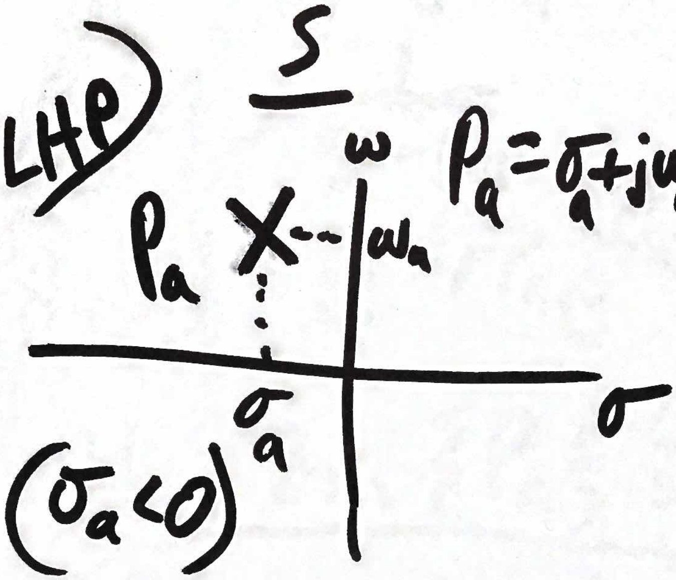
Complex Resonator

s

$p_a = \sigma_a + j\omega_a$



$$H_a(s) = \frac{1}{s - p_a} = \int_0^\infty h_a(t)e^{-st}\,dt$$

$$\Rightarrow \boxed{h_a(t) = e^{p_a t}}$$

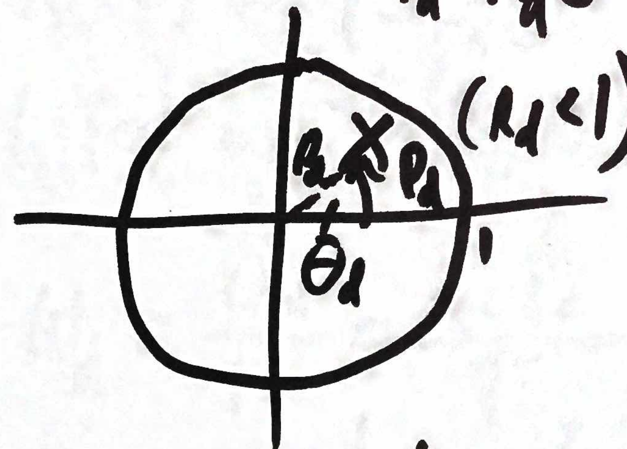
$$= e^{(\sigma_a + j\omega_a)t}$$

$$= e^{\sigma_a t} e^{j\omega_a t}$$

$$= e^{-t/\tau_a}\left[\cos\omega_a t + j\sin\omega_a t\right]$$

z

$p_d = R_d e^{j\theta_d}$



$$H_d(z) = \frac{1}{1 - p_d z^{-1}}$$

$$= 1 + p_d z^{-1} + p_d^2 z^{-2} + \cdots$$

$$= \sum_{n=0}^{\infty} p_d^n z^{-n}$$

$$h_d(n) = p_d^n = R_d^n e^{jn\theta_d}$$

$$= \left[e^{-T/\tau_d}\right]^n e^{jn\omega_d T}$$

$$= e^{-t_n/\tau_d} e^{j\omega_d t_n}\Big|_{t_n = nT}$$

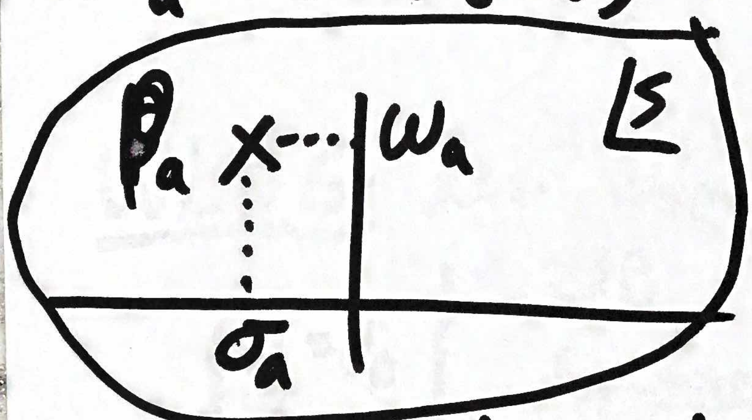
s

$$P_a = \sigma_a + j\omega_a$$

$$H_a(s) = \frac{1}{s - P_a}$$

$$h_a(t) = e^{-t/\tau_a} e^{j\omega_a t}$$

$$\omega_a = \Im\{P_a\} \ \frac{\text{rad}}{\text{sec}}$$



$$\tau_a = -\frac{1}{\sigma_a} = \frac{-1}{\Re\{P_a\}}$$

z

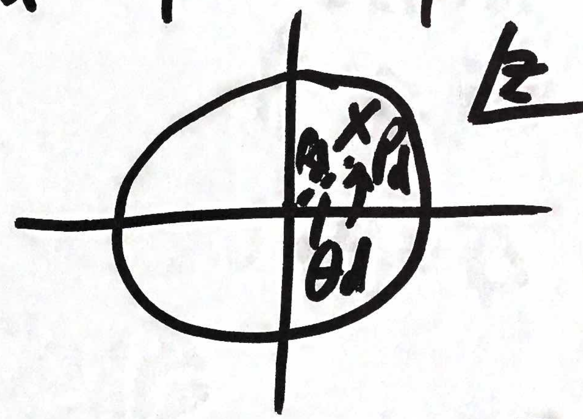
$\omega_d T$

$$P_d = R_d e^{j\theta_d}$$

$$H_d(z) = \frac{1}{1 - P_d z^{-1}}$$

$$h_d(n) = e^{-t_n/\tau_d} e^{j\omega_d t_n}$$

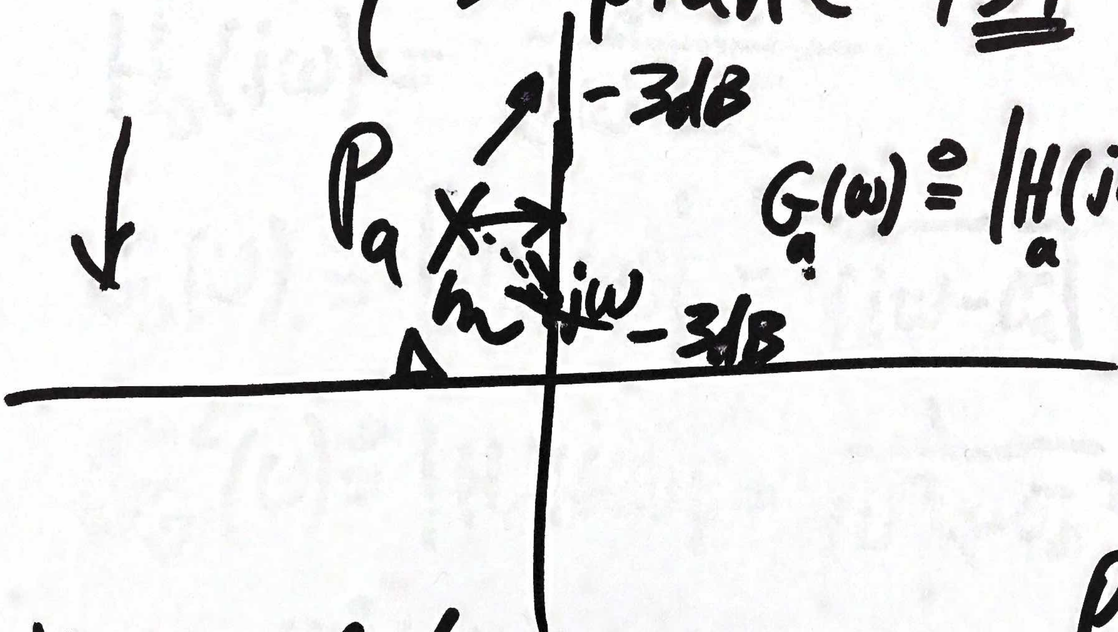
$$\omega_d = \frac{\theta_d}{T} = \frac{\angle P_d}{T}$$



$$\tau_d = \frac{-T}{\ln R_d} = \frac{-T}{\ln |P_d|}$$

Pole Bandwidth

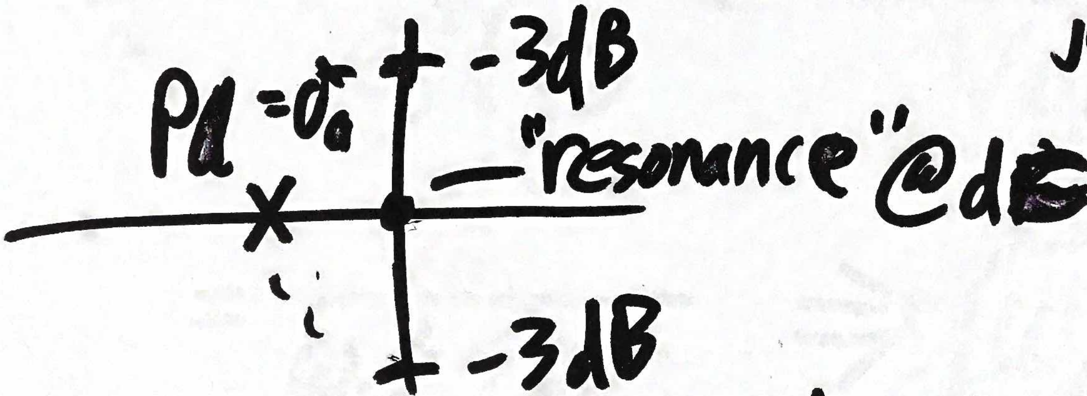
(s-plane 1st)



$G_a(\omega) \triangleq |H_a(j\omega)| = \left|\frac{1}{j\omega - P_a}\right|$ length of vector $P_a \to j\omega$

$j\omega = P_a + \Delta$

WLOG let $\omega_a = 0$



Find $\dfrac{|H_0(j\omega_c)|}{|H_0(0)|} = \dfrac{1}{\sqrt{2}}$ (-3dB)

$$H_0(s) = \frac{1}{s - p_a} = \frac{1}{s - \sigma_a}$$

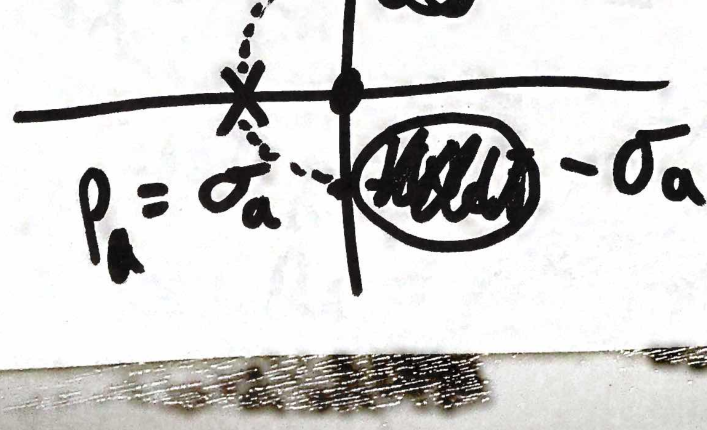
$$H_0(j\omega) = \frac{1}{j\omega - \sigma_a}$$

$$G_0(\omega) \triangleq \left| H_0(j\omega) \right| = \frac{1}{|j\omega - \sigma_a|} = \frac{1}{\sqrt{\omega^2 + \sigma_a^2}}$$

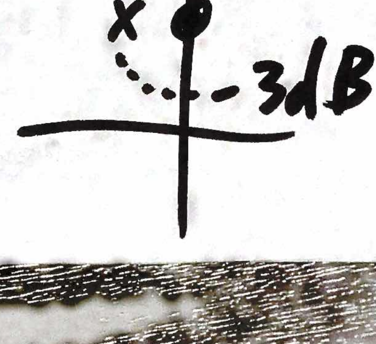
$$G_0^2(\omega) \triangleq \left| H_0(j\omega) \right|^2 = \frac{1}{\omega^2 + \sigma_a^2}$$

$$\frac{1}{2} = \frac{G_0^2(\omega_c)}{G_0^2(0)} = \frac{1/(\omega_c^2 + \sigma_a^2)}{1/\sigma_a^2}$$

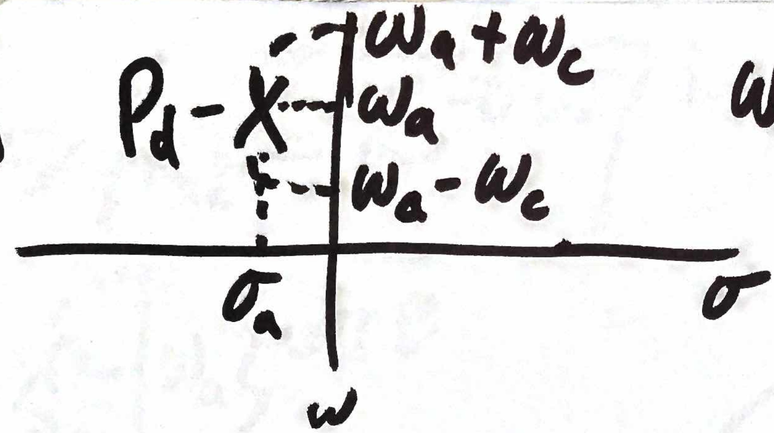
$$= \frac{\sigma_a^2}{\omega_c^2 + \sigma_a^2} \Rightarrow \boxed{\omega_c = \pm \sigma_a}$$



$$\Rightarrow \omega_c \triangleq |\sigma_a|$$



S]



$\omega_c = |\sigma_a|$

$S \to Z$

1. Sampling: $z = e^{sT}$
2. BE
3. FE
4. BLT

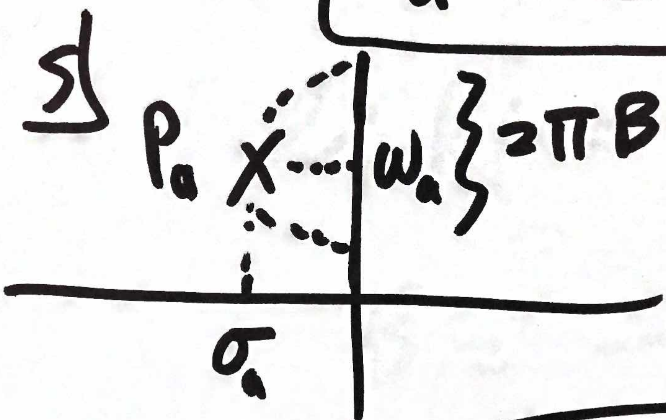
$$e^{j\omega_c T} \overset{?}{=} e^{j\omega_c T}$$

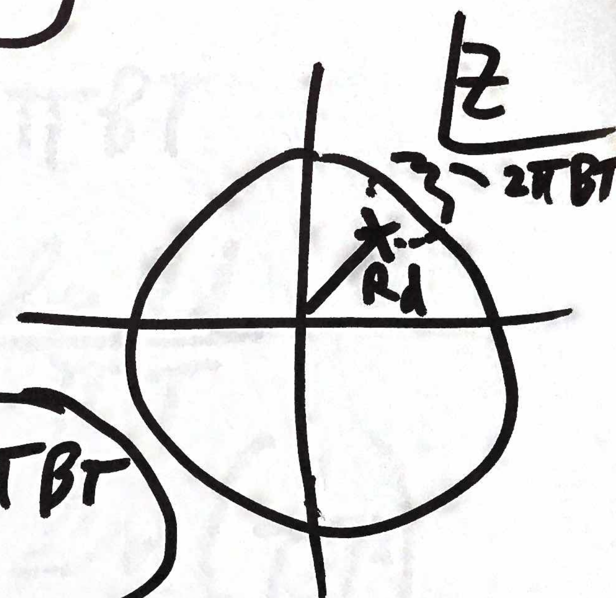
$$P_d = R_d e^{j\theta_d} \overset{?}{=} e^{p_a T}$$

$$\Rightarrow R_d e^{j\theta_d} = e^{\sigma_a T} e^{j\omega_a T}$$

$$\Rightarrow R_d = e^{\sigma_a T} = e^{-T/\tau_a}$$

$$\Rightarrow \boxed{R_d = e^{-\pi B T}} \quad \text{Recall: } B = \frac{|\sigma_a|}{\pi}$$





$$R_d = e^{-\pi B T}$$

$$= 1 - \pi B T + \frac{(-\pi B T)^2}{2!} + \dots$$

$$\approx 1 - \pi B T$$

$$= 1 - \frac{\cancel{2}\pi B}{\cancel{2} f_s}$$

$$\Rightarrow B = \frac{1 - R_d}{\pi T}$$

$$\frac{\overbrace{(2\pi B)}^{\Delta\omega} T}{2} \approx 1 - R_d$$

$$R_d = e^{-\pi B T}$$

$$\ln R_d = -\pi B T$$

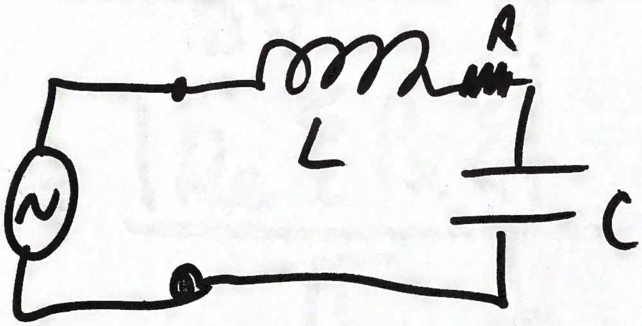
$$B = -\frac{\ln R_d}{\pi T}$$

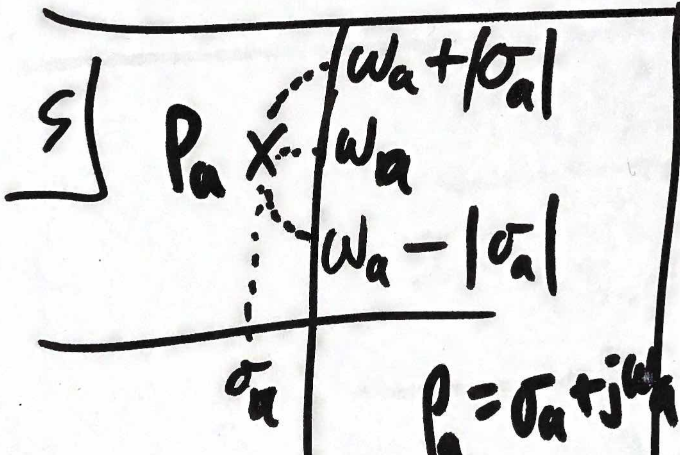
$$\frac{2\pi(B/f_s)}{2} = \ln\left(\frac{1}{R_d}\right)$$

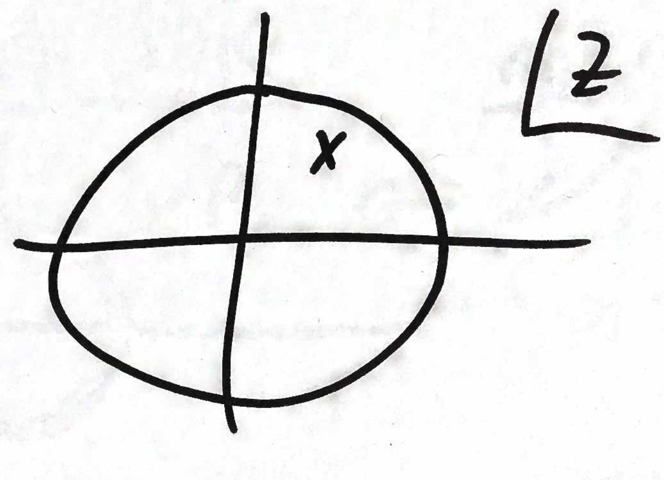
Q

Quality Factor (Q)

$$\boxed{Q} \triangleq \frac{\omega_r}{\Delta\omega_r} = \frac{f_r\,(Hz)}{B_r\,(Hz)}$$

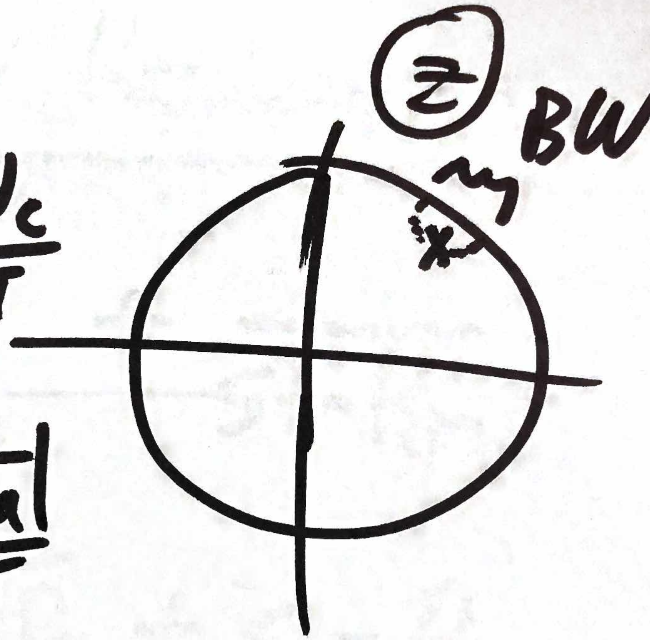






$$\boxed{Q = \frac{\omega_a}{2|\sigma_a|}}$$

Bandwidth

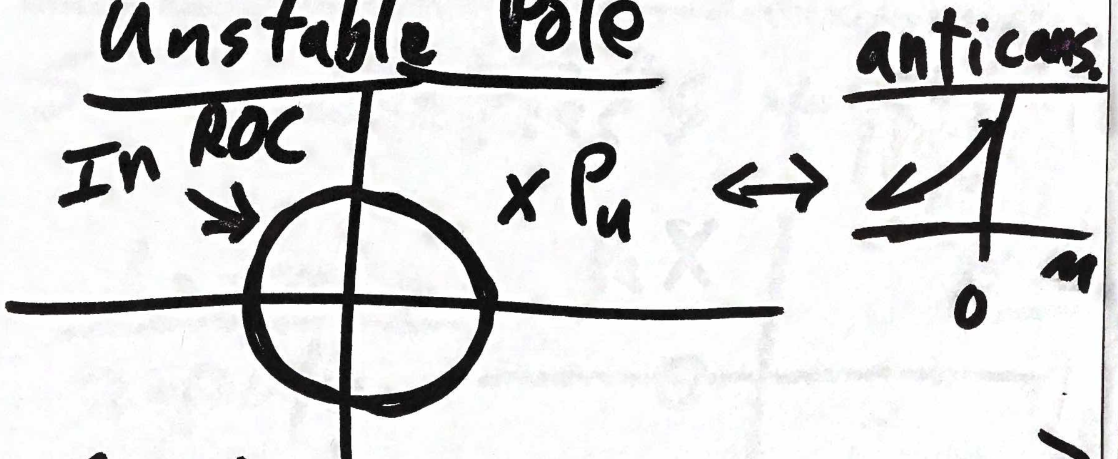


$$B \triangleq \frac{2\omega_c}{2\pi} = \frac{\omega_c}{\pi}$$

$$= \frac{2|\sigma_a|}{2\pi} = \frac{|\sigma_a|}{\pi}$$

$$= \frac{|Re\{p_a\}|}{\pi} = -\frac{Re\{p_a\}}{\pi}$$

Unstable Pole



$$ROC \triangleq \left\{ z \;\middle|\; \sum_{n=-\infty}^{\infty} h(n)\, z^{-n} < \infty \right\}$$

2-Pole Resonator (Real)

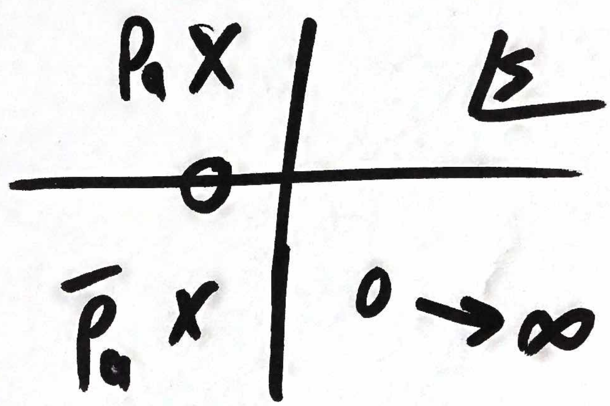
$$H_2(s) = \frac{1}{s-p_a} + \frac{1}{s-\bar{p}_a}$$

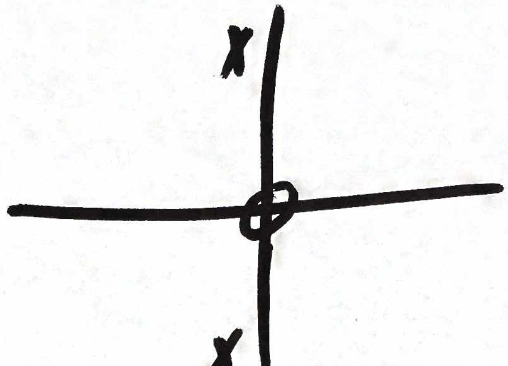
$$= \frac{s-\bar{p}_a + s - p_a}{(s-p_a)(s-\bar{p}_a)}$$

0 @ $s=\sigma_a$

$$= \frac{2s - 2\,\mathrm{Re}\{p_a\}}{s^2 - 2\,\mathrm{Re}\{p_a\}\,s + |p_a|^2}$$

$$= \frac{\text{1-zero}}{\text{2-pole}}$$





$$\frac{\sigma_a s}{s^2 + 2\sigma_a s + \underbrace{|p_a|^2}_{\omega_a^2}}$$