Sinusoidal Modulation of Sinusoids*

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1.1 Sinusoid Magnitude Spectra

A sinusoid's frequency content may be graphed in the *frequency domain* as shown in Fig. 1.

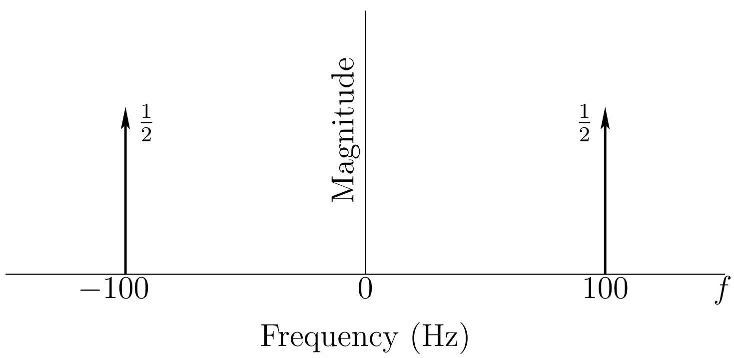

Figure 1: Spectral magnitude representation of a unit-amplitude sinusoid at frequency 100 Hz such as $\cos(200\pi t)$ or $\sin(200\pi t)$. (Phase is not shown.)

An example of a particular sinusoid graphed in Fig. 1 is given by

$$x(t) = \cos(\omega_x t) = \frac{1}{2}e^{j\omega_x t} + \frac{1}{2}e^{-j\omega_x t}$$

where

$$\omega_x = 2\pi 100.$$

That is, this sinusoid has amplitude 1, frequency 100 Hz, and phase zero (or $\pi/2$, if $\sin(\omega_x t)$ is defined as the zero-phase case).

*This is a new section slated for the second edition of **Mathematics of the DFT** [7].

Figure 1 can be viewed as a graph of the *magnitude spectrum* of $x(t)$, or its *spectral magnitude representation* [4]. Note that the spectrum consists of two components with amplitude $1/2$, one at frequency 100 Hz and the other at frequency -100 Hz.

Phase is not shown in Fig. 1 at all. The phase of the components could be written simply as labels next to the magnitude arrows, or the magnitude arrows can be rotated "into or out of the page" by the appropriate phase angle as illustrated in [7, Fig. 4.8 on p. 43].

1.2 Sinusoidal Amplitude Modulation (AM)

It is instructive to study the *modulation* of one sinusoid by another. In this section, we will look at sinusoidal *Amplitude Modulation (AM)*. The general AM formula is given by

$$x_\alpha(t) = [1 + \alpha \cdot a_m(t)] \cdot A_c \sin(\omega_c t + \phi_c),$$

where (A_c, ω_c, ϕ_c) are parameters of the sinusoidal *carrier wave*, $\alpha \in [0, 1]$ is called the *modulation index* (or *AM index*), and $a_m(t) \in [-1, 1]$ is the *amplitude modulation signal*. In AM radio broadcasts, $a_m(t)$ is the audio signal being transmitted (usually bandlimited to less than 10 kHz), and ω_c is the channel center frequency that one dials up on a radio receiver. The modulated signal $x_\alpha(t)$ can be written as the sum of the unmodulated carrier wave plus the product of the carrier wave and the modulating wave:

$$x_\alpha(t) = x_0(t) + \alpha \cdot a_m(t) \cdot A_c \sin(\omega_c t + \phi_c) \tag{1}$$

In the case of *sinusoidal* AM, we have

$$a_m(t) = \sin(\omega_m t + \phi_m). \tag{2}$$

Periodic amplitude modulation of this nature is often called the *tremolo effect* when $\omega_m < 20\pi$ or so ($<$ 10 Hz).

Let's analyze the second term of Eq. (1) for the case of sinusoidal AM with $\alpha = 1$ and $\phi_m = \phi_c = 0$:

$$x_m(t) \stackrel{\Delta}{=} \sin(\omega_m t) \sin(\omega_c t) \tag{3}$$

An example waveform is shown in Fig. 2 for $f_c = 100$ Hz and $f_m = 10$ Hz. Such a signal may be produced on an analog synthesizer by feeding two differently tuned sinusoids to a *ring modulator*, which is simply a "four-quadrant multiplier" for analog signals.

When ω_m is small (say less than 20π radians per second, or 10 Hz), the signal $x_m(t)$ is heard as a "beating sine wave" with $\omega_m/\pi = 2f_m$ beats per second. The beat rate is twice the modulation frequency because both the positive and negative peaks of the modulating sinusoid cause an "amplitude swell" in $x_m(t)$. (One period of modulation—$1/f_m$ seconds—is shown in Fig. 2.) The sign inversion during the negative peaks is not normally audible.

Recall the trigonometric identity for a sum of angles:

$$\cos(A + B) = \cos(A)\cos(B) - \sin(A)\sin(B)$$

Subtracting this from $\cos(A - B) = \cos(A)\cos(B) + \sin(A)\sin(B)$ leads to the identity

$$\sin(A)\sin(B) = \frac{\cos(A - B) - \cos(A + B)}{2}.$$

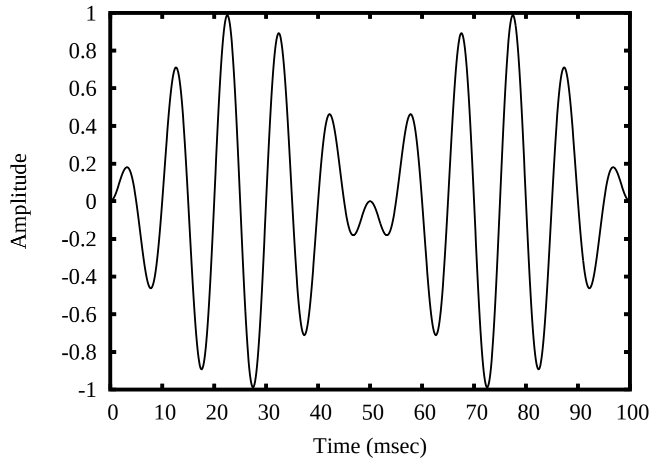

Figure 2: Sinusoidal amplitude modulation as in Eq. (3)—time waveform.

Setting $A = \omega_m t$ and $B = \omega_c t$ gives us an alternate form for our "ring-modulator output signal":

$$x_m(t) \triangleq \sin(\omega_m t)\sin(\omega_c t) = \frac{\cos[(\omega_m - \omega_c)t] - \cos[(\omega_m + \omega_c)t]}{2} \tag{4}$$

These two sinusoidal components at the *sum and difference frequencies* of the modulator and carrier are called *side bands* of the carrier wave at frequency ω_c (since typically $\omega_c \gg \omega_m > 0$).

Equation (3) expresses $x_m(t)$ as a "beating sinusoid", while Eq. (4) expresses as it two *unmodulated* sinusoids at frequencies $\omega_c \pm \omega_m$. Which case do we hear?

It turns out we hear $x_m(t)$ as two separate tones (Eq. (4)) whenever the side bands are *resolved* by the ear. As mentioned in [7, Section 4.1.2], the ear performs a "short time Fourier analysis" of incoming sound (the basilar membrane in the cochlea acts as a mechanical filter bank). The *resolution* of this filterbank—its ability to discern two separate spectral peaks for two sinusoids closely spaced in frequency—is determined by the *critical bandwidth of hearing* [5, 9, 11]. A critical bandwidth is roughly 15-20% of the band's center-frequency, over most of the audio range [8]. Thus, the side bands in sinusoidal AM are heard as separate tones when they are both in the audio range and separated by at least one critical bandwidth. When they are well inside the same critical band, "beating" is heard. In between these extremes, near separation by a critical-band, the sensation is often described as "roughness" [3].

1.2.1 Example AM Spectra

Equation (4) can be used to write down the spectral representation of $x_m(t)$ by inspection, as shown in Fig. 3. In the example of Fig. 3, we have $f_c = 100$ Hz and $f_m = 20$ Hz, where, as always, $\omega = 2\pi f$. For comparison, the spectral magnitude of an *unmodulated* 100 Hz sinusoid is shown in Fig. 1 on page 1. Note in Fig. 3 how each of the two sinusoidal components at ± 100 Hz have been "split" into two "side bands", one 20 Hz higher and the other 20 Hz lower, that is, $\pm 100 \pm 20 = \{-120, -80, 80, 120\}$. Note also how the *amplitude* of the split component is divided equally among its two side bands.

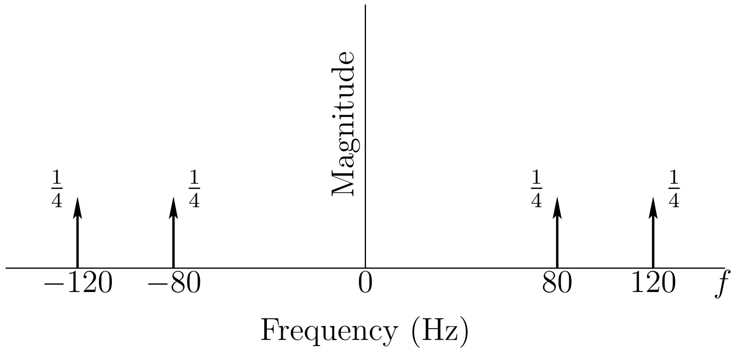

Figure 3: Spectral magnitude representation of the sinusoidally modulated sinusoid $\sin(40\pi t)\sin(200\pi t)$ defined in Eq. (3). Phase is not shown.

Recall that $x_m(t)$ was defined as the *second term* of Eq. (1). The first term is simply the original unmodulated signal. Therefore, we have effectively been considering AM with a "very large" modulation index. In the more general case of Eq. (1) with $a_m(t)$ given by Eq. (2), the magnitude of the spectral representation appears as shown in Fig. 4.

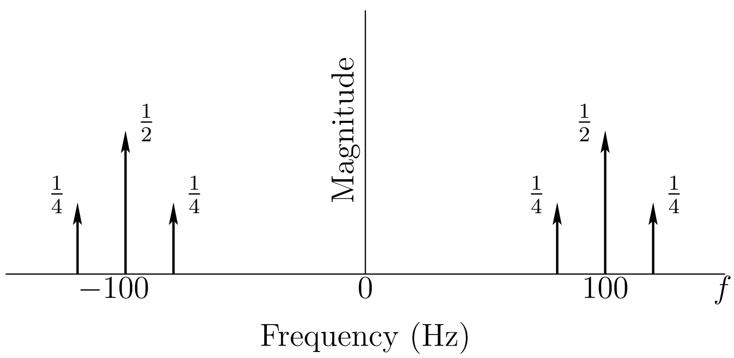

Figure 4: Spectral representation of the sinusoidally modulated sinusoid $[1+\sin(40\pi t)]\sin(200\pi t)$ from Eq. (1), with $\alpha = 1$, and $a_m(t)$ given by Eq. (2).

1.3 Sinusoidal Frequency Modulation (FM)

Frequency Modulation (FM) is well known as the broadcast signal format for FM radio. It is also the basis of the first commercially successful method for digital sound synthesis. Invented by John Chowning [1], it was the method used in the the highly successful Yamaha DX-7 synthesizer, and later the Yamaha OPL chip series, which was used in all “SoundBlaster compatible” multimedia sound cards for many years. At the time of this writing, descendants of the OPL chips remain the dominant synthesis technology for “ring tones” in cellular telephones.

A general formula for frequency modulation of one sinusoid by another can be written as

$$x(t) = A_c \cos[\omega_c t + \phi_c + A_m \sin(\omega_m t + \phi_m)], \tag{5}$$

where the parameters (A_c, ω_c, ϕ_c) describe the *carrier sinusoid*, while (A_m, ω_m, ϕ_m) specify the *modulator* sinusoid. Note that, strictly speaking, it is not the frequency of the carrier that is modulated sinusoidally, but rather the *instantaneous phase* of the carrier. Therefore, *phase modulation* would be a better term (which is in fact used). Potential confusion aside, any modulation of phase implies a modulation of frequency, and vice versa, since the instantaneous frequency is always defined as the time-derivative of the instantaneous phase. In this course, only phase modulation will be considered, and we will call it FM, following common practice.[1]

It is well known that sinusoidal FM has a harmonic spectrum with harmonic amplitudes given by *Bessel functions* of the first kind [1]. We will derive this in the next section.[2]

1.3.1 Bessel Functions

The *Bessel functions of the first kind* may be defined as the coefficients $J_k(\beta)$ in the two-sided *Laurent expansion* of the so-called *generating function* [10, p. 14],[3]

$$e^{\frac{1}{2}\beta\left(z-\frac{1}{z}\right)} = \sum_{k=-\infty}^{\infty} J_k(\beta) z^k \tag{6}$$

where k is the integer *order* of the Bessel function, and β is its argument (which can be complex, but we will only consider real β). Setting $z = e^{j\omega_m t}$, where ω_m will interpreted as the *FM modulation frequency* and t as time in seconds, we obtain

$$x_m(t) \triangleq e^{j\beta\sin(\omega_m t)} = \sum_{k=-\infty}^{\infty} J_k(\beta) e^{jk\omega_m t}. \tag{7}$$

[1] An important variant of FM called *feedback FM*, in which a single oscillator phase-modulates itself, simply does not work if true frequency modulation is implemented.

[2] The mathematical derivation of FM spectra is included here as a side note. No further use will be made of it in this course.

[3] Existence of the Laurent expansion follows from the fact that the generating function is a product of an exponential function, $\exp(\beta z/2)$, and an exponential function inverted with respect to the unit circle, $\exp(-0.5\beta/z)$. It is readily verified by direct differentiation in the complex plane that the exponential is an *entire function* of z (analytic at all finite points in the complex plane) [2], and therefore the inverted exponential is analytic everywhere except at $z = 0$. The desired Laurent expansion may be obtained, in principle, by multiplying one-sided series for the exponential and inverted exponential together. The exponential series has the well known form $\exp(z) = 1 + z + z^2/2! + z^3/3! + \cdots$. The series for the inverted exponential can be obtained by inverting again ($z \leftarrow 1/z$), obtaining the appropriate exponential series, and inverting each term.

The last expression can be interpreted as the Fourier superposition of the sinusoidal harmonics of $\exp[j\beta\sin(\omega_m t)]$, *i.e.*, an *inverse Fourier series* sum. In other words, $J_k(\beta)$ is the amplitude of the kth harmonic in the Fourier-series expansion of the periodic signal $x_m(t)$.

Note that $J_k(\beta)$ is real when β is real. This can be seen by viewing Eq. (6) as the product of the series expansion for $\exp[(\beta/2)z]$ times that for $\exp[-(\beta/2)/z]$ (see footnote pertaining to Eq. (6)).

Figure 5 illustrates the first eleven Bessel functions of the first kind for arguments up to $\beta = 30$. It can be seen in the figure that when the FM index β is zero, $J_0(0) = 1$ and $J_k(0) = 0$ for all $k > 0$. Since $J_0(\beta)$ is the amplitude of the carrier frequency, there are no side bands when $\beta = 0$. As the FM index increases, the sidebands begin to grow while the carrier term diminishes. This is how FM synthesis produces an expanded, brighter bandwidth as the FM index is increased.

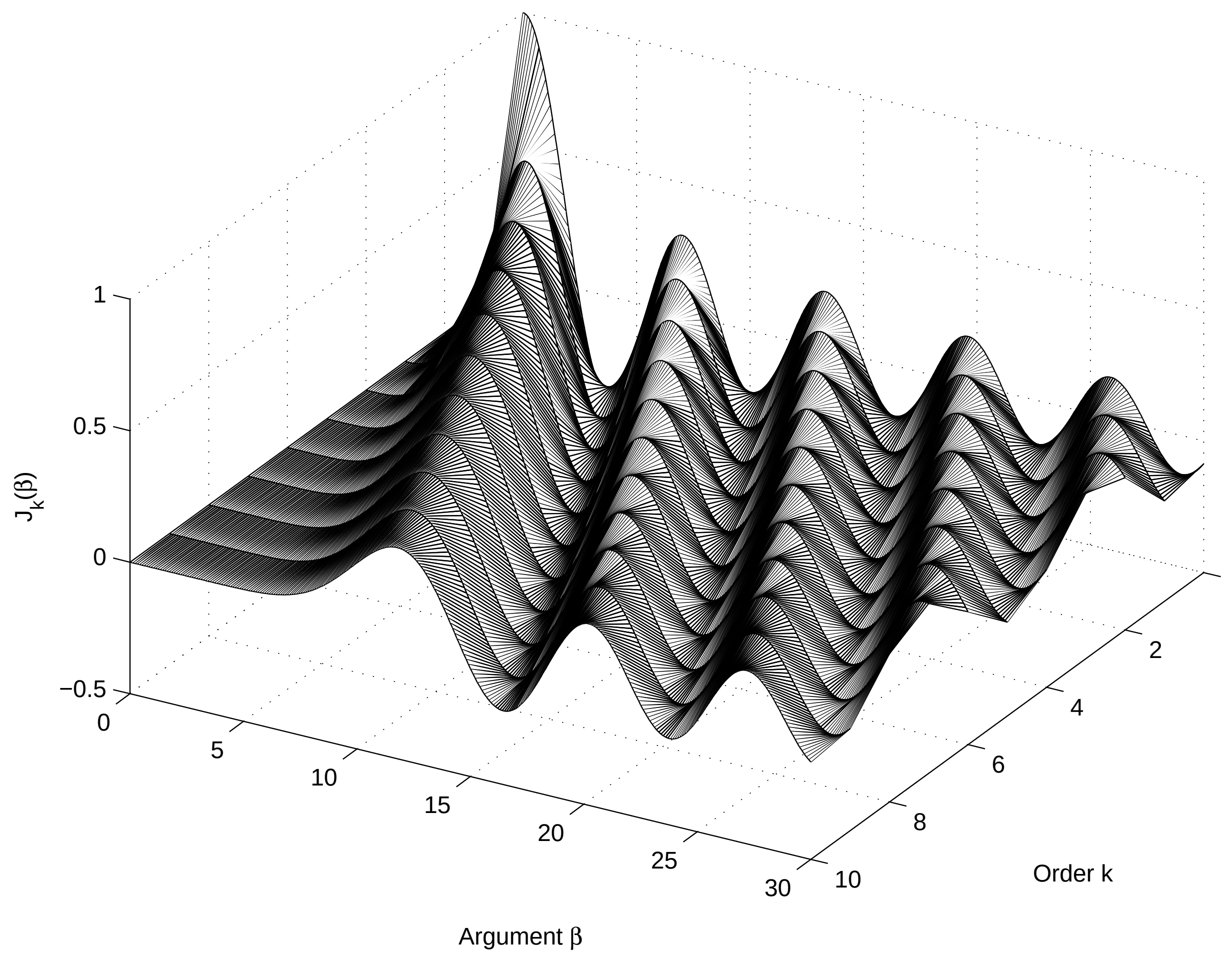

Figure 5: Bessel functions of the first kind for a range of orders k and argument β.

1.3.2 FM Spectra

Using the expansion in Eq. (7), it is now easy to determine the spectrum of sinusoidal FM. Eliminating scaling and phase offsets for simplicity in Eq. (5) yields

$$x(t) = \cos[\omega_c t + \beta \sin(\omega_m t)]. \tag{8}$$

where we have changed the modulator amplitude A_m to the more traditional symbol β, called the *FM index* in FM sound synthesis contexts. Using *phasor analysis*,

$$\begin{aligned}
x(t) &= \mathrm{re}\left\{e^{j[\omega_c t + \beta \sin(\omega_m t)]}\right\} \\
&= \mathrm{re}\left\{e^{j\omega_c t} e^{j\beta \sin(\omega_m t)}\right\} \\
&= \mathrm{re}\left\{e^{j\omega_c t} \sum_{k=-\infty}^{\infty} J_k(\beta) e^{jk\omega_m t}\right\} \\
&= \mathrm{re}\left\{\sum_{k=-\infty}^{\infty} J_k(\beta) e^{j(\omega_c + k\omega_m)t}\right\} \\
&= \sum_{k=-\infty}^{\infty} J_k(\beta) \cos[(\omega_c + k\omega_m)t]
\end{aligned} \tag{9}$$

where we used the fact that $J_k(\beta)$ is real when β is real. We can now see clearly that the sinusoidal FM spectrum consists of an infinite number of side-bands about the carrier frequency ω_c (when $\beta \neq 0$). The side bands occur at multiples of the modulating frequency ω_m away from the carrier frequency ω_c

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