Efficient computational modeling of piano strings for real-time synthesis using mass-spring chains, coupled finite differences, and digital waveguide sections

Smith, Kuroda, Perng, Van Heusen, Abel
CCRMA, Stanford University

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Goals

- **Overall Goal:** *Ultimate Virtual Piano*
- **Current Focus:** *Audibly Perfect Piano-String Synthesis*
- **Method:** *Start with High Accuracy*, then *Simplify*

- Goals
- Outline

String Synth History

Mass-Spring Model

Finite Differences

Outline

- Brief History of Virtual Strings
  - Ideal Strings
  - Stiff Piano Strings
  - Nonlinear Piano Strings
- General Mass-Spring String Model
- Finite Difference Implementation

String Synth History
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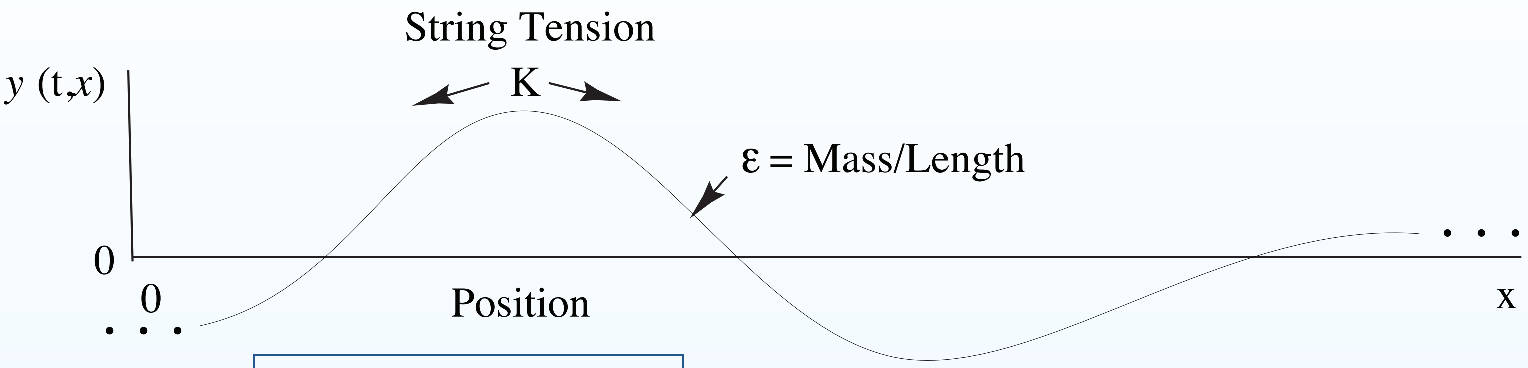
Finite Differences

Brief History of String Sound Synthesis

D'Alembert's PDE for the Ideal String (1747)



$$\boxed{K\frac{\partial^2 y}{\partial x^2} = \epsilon\frac{\partial^2 y}{\partial t^2}}$$

i.e.,

$$\boxed{Ky'' = \epsilon\ddot{y}}$$

$K \triangleq$ string tension

$\epsilon \triangleq$ linear mass density

$t =$ time (s)

$y \triangleq y(t,x)$

$=$ transverse displacement

$x =$ position along string axis

Wave equation PDE is derived from Newton's second law ($f = ma$) by equating Brook Taylor's "restoring force" Ky'' (1713) to mass-density times acceleration $\epsilon\ddot{y}$

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- Ideal String
- Traveling Waves
- Digital Waveguide
- Plucked String
- Struck String
- Stiff Strings
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Finite Differences

D’Alembert’s Traveling-Wave Solution (1747)

- In the same paper, d’Alembert also showed that the ideal-string PDE was satisfied by *traveling waves* in either direction:

$$y(t,x) \;=\; y_r\left(t-\frac{x}{c}\right)+y_l\left(t+\frac{x}{c}\right)$$

where

$$c \;=\; \sqrt{\frac{K}{\epsilon}} \;=\; \text{wave propagation speed}$$

- These ideas were developed into essentially modern form by Euler (1707–1783)

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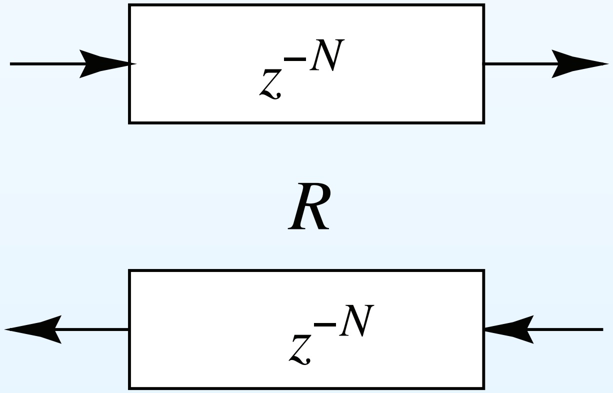
Digital Waveguide Models (1985)

We *sample* d'Alembert's traveling-wave components to make *digital waveguide models.*

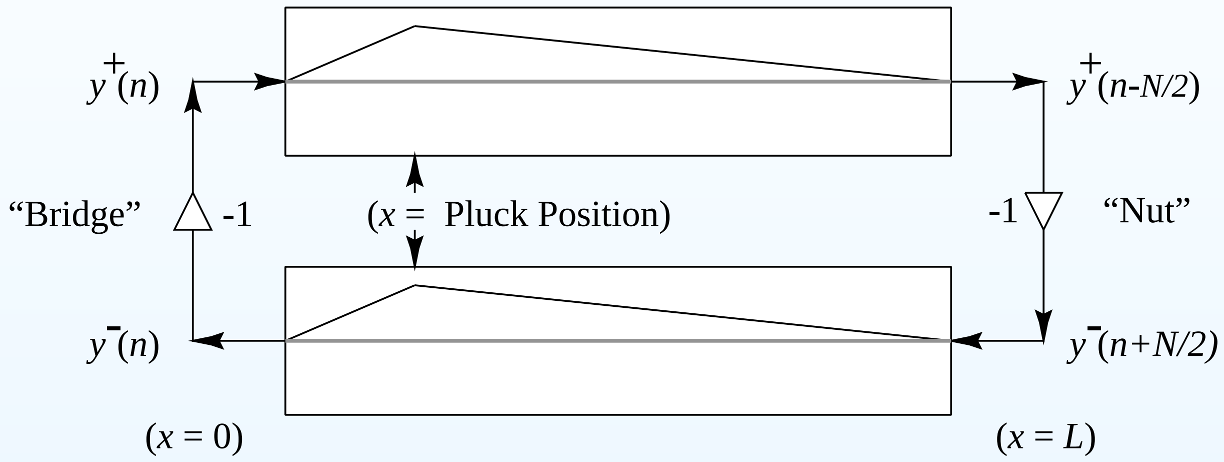
Lossless digital waveguide $\stackrel{\Delta}{=}$ *bidirectional delay line*
at some *wave impedance* $R = \sqrt{K\epsilon}$



Useful for *efficient* models of strings, bores, plane waves, conical waves, and more

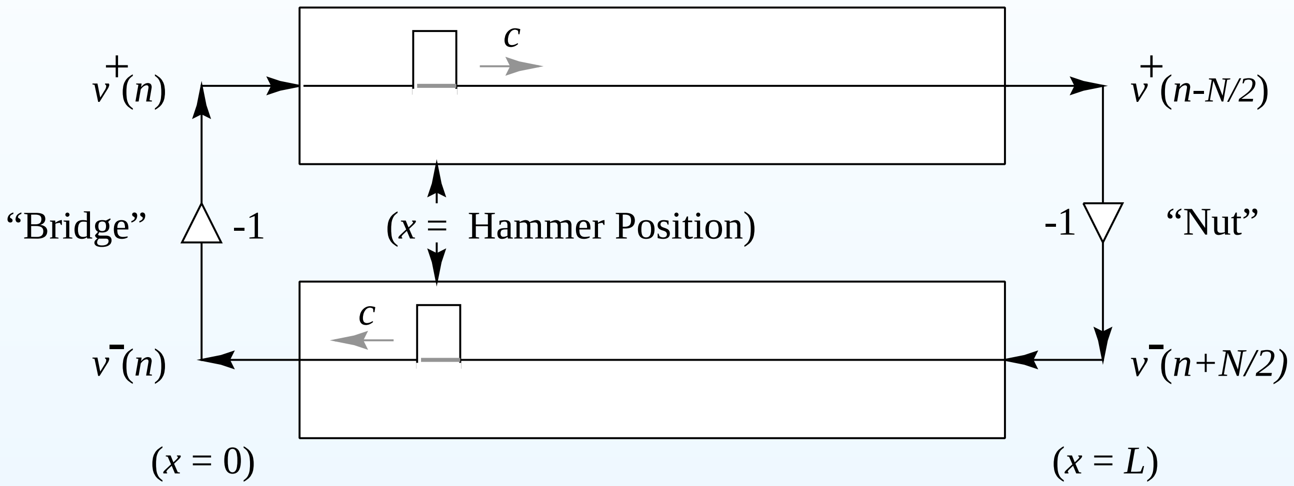
Ideal Plucked String (Displacement Waves)

Digital waveguide string models are *extremely efficient* computationally [$\mathcal{O}(1)$ complexity per sample of output]



- Load each delay line with *half* of initial string displacement
- Sum of upper and lower delay lines = string displacement
- Insert a linear, time-invariant filter in the loop for any desired *attenuation* and *dispersion* as a function of frequency

Ideal Struck String (Velocity Waves)



Velocity waves v easily converted to *force* waves f which are proportional to *string slope* y':

$$-Ky' \overset{\Delta}{=} f = f^+ + f^- = Rv^+ - Rv^-$$

Plucked/Struck Sound Example (1988)

Bach A-Minor Concerto—Orchestra Part: (WAV) (MP3)

- Executed in real time on one Motorola DSP56001 (20 MHz clock, 128K SRAM)

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- Executed in real time on one Motorola DSP56001 (20 MHz clock, 128K SRAM)
- Developed for the NeXT Computer introduction at Davies Symphony Hall, San Francisco, 1988
- Solo violin part was played live by Dan Kobialka of the San Francisco Symphony

Stiff Strings

- Piano strings are *stiff*, resulting in significant *dispersion*:

$$\epsilon\ddot{y} = Ky'' - \kappa y'''' \qquad \text{(Stiff string PDE)}$$

The new “stiffness term” is proportional to

$$y'''' \triangleq \frac{\partial^4}{\partial x^4} y(t,x) \qquad \Rightarrow$$

1. *Faster wave propagation at higher frequencies* (hence the “dispersion” of traveling-wave shapes)
    - Provided in digital waveguide strings using an *allpass filter* having a delay that decreases with frequency (typically on the order of 10 poles and zeros)
2. A *spring-resistance to corner formation* in the string (usually left out in string synthesis models!) See, *e.g.*, Cremer 1984

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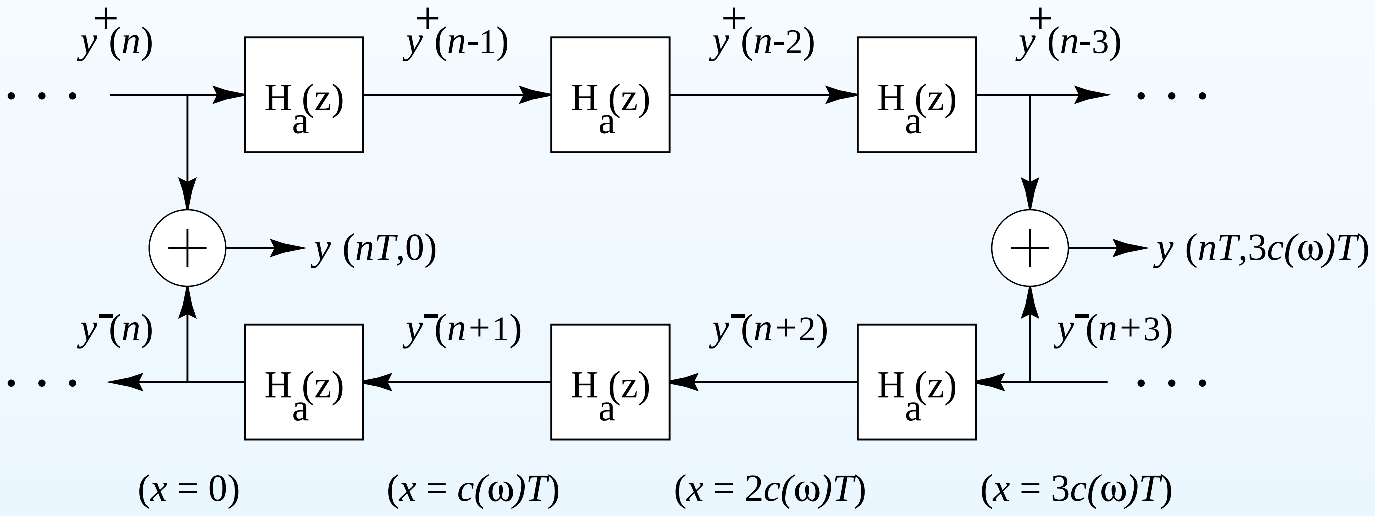
Digital Waveguide Model for Stiff Strings

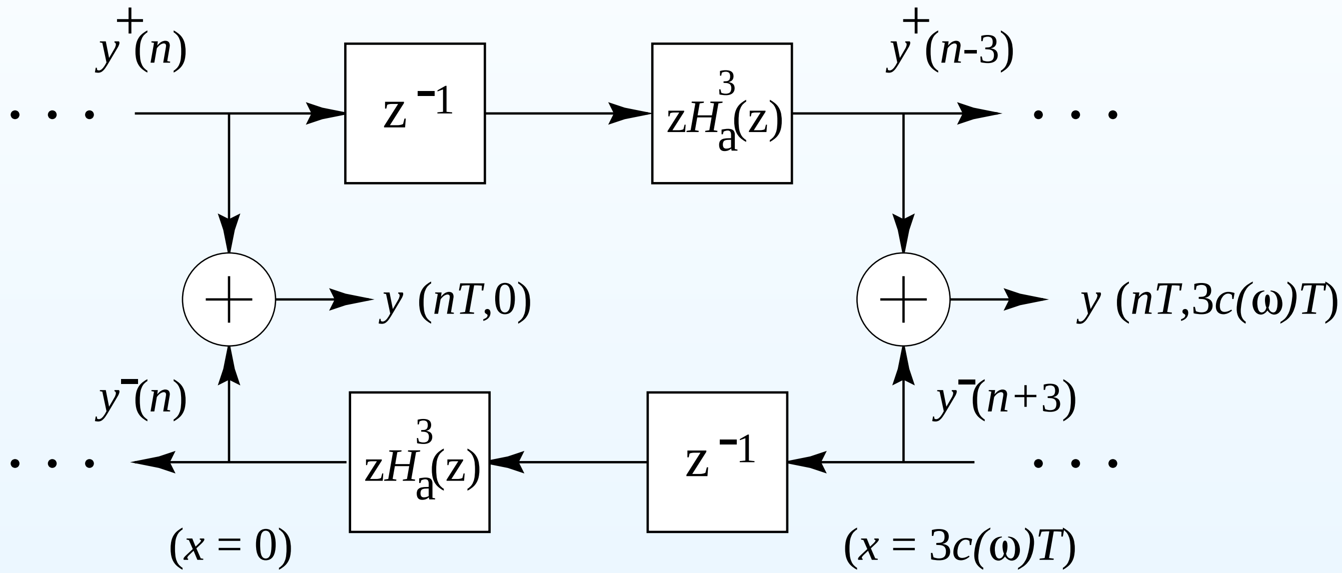
Example waveguide string, three spatial-samples long:



$H_a(z) =$ allpass giving desired delay vs. frequency for one sample

Consolidated Allpasses

In practice, we pull out one or more samples of pure delay:



- $zH_a^3(z)$ (or $z^2H_a^6(z)$) is designed as a single allpass filter
- ≈ 10 poles and zeros can handle hundreds of samples of dispersion with perceptual equivalence
- Not valid for distributed nonlinear behavior (discussed later)

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Digital Waveguide Piano Sound Example

- Piano: (WAV) (MP3) (Stanford Sondius Project, ca. 1995)

Digital Waveguide Piano Sound Example

- Piano: (WAV) (MP3) (Stanford Sondius Project, ca. 1995)
- Uses the *commuted synthesis* technique

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Nonlinear Piano Strings

- Vertical excitation by hammer *couples nonlinearly* to the *longitudinal* direction:[1]

$$\epsilon\ddot{\xi} = ES\xi'' + \frac{1}{2}ES[(y')^2]' \qquad \text{(longitudinal wave PDE)}$$

where $\xi =$ *longitudinal displacement*

- Nonlinear coupling driven by

$$[(y')^2]' \triangleq \frac{\partial}{\partial x}\left[\frac{\partial}{\partial x}y(t,x)\right]^2$$

- Slope of *square* of string slope drives longitudinal waves
- Coupling from longitudinal back to transverse is smaller and typically *neglected*

---

[1] *E.g.*, Morse & Ingard 1968

Effects of Nonlinear Mode Coupling in Piano Strings

Main *audible effects* of nonlinear transverse-to-longitudinal coupling:

1. Initial *longitudinal attack pulse*
   (the initial “shock noise” audible in a piano tone)

2. Inharmonic longitudinal modes

3. *“Phantom partials”*
   (ongoing intermodulation products from transverse partials)

See Conklin lecture in “Five Lectures on the Acoustics of the Piano,” edited by Anders Askenfelt (and listen to the sound examples)

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Nonlinear Piano-String Synthesis

If one-way coupling is accurate enough (transverse to longitudinal but *not* vice versa), we can model its effects *separately* based on *observations* of transverse waves (Bank and Sujbert, JASA 2005):

- Longitudinal modes implemented as *second-order resonators* ("modal synthesis")
- Slope of squared slope *projected* onto longitudinal modes

Up-to-date summary:

"A Modal-Based Real-Time Piano Synthesizer"
Balázs Bank, Stefano Zambon, and Federico Fontana
IEEE Trans. Audio, Speech, and Language Processing
May, 2010 (IEEE-ASLP)

Synthesis Strategies

Model complexity grows with dynamic level:

1. Linear Superposition: Transverse and longitudinal waveguides *decouple* into separate modes
2. Transverse → Longitudinal Coupling
3. Transverse ↔ Longitudinal Coupling

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3. Transverse $\leftrightarrow$ Longitudinal Coupling

Proposed Synthesis Strategy:

- Initial striking force determines the starting regime (1, 2, or 3)
- Maximum slope $|y'|$ over one or more periods indicates regime transitions
- String model simplifies as it decays

Checking the Approximations

The preceding involved *several* approximations:

- Neglected terms in PDEs
- Simplified synthesis models

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The following questions naturally arise:

- How do we know for sure our approximations are inaudible?
- We can listen, but could we miss an audible effect?
- Could a difference become audible after more listening?

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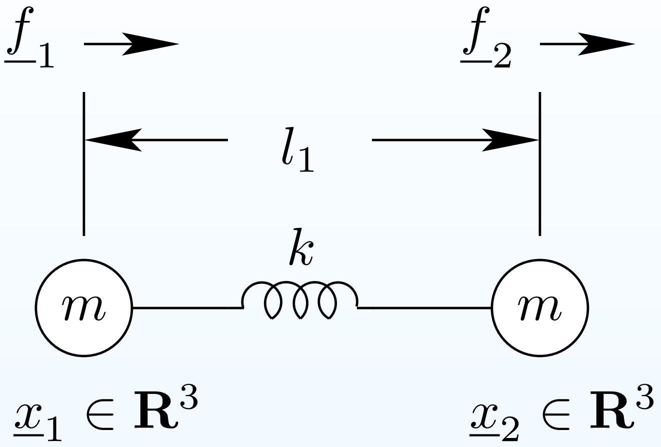
- There are software tools (*e.g.*, from *perceptual audio coding*) for measuring the *audible equivalence* of two sounds:
    - “Original” and “Encoded” for a CODEC
    - “Exact” and “Computationally Efficient” for piano models

Mass-Spring String Model

Mass-Spring Model in 3D Space



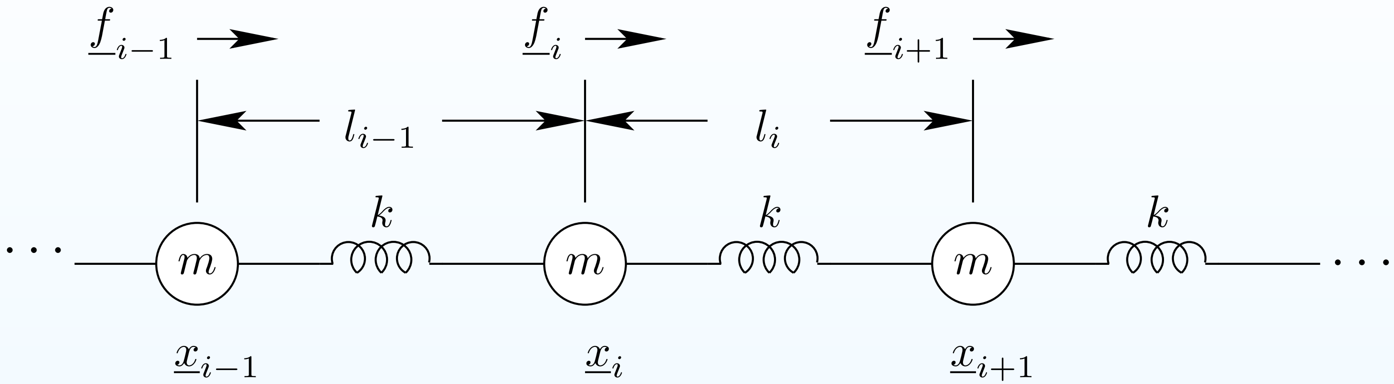
- **Hooke's Law:**

$$\|\,\underline{f}_1\,\| = k\cdot|l_1 - l_0| \qquad (l_0 = \text{spring rest length})$$

- **Vector Equation of Motion** $\left(\underline{f}_i \in \mathbf{R}^3, \quad \underline{x}_i \in \mathbf{R}^3\right)$:

$$\begin{aligned} \underline{f}_1 &= k\cdot\left(\|\,\underline{x}_2-\underline{x}_1\,\| - l_0\right)\cdot\frac{\underline{x}_2-\underline{x}_1}{\|\,\underline{x}_2-\underline{x}_1\,\|} \\ &= k\left[1-\frac{l_0}{\|\,\underline{x}_2-\underline{x}_1\,\|}\right](\underline{x}_2-\underline{x}_1) \;=\; m_1\,\ddot{\underline{x}}_1 \end{aligned}$$

Mass-Spring String Model

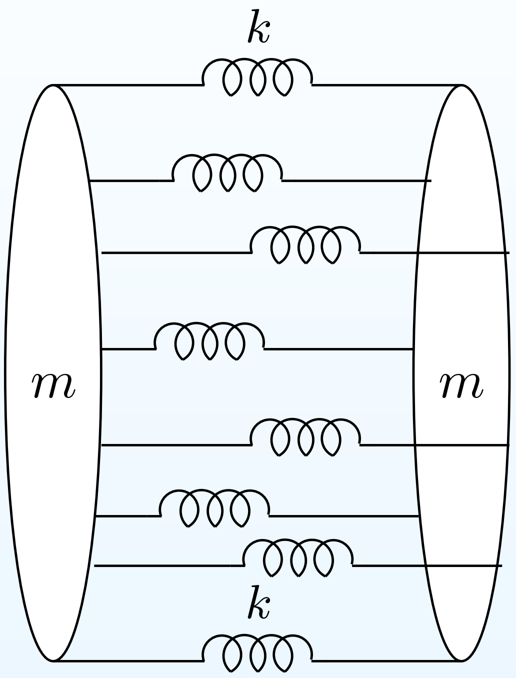


$$
\begin{aligned}
\underline{f}_i &= \alpha_i \cdot \left(\underline{x}_{i+1} - \underline{x}_i\right) + \alpha_{i-1} \cdot \left(\underline{x}_{i-1} - \underline{x}_i\right) \\
&= \alpha_{i-1}\,\underline{x}_{i-1} - \left(\alpha_{i-1} + \alpha_i\right)\underline{x}_i + \alpha_i\,\underline{x}_{i+1}
\end{aligned}
$$

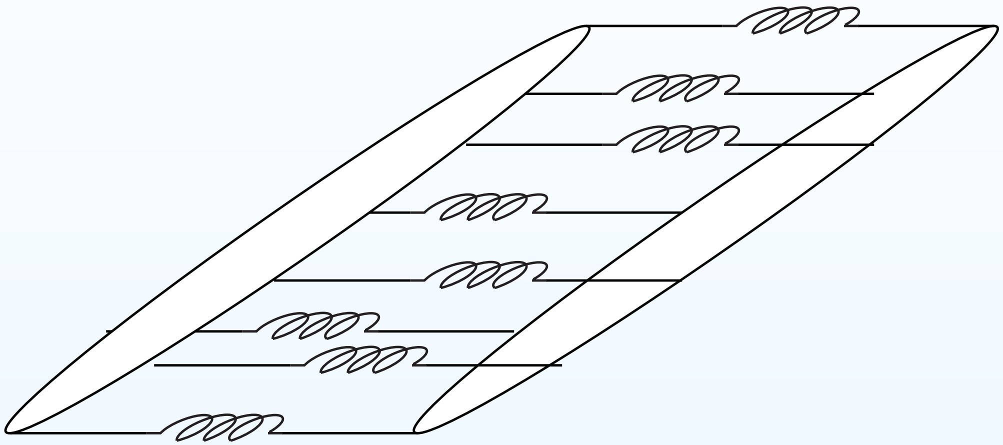
where

$$
\alpha_i \triangleq k \cdot \left[1 - \frac{l_0}{\left\|\,\underline{x}_{i+1} - \underline{x}_i\,\right\|}\right]
$$

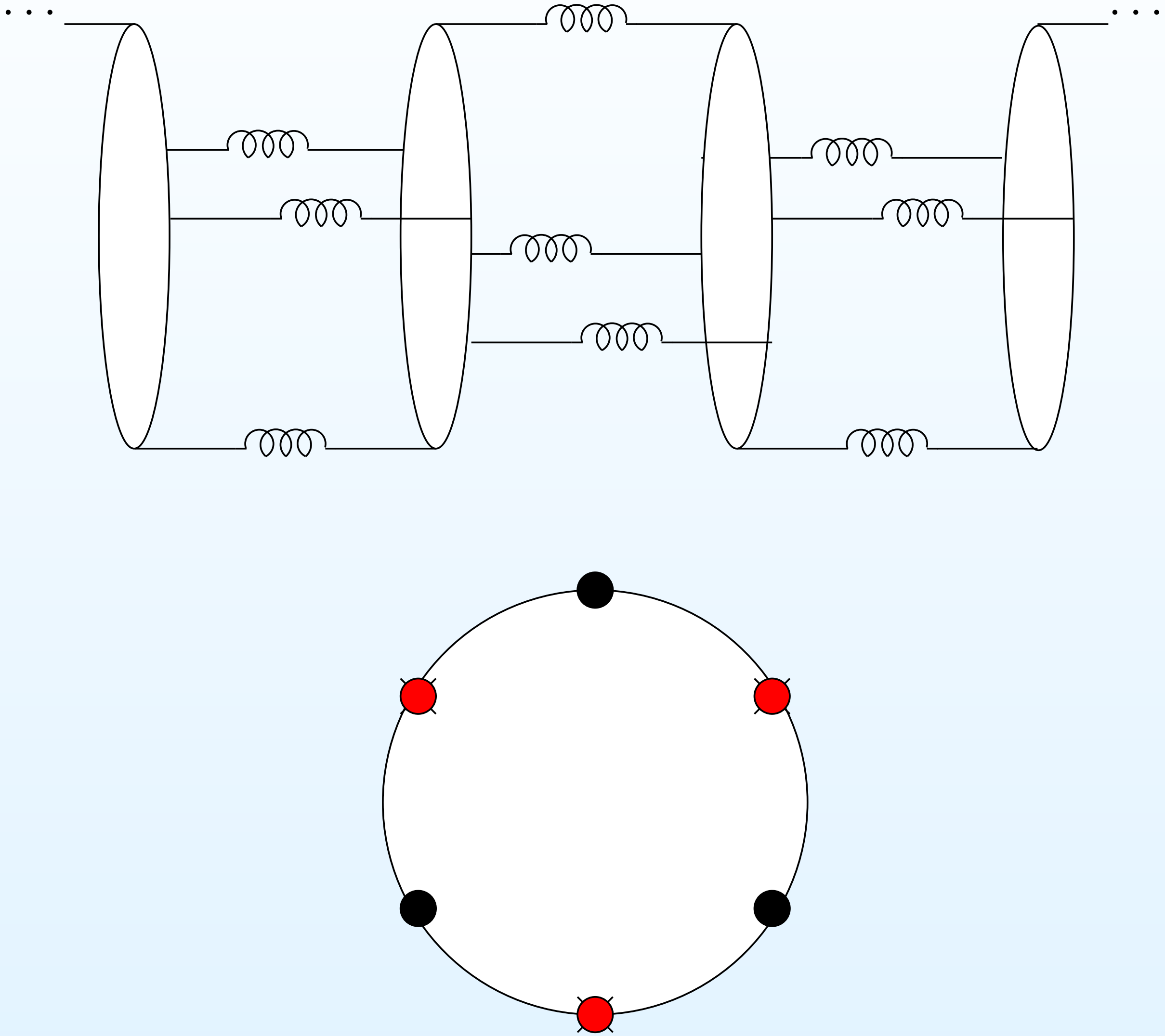
Adding Stiffness



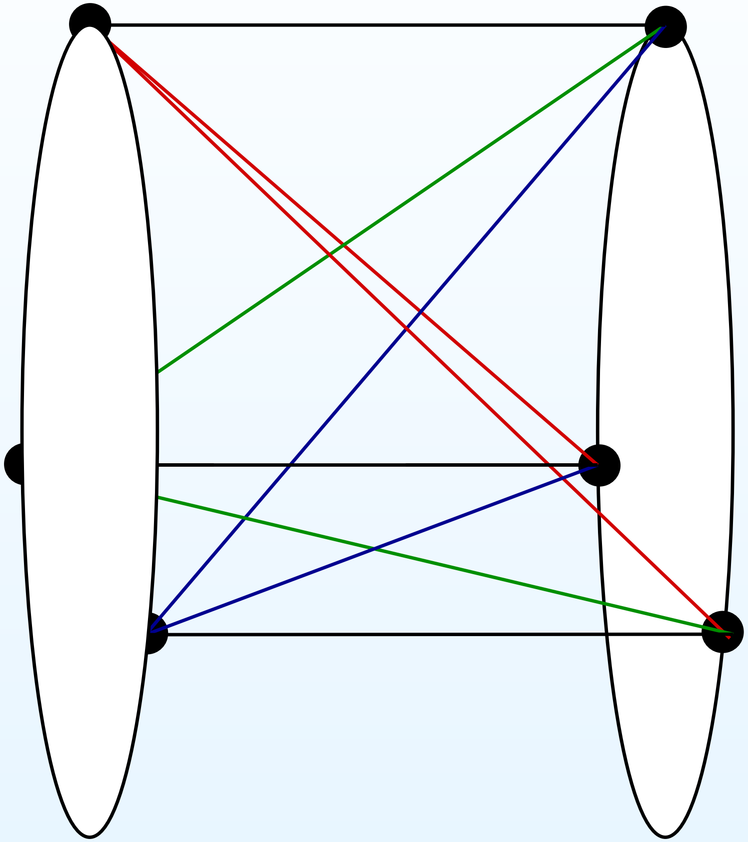
Need Shear Stiffness Too



Three-Spring Design (Before Shear Springs Added)



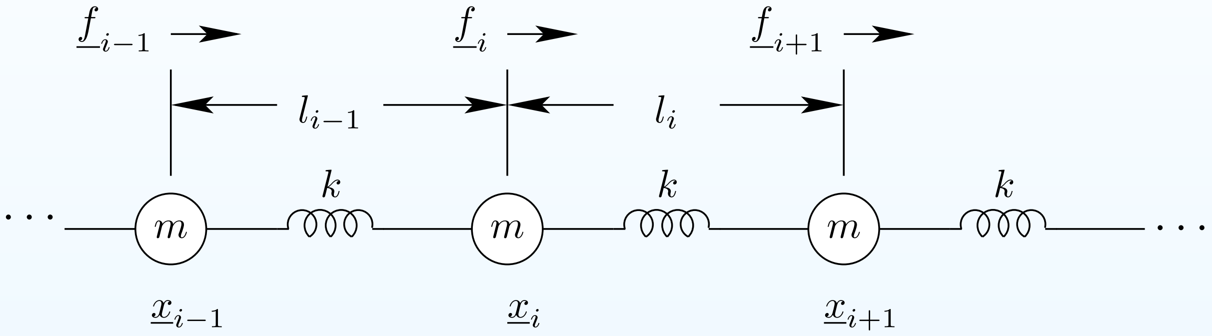
Shear Springs



Finite Difference Implementation

Digitizing the Flexible String

Recall:



$$\begin{aligned} \underline{f}_i &= \alpha_i \cdot \left(\underline{x}_{i+1} - \underline{x}_i\right) + \alpha_{i-1} \cdot \left(\underline{x}_{i-1} - \underline{x}_i\right) \\ &= \alpha_{i-1}\, \underline{x}_{i-1} - \left(\alpha_{i-1} + \alpha_i\right) \underline{x}_i + \alpha_i\, \underline{x}_{i+1} \end{aligned}$$

where

$$\alpha_i \triangleq k \cdot \left[1 - \frac{l_0}{\|\,\underline{x}_{i+1} - \underline{x}_i\,\|}\right]$$

Equations of Motion, State-Space Form

$$
\begin{array}{rcl}
\underline{f}_1 & = & m_1\,\ddot{\underline{x}}_1 \;=\; \alpha_1 \cdot (\underline{x}_2 - \underline{x}_1) \\
\vdots & & \vdots \\
\underline{f}_i & = & m_i\,\ddot{\underline{x}}_i \;=\; \alpha_{i-1}\,\underline{x}_{i-1} - (\alpha_{i-1} + \alpha_i)\,\underline{x}_i + \alpha_i\,\underline{x}_{i+1} \\
\vdots & & \vdots \\
\underline{f}_M & = & m_M\,\ddot{\underline{x}}_M \;=\; \alpha_{M-1} \cdot (\underline{x}_{M-1} - \underline{x}_M)
\end{array}
$$

or ($3M \times 1$):

$$\underline{F} = \mathbf{M}\,\ddot{\underline{X}} = \mathbf{A}\,\underline{X}$$

Digitization (Explicit Finite Difference Scheme):

$$\underline{X}_{n+1} = \left[2\mathbf{I} + \mathbf{M}^{-1}\mathbf{A}\right]\underline{X}_n - \underline{X}_{n-1} + B\underline{u}_n$$

where $\underline{u}_n$ is the external input vector, and digitization is based on

$$\ddot{\underline{x}}_n \triangleq \underline{x}_{n+1} - 2\underline{x}_n + \underline{x}_{n-1}$$

Results to Date

Presented this morning by Junji Kuroda

[.toc=]Results

Summary

- String-Synthesis History Reviewed
  - Ideal Strings
  - Stiff Piano Strings
  - Nonlinear Piano Strings
- Mass-Spring Model = Accurate Benchmark Reference
  - Accuracy $\sim$ time/space sampling density
  - Approximations can be quantified in practical cases
  - Given faster and more parallel computing, maybe we'll just go ahead and use it for real-time sound synthesis from performance!