 Bilinear Transform

Substituting the analog filter coefficients into the expressions for the digital filter coefficients, the digital filter coefficients are obtained in terms of the circuit's component values and the sampling period:

$$\begin{aligned} b_0 &= 4\beta_2 + 2T\beta_1 + \beta_0 T^2 \\ &= 4R_{44}R_L C_{31} C_{32} \end{aligned}$$

$$\begin{aligned} b_1 &= -8\beta_2 + 2\beta_0 T^2 \\ &= -8R_{44}R_L C_{31} C_{32} \end{aligned}$$

$$\begin{aligned} b_2 &= 4\beta_2 - 2T\beta_1 + \beta_0 T^2 \\ &= 4R_{44}R_L C_{31} C_{32} \end{aligned}$$

$$\begin{aligned} a_0 &= 4\alpha_2 + 2T\alpha_1 + T^2\alpha_0 \\ &= 4\left(R_{44}\left(R_{43}+R_L\right)C_{31}C_{32}\right) + 2T\left(\left(R_{43}+R_{44}+R_L\right)C_{31} + R_{44}C_{32}\right) + T^2 \end{aligned}$$

$$\begin{aligned} a_1 &= -8\alpha_2 + 2T^2\alpha_0 \\ &= -8\left(R_{44}\left(R_{43}+R_L\right)C_{31}C_{32}\right) + 2T^2 \end{aligned}$$

$$\begin{aligned} a_2 &= 4\alpha_2 - 2T\alpha_1 + T^2\alpha_0 \\ &= 4\left(R_{44}\left(R_{43}+R_L\right)C_{31}C_{32}\right) - 2T\left(\left(R_{43}+R_{44}+R_L\right)C_{31} + R_{44}C_{32}\right) + T^2 \end{aligned}$$

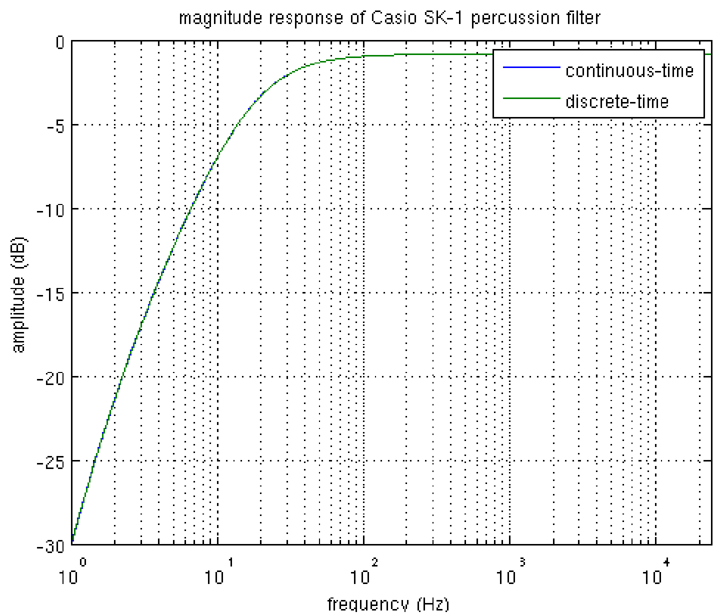
For most implementations, $\alpha_0$ should equal 1, so the whole filter can be normalized by dividing by $\alpha_0$:

$$\begin{aligned} b_0 &= \frac{4R_{44}R_L C_{31}C_{32}}{4\left(R_{44}\left(R_{43}+R_L\right)C_{31}C_{32}\right) + 2T\left(\left(R_{43}+R_{44}+R_L\right)C_{31} + R_{44}C_{32}\right) + T^2} \\ b_1 &= \frac{-8R_{44}R_L C_{31}C_{32}}{4\left(R_{44}\left(R_{43}+R_L\right)C_{31}C_{32}\right) + 2T\left(\left(R_{43}+R_{44}+R_L\right)C_{31} + R_{44}C_{32}\right) + T^2} \\ b_2 &= \frac{4R_{44}R_L C_{31}C_{32}}{4\left(R_{44}\left(R_{43}+R_L\right)C_{31}C_{32}\right) + 2T\left(\left(R_{43}+R_{44}+R_L\right)C_{31} + R_{44}C_{32}\right) + T^2} \end{aligned}$$

$$
\begin{aligned}
a_0 &= 1 \\
a_1 &= \frac{-8\left(R_{44}\left(R_{43}+R_L\right)C_{31}C_{32}\right)+2T^2}{4\left(R_{44}\left(R_{43}+R_L\right)C_{31}C_{32}\right)+2T\left(\left(R_{43}+R_{44}+R_L\right)C_{31}+R_{44}C_{32}\right)+T^2} \\
a_2 &= \frac{4\left(R_{44}\left(R_{43}+R_L\right)C_{31}C_{32}\right)-2T\left(\left(R_{43}+R_{44}+R_L\right)C_{31}+R_{44}C_{32}\right)+T^2}{4\left(R_{44}\left(R_{43}+R_L\right)C_{31}C_{32}\right)+2T\left(\left(R_{43}+R_{44}+R_L\right)C_{31}+R_{44}C_{32}\right)+T^2}
\end{aligned}
$$

## 4.2 Magnitude Response

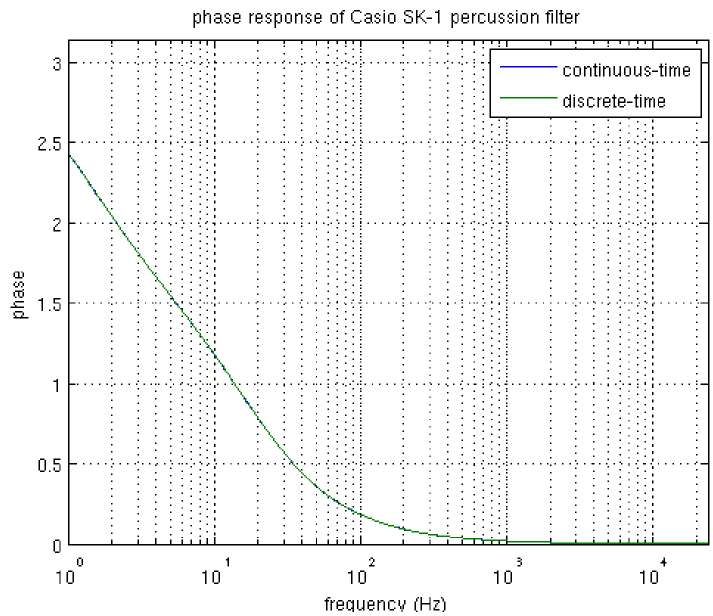
The magnitude response of the filter (with sampling rate $f_s = 48000\,Hz$) is plotted in both in continuous time and in discrete time.




The filter is clearly a high-pass filter. By inspection, the cutoff frequency is around $200\,Hz$ and the filter has a low-frequency roll-off of $12\,dB/octave$ or $40\,dB/decade$.

## 4.3 Phase Response

The phase response of the filter (with $f_s = 48000\,Hz$) is plotted in both in continuous time and in discrete time.




For both the magnitude and phase responses, there is no visible difference between the plots for the continuous-time and discrete-time filters, so the bilinear transform is assumed to be an approximation of high enough order to well-describe this system. Presumably, frequency warping is not an issue for low-order filters.

# 5 Parameterizing

## 5.1 Filter Parameters

It will be helpful to understand the Percussion Filter's behavior in terms of filter parameters ($Q$, $\omega_c$). This is easier done in continuous-time, so the analog filter coefficients are revisited:

$$\begin{aligned}
\beta_2 &= R_{44} R_L C_{31} C_{32} \\
\beta_1 &= 0 \\
\beta_0 &= 0 \\
\alpha_2 &= R_{44} \left( R_{43} + R_L \right) C_{31} C_{32} \\
\alpha_1 &= \left( R_{43} + R_{44} + R_L \right) C_{31} + R_{44} C_{32} \\
\alpha_0 &= 1
\end{aligned}$$

The Percussion Filter closely resembles the case of a resonant high-pass filter, as defined in the course notes for *MUS424/EE367D: Signal Processing Techniques for Digital Audio Effects* (Abel/Berners, 2011), whose transfer function is given as:

$$H(s) = \frac{\left(\frac{s}{\omega_c}\right)^2}{\left(\frac{s}{\omega_c}\right)^2 + \frac{1}{Q}\left(\frac{s}{\omega_c}\right) + 1}$$

And whose filter coefficients are:

$$\begin{aligned}
\beta_2 &= \left(\frac{1}{\omega_c}\right)^2 \\
\beta_1 &= 0 \\
\beta_0 &= 0 \\
\alpha_2 &= \left(\frac{1}{\omega_c}\right)^2 \\
\alpha_1 &= \frac{1}{Q\omega_c} \\
\alpha_0 &= 1
\end{aligned}$$

In this case, the denominator coefficients will be used to find the values for $Q$ and $\omega_c$ in terms of the component values, and the numerator coefficients will give the high-frequency gain.

## 5.2 Cutoff Frequency

First, solving for $\omega_c$ using the analog filter coefficient $\alpha_2$:

$$\begin{aligned}
\left(\frac{1}{\omega_c}\right)^2 &= R_{44}\left(R_{43} + R_L\right) C_{31} C_{32} \\
\omega_c &= \frac{1}{\sqrt{R_{44}\left(R_{43} + R_L\right) C_{31} C_{32}}}
\end{aligned}$$

$\omega_c$ is in radians, to get the cutoff frequency in Hertz ($f_c$), $\omega_c$ is multiplied by $2\pi$:

$$f_c = \frac{2\pi}{\sqrt{R_{44}\left(R_{43} + R_L\right) C_{31} C_{32}}}$$

Using the "reasonable" value of $1\,M\Omega$ for $R_L$, the cutoff frequency of the Percussion Filter is $189.45\,Hz$.

## 5.3 Gain at Center Frequency (Q)

Now, solving for $Q$ using the analog filter coefficient $\alpha_1$ and the expression for $\omega_c$:

$$\begin{aligned}
\frac{1}{Q\omega_c} &= (R_{43}+R_{44}+R_L)\,C_{31}+R_{44}C_{32} \\
Q &= \frac{1}{((R_{43}+R_{44}+R_L)\,C_{31}+R_{44}C_{32})\,\omega_c} \\
&= \frac{\sqrt{R_{44}\,(R_{43}+R_L)\,C_{31}C_{32}}}{(R_{43}+R_{44}+R_L)\,C_{31}+R_{44}C_{32}}
\end{aligned}$$

Using the "reasonable" value of $1\,M\Omega$ for $R_L$, the value of $Q$ is 0.2551.

### 5.4 High-Frequency Gain

The high-frequency gain is found as $s \to \infty$. As $s$ gets large, the $s^1$ and $s^0$ terms in the numerator and denominator cease to contribute to the value of $H(s)$:

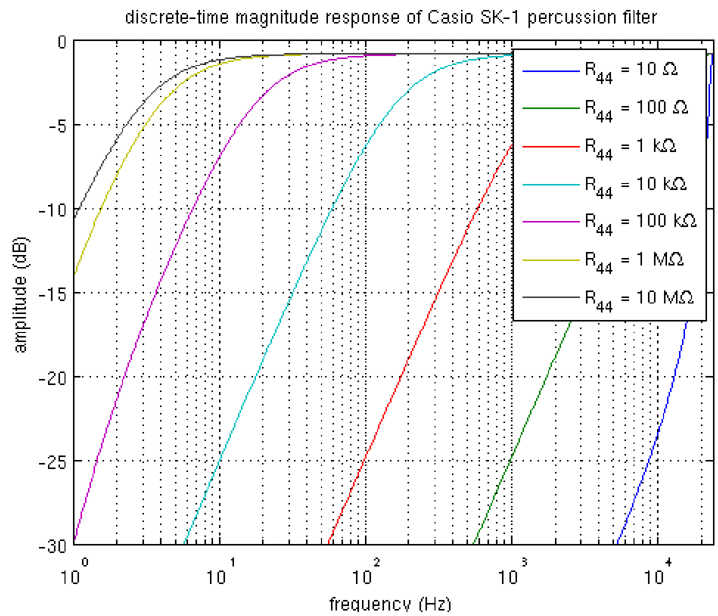
$$\begin{aligned}
\lim_{s\to\infty} H(s) &= \frac{\beta_2}{\alpha_2} \\
&= \frac{R_{44}R_LC_{31}C_{32}}{R_{44}\,(R_{43}+R_L)\,C_{31}C_{32}} \\
&= \frac{R_L}{R_{43}+R_L}
\end{aligned}$$

The high frequency gain depends on $R_{43}$ and $R_L$. For large values of $R_L$, it approaches unity. For smaller values, it is dependent on $R_{43.}$ Using the "reasonable" value of $1\,M\Omega$ for $R_L$, the high-frequency gain of the Percussion Filter is $0.\overline{90}$.

### 5.5 "Circuit Bending"

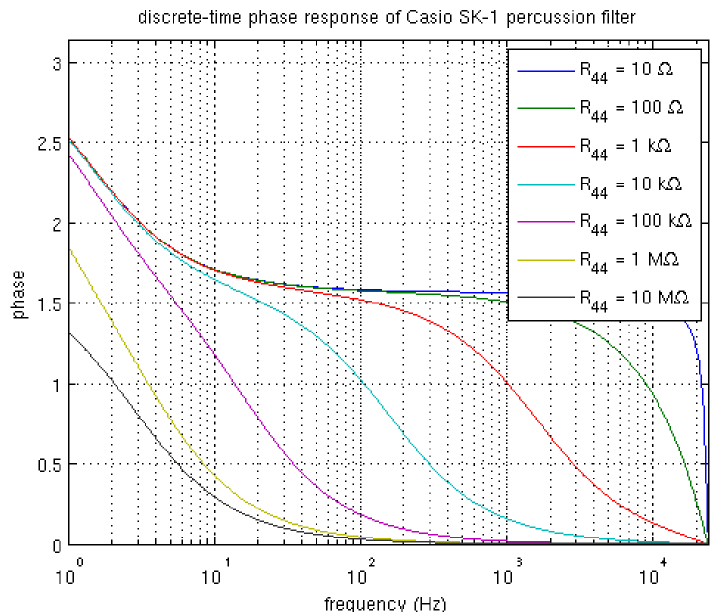
The Casio SK-1 Percussion Filter cirucit has static component values. However, it may be desirable to modify the circuit to allow some of the parameters (for instance $\omega_c, Q$) to be controlled (essentially, a digital implementation of circuit-bending). By experimentation, it is found that changing any of the component values of the filter (except resistor $R_{45}$) will affect the magnitude and phase response of the filter, but that changing the value of Resistor $R_{44}$ in particular has the result of changing the cutoff frequency of the filter.

The magnitude response of the filter (with $f_s = 48000\,Hz$) is plotted in discrete time, with the resistance of $R_{44}$ varying between $10\,\Omega$ and $10\,M\Omega$.




By inspection, a ten-fold increase in the resistance of $R_{44}$ generally corresponds to a ten-fold decrease of the cutoff frequency of the filter, and a ten-fold decrease in the resistance of $R_{44}$ generally corresponds to a ten-fold increase of the cutoff frequency. This behavior is somewhat different around DC and the Nyquist frequency.

The phase response of the filter (with $f_s = 48000\,Hz$) is plotted in discrete time, with the resistance of $R_{44}$ varying between $10\,\Omega$ and $10\,M\Omega$.

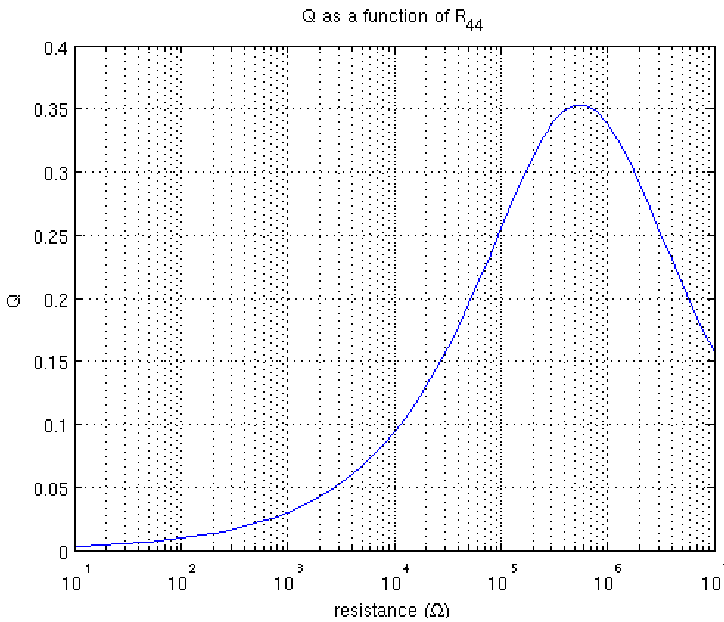



It will be useful to find the value of resistor $R_{44}$ that is required to produce a filter with a desired cutoff frequency. The equation for the cutoff frequency in Hertz ($f_c$) is revisited, solving for $R_{44}$ in terms of the desired cutoff frequency and component values:

$$\begin{aligned} f_c &= \frac{2\pi}{\sqrt{R_{44}\left(R_{43}+R_L\right)C_{31}C_{32}}} \\ f_c^2 &= \frac{4\pi^2}{R_{44}\left(R_{43}+R_L\right)C_{31}C_{32}} \\ R_{44} &= \frac{4\pi^2}{f_c^2\left(R_{43}+R_L\right)C_{31}C_{32}} \end{aligned}$$

This allows the filter model to include a control for the center frequency of the filter, which is analagous to replacing resistor $R_{44}$ in the physical circuit with a potentiometer - *a digital implementation of circuit-bending*!

This will also affect the value of $Q$, subtly changing the magnitude response of the filter around the cutoff frequency (keep in mind, the original value of $Q$ is 0.2551, corresponding to $R_{44} = 100\,k\Omega$):




However, since the value of $Q$ never gets above 1.0, this effect is not very noticable.

## 6 Implementation

A test implementation of the Casio SK-1 Percussion Filter is made in "PercussionFilter.vst," a Steinberg Virtual Studio Technology plugin. The plugin has a control for cuttoff frequency that will select the proper value of $R_{44}$ for the desired cutoff frequency (defaulting to $189.45\,Hz$, corresponding to $R_{44} = 100\,k\Omega$). The transfer function $H\left(z\right)$ is implemented mostly as it is described in this write-up, with the biquad filter arranged in a Direct Form II Transposed topology. The only difference is that smoothing (via an RMS level estimator with a very short time constant, $\tau = 10\,milliseconds$) is done on the cutoff frequency to avoid "pops" when the parameter is changed.