

MUS420/EE367A Lecture 7 Wave Digital Filters

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A *Wave digital filter* (WDF) is a particular kind of *digital filter* (or finite difference scheme) based on physical modeling principles.

- Developed to digitize lumped *electrical* circuit elements:
 - inductors
 - capacitors
 - resistors
 - gyrators, circulators, etc., (classical circuit theory)
- Each element is digitized by the *bilinear transform*
- *Wave variables* are used in place of physical variables (new), yielding superior numerical properties.
- Element connections involve *wave scattering*

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Wave Digital Filter (WDF) Construction

Wave digital elements may be derived from their describing differential equations (in continuous time) as follows:

1. Express forces and velocities as *sums of traveling-wave components* (“*wave variables*”):

$$f(t) = f^+(t) + f^-(t)$$

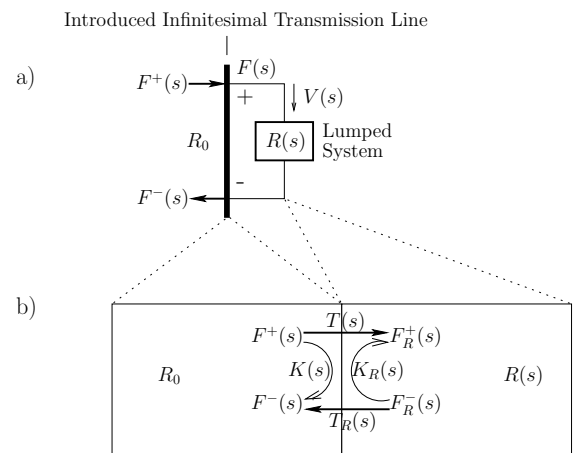
$$v(t) = v^+(t) + v^-(t)$$

The actual “travel time” is always *zero*.
(For historical reasons, WDFs typically use traveling-wave components scaled by 2.)

2. Digitize via the *bilinear transform* (trapezoid rule)
3. Use *scattering junctions* (“*adaptors*”) to connect elements together in
 - *series* (common velocity, summing forces), or
 - *parallel* (common force, summing velocities).

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Wave Variable Decomposition



- The inserted waveguide impedance R_0 is *arbitrary* because it was *physically introduced*.
- The element now interfaces to other elements by abutting its waveguide (transmission line) to that of other element(s).
- Such junctions involve *lossless wave scattering*:

$$F_R^+(s) = T(s)F^+(s) + K_R(s)F_R^-(s)$$

$$F^-(s) = T_R(s)F_R^-(s) + K(s)F^+(s)$$

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Element Reflectance

Imposing *physical continuity constraints* across the junction:

$$\begin{aligned} F(s) &= F_R(s) \\ 0 &= V(s) + V_R(s) \end{aligned}$$

with

$$\begin{aligned} F(s) &= F^+(s) + F^-(s) \\ F_R(s) &= F_R^+(s) + F_R^-(s) \\ V(s) &= V^+(s) + V^-(s) = \frac{F^+(s)}{R_0} - \frac{F^-(s)}{R_0} \\ V_R(s) &= V_R^+(s) + V_R^-(s) = \left[\frac{F_R^+(s)}{R(s)} - \frac{F_R^-(s)}{R(s)} \right] \end{aligned}$$

we obtain the *reflection transfer function* (“reflectance”) of the element with impedance $R(s)$:

$$S_R(s) \triangleq \frac{F^-(s)}{F^+(s)} = \frac{R(s) - R_0}{R(s) + R_0}$$

This is the *impedance step over the impedance sum*, the usual force-wave reflectance at an impedance discontinuity, but now in the Laplace domain.

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Bilinear Transformation

To digitize via the bilinear transform, we make the substitution

$$s = c \frac{1 - z^{-1}}{1 + z^{-1}}$$

where c is any positive real constant (typically $2/T$).

For the ideal mass reflectance

$$S_m(s) = \frac{ms - R_0}{ms + R_0}$$

the bilinear transform yields

$$\tilde{S}_m(z) = \frac{p_m - z^{-1}}{1 - p_m z^{-1}}$$

with

$$p_m \triangleq \frac{mc - R_0}{mc + R_0}$$

Note that $|p_m| < 1$ and $|\tilde{S}_m(e^{j\omega T})| = 1$. The stable allpass nature of the digitized mass reflectance is preserved by the bilinear transform, as always.

Important Observation:

If we choose $R_0 = mc$, then $p_m = 0$ and $\tilde{S}_m(z) = -z^{-1} \Rightarrow$ *no delay-free path through the mass reflectance*

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Reflectance of Ideal Mass, Spring, and Dashpot

For a mass m kg, the impedance and reflectance are respectively

$$\begin{aligned} R_m(s) &= ms \\ \Rightarrow S_m(s) &= \frac{ms - R_0}{ms + R_0} \end{aligned}$$

This reflectance is a *stable first-order allpass filter*, as expected, since energy is not dissipated by a mass.

For a spring k N/m, we have

$$\begin{aligned} R_k(s) &= \frac{k}{s} \\ \Rightarrow S_k(s) &= \frac{\frac{k}{s} - R_0}{\frac{k}{s} + R_0} \end{aligned}$$

also allpass as expected.

For a dashpot μ N s/m, we have

$$\begin{aligned} R_\mu(s) &= \mu \\ \Rightarrow S_\mu(s) &= \frac{\mu - R_0}{\mu + R_0} \end{aligned}$$

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Digitized Reflectances Without Delay-Free Paths

Plan:

1. Fix the bilinear-transform frequency-scaling parameter c once for the whole system (so there is only one frequency-warping)
2. Set the “connector” wave impedance R_0 separately for each circuit element to eliminate the delay-free path in its reflectance
3. We will then get scattering when we connect different elements together

This yields the following elementary reflectances:

<u>Element</u>	<u>Reflectance</u>
<i>ideal spring (capacitor)</i>	\leftrightarrow <i>unit delay</i>
<i>ideal mass (inductor)</i>	\leftrightarrow <i>unit delay and sign inversion</i>
<i>ideal dashpot (resistor)</i>	\leftrightarrow 0

The original element values remain only in the waveguide-interface impedances $R_0 = k/c, mc, \mu$

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Wave Digital Elements

In summary, our chosen digital element reflectances (and their connecting wave impedances R_0) are

- “Wave digital mass” (interface impedance $R_0 = mc$)

$$\tilde{S}_m(z) = -z^{-1} \quad (\text{mass reflectance})$$

- “Wave digital spring” ($R_0 = k/c$)

$$\tilde{S}_k(z) = z^{-1} \quad (\text{spring reflectance})$$

- “Wave digital dashpot” ($R_0 = \mu$)

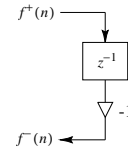
$$\tilde{S}(z) = 0 \quad (\text{dashpot [non-]reflectance})$$

(In this case, the interface is the element itself.)

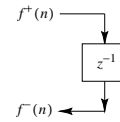
These are the *discrete-time reflectances* of the basic circuit building-blocks as seen from their interface-waveguides

We still have the usual freedom in choosing our bilinear-transform frequency-scaling constant c

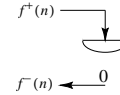
Elementary Wave Flow Diagrams



Wave digital mass



Wave digital spring



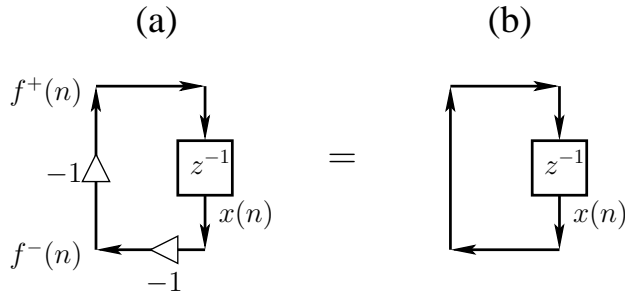
Wave digital dashpot

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Example: “Piano hammer in flight”

Mass m at constant velocity, force-wave simulation:



- The reflecting termination on the left corresponds to zero force on the mass
- A nonzero state variable $x(n)$ corresponds to a nonzero *velocity* for the mass:

$$\begin{aligned} v(n) &= v^+(n) + v^-(n) = \frac{f^+(n)}{R_0} - \frac{f^-(n)}{R_0} \\ &= \frac{f^+(n)}{mc} + \frac{f^+(n-1)}{mc} = \frac{x(n+1) + x(n)}{mc} \\ &= \frac{2}{mc}x(n) = \frac{T}{m}x(n) \end{aligned}$$

when $c = 2/T$ is chosen for the bilinear transform

Mass Momentum and Energy

- Above we found the mass *velocity* to be

$$v(n) = \frac{2}{mc}x(n) = \frac{T}{m}x(n)$$

when $c = 2/T$ is chosen for the bilinear transform

- The *momentum* of the mass is therefore

$$p(n) \triangleq mv(n) = \frac{2}{c}x(n) = Tx(n)$$

when $c = 2/T$

- State variable $x(n) = p(n)/T$ is *mass momentum per sample*

- Since momentum is conserved, *momentum waves* are good to consider in place of velocity waves

- The *kinetic energy* of the mass is given by

$$\mathcal{E}_m = \frac{1}{2}mv^2(n) = \frac{p^2(n)}{2m} = \frac{2}{mc^2}x^2(n) \rightarrow \frac{[Tx(n)]^2}{2m}$$

for $c \rightarrow 2/T$

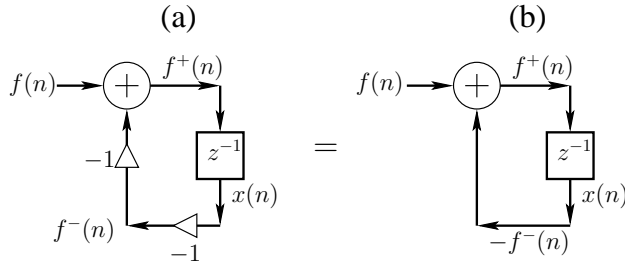
- The *potential energy* of the mass-in-flight is of course zero ($f(n) \equiv 0$)

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Force Driving a Mass

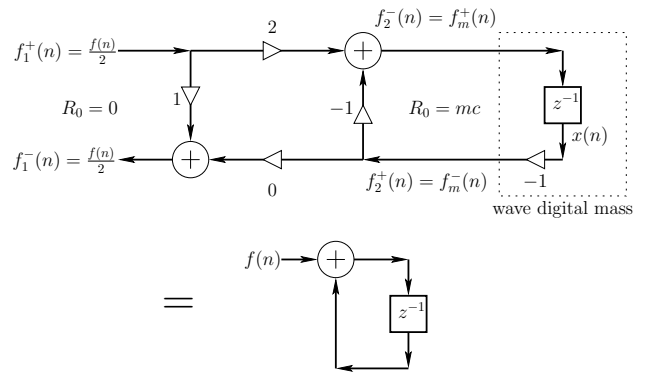
$$f(n) = f^+(n) + f^-(n) \Rightarrow f^+(n) = f(n) - f^-(n)$$



Wave digital mass driven by external force $f(n)$.

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Traveling-Wave View of Driving Force

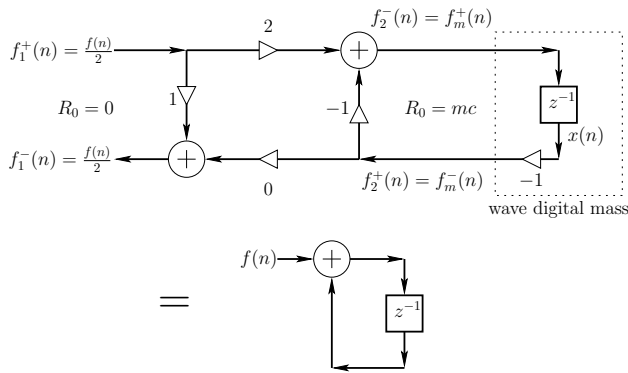


- Parallel junction with $R_0 = 0$ on the force side and $R_0 = mc$ on the mass side
- Impedance step over impedance sum is $R = (mc - 0)/(mc + 0) = 1$
- Obviously *non-physical* (see next page)

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Zero Source-Impedances are Non-Physical

We postulated the following driving-source interface:



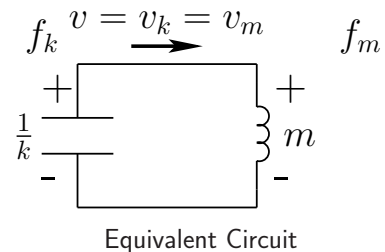
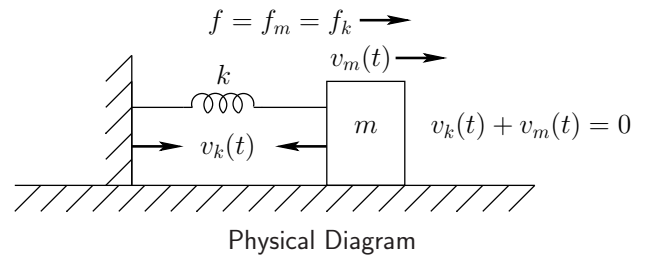
Non-physical because:

- Velocity transmission is zero \Rightarrow *no power* delivered
- There can be no traveling force (voltage) wave in a zero impedance (which would “short it out”)
- Recall power waves: $[f^+(n)]^2/R_0 = \infty$ if $f^+(n) \neq 0$
- Zero source-impedances can be a useful idealization, but be careful
- **Exercise:** Study the case of small $R_0 = \epsilon > 0$.

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Spring-Driven Mass

To keep the model physical, let's use a pre-compressed *spring* as our force-source for driving the mass:



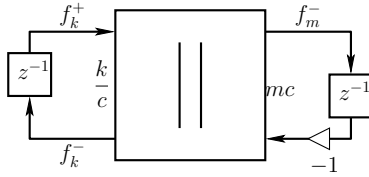
- The mass and spring form a *loop*, so the connection can be defined as either parallel or series (as determined by the element reference directions)

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- We arbitrarily choose a *parallel* junction, giving the following physical constraints:

- $f_k(n) = f_m(n)$ (common force)
- $v_k(n) + v_m(n) = 0$ (sum of spring-compression-velocity and rightgoing-mass velocity is zero)

- **Exercise:** Work out the case for a series junction and verify everything comes out the same physically
- Connecting our wave digital spring and mass at a parallel force-wave junction is depicted as follows:



WDF Diagram

Note the WDF symbol “||” for a *parallel adaptor* (scattering junction)

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Low-Frequency Spring-Driven-Mass Analysis

Referring to the previous figure:

- We found earlier that $x_2(n) \approx p_m(n)/T$ where $p_m(n)$ is the mass momentum at time n , and T is the sampling interval
- We similarly find that $x_1(n) = f_k^-(n) \approx f(n)/2$, so that the mass sees $(1+s)f(n)/2 \approx f(n)$ coming in each sample from the summer, *i.e.*,

$$\frac{p_m(n)}{T} \approx \frac{p_m(n-1)}{T} + f(n)$$

- Multiplying through by T gives the *momentum update* per sample:

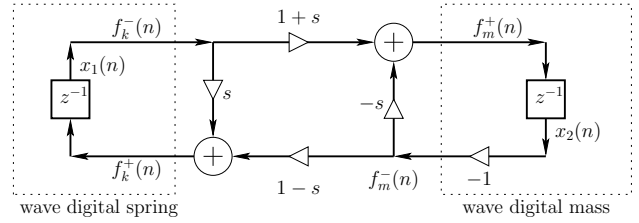
$$p_m(n) \approx p_m(n-1) + f(n)T \triangleq p_m(n-1) + \Delta p(n)$$

where $\Delta p(n) \triangleq f(n)T$ is the momentum transferred to the mass by constant force $f(n)$ during one sampling interval T

- This makes physical sense and suggests *momentum* and *momentum-increment* samples as an appealing choice of wave variables

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Expanded Wave Digital Spring-Mass System



State variables labeled $x_1(n)$ and $x_2(n)$

Low-Frequency Analysis:

- Assume sampling rate $f_s = 1/T$ is large \Rightarrow
- Bilinear transform constant $c = 2/T$
- Frequency warping not an issue
- Physical simulation should be very accurate

The reflection coefficient for our parallel force-wave connection is given as usual by the *impedance step over the impedance sum*:

$$s = \frac{mc - k/c}{mc + k/c} = \frac{m2/T - kT/2}{m2/T + kT/2} = \frac{m - kT^2/4}{m + kT^2/4} \approx 1$$

We can now see what's going physically at low frequencies relative to the sampling rate:

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Classic WDF Wave Variables

We have been using our usual traveling-wave decomposition of force and velocity waves:

$$f(t) = f^+(t) + f^-(t) = R_0 v^+(t) - R_0 v^-(t)$$

$$v(t) = v^+(t) + v^-(t) = \frac{f^+(t)}{R_0} - \frac{f^-(t)}{R_0}$$

where R_0 is the wave impedance of the medium, or

$$\begin{bmatrix} f(t) \\ v(t) \end{bmatrix} = \begin{bmatrix} R_0 & -R_0 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} v^+(t) \\ v^-(t) \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ \frac{1}{R_0} & -\frac{1}{R_0} \end{bmatrix} \begin{bmatrix} f^+(t) \\ f^-(t) \end{bmatrix}$$

Inverting these gives

$$\begin{bmatrix} v^+(t) \\ v^-(t) \end{bmatrix} = \frac{1}{2} \begin{bmatrix} 1/R_0 & 1 \\ -1/R_0 & 1 \end{bmatrix} \begin{bmatrix} f(t) \\ v(t) \end{bmatrix}$$

$$\begin{bmatrix} f^+(t) \\ f^-(t) \end{bmatrix} = \frac{1}{2} \begin{bmatrix} 1 & R_0 \\ 1 & -R_0 \end{bmatrix} \begin{bmatrix} f(t) \\ v(t) \end{bmatrix}$$

In the WDF literature, the second case is typically used, multiplied by 2, and replacing force and velocity by voltage and current:

$$a(t) = v(t) + R_0 i(t)$$

$$b(t) = v(t) - R_0 i(t)$$

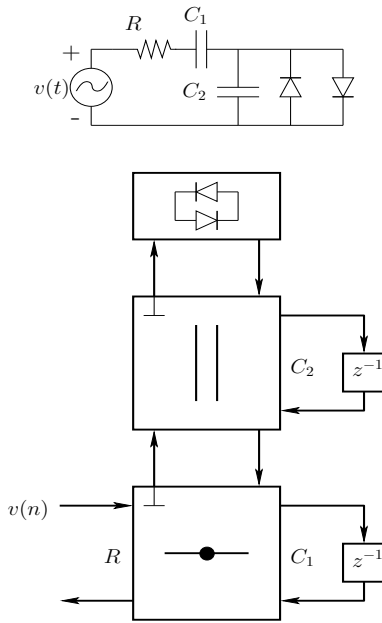
where $v(t)$ is now *voltage* and $i(t)$ denotes *current*.

Thus, $a(t) = 2v^+(t)$ and $b(t) = 2v^-(t)$ (doubled voltage traveling-wave components)

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Binary Connection Tree

It has become common practice to organize WDF elements into a *Binary Connection Tree* (BCT):



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Reflection-Free Ports

- The symbol \perp on a WDF adaptor port denotes a *reflection-free port* (RFP)
- To make a port reflection-free, its wave-impedance must be the
 - *parallel combination* of the other port impedances for a parallel adaptor, or
 - *series combination* of the other port impedances for a series adaptor

This choice of port impedance zeros the impedance step “seen” by waves in the RFP, thus suppressing instantaneous reflection from it

- All ports *outgoing* from the BCT root must be RFPs, for computability (no delay-free loops)
- Computations propagate (each sample) from the leaves of the tree (delay element outputs) up to the root, where there is a final reflection which then propagates back down to all of the reflection-free ports, thereby updating all of the delay elements (capacitor/spring and inductor/mass states)
- When an element value changes (typically a resistor), RFPs must be *recalculated up to the root*.

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Reflection-Free Port Coefficients

For an N -port adaptor, with port wave-impedances R_i , $i = 1, 2, \dots, N$, let's arbitrarily designate port N as the *reflection-free port* (the one on top). It is convenient to define the port *conductances* $G_i \triangleq 1/R_i$. To suppress reflection on port N , we need, for a *parallel adaptor*,

$$R_N = R_1 \parallel R_2 \parallel \dots \parallel R_{N-1} \Leftrightarrow$$

$$G_N = G_1 + G_2 + \dots + G_{N-1}$$

and, for a *series adaptor*,

$$R_N = R_1 + R_2 + \dots + R_{N-1}.$$

Recall the *alpha parameters* for an N -port series scattering junction, derived from the physical constraints that the velocities be equal and the forces sum to zero at the (series) junction:

$$\alpha_i \triangleq \frac{2R_i}{R_1 + R_2 + \dots + R_N} = \frac{R_i}{R_N}$$

when port N is reflection free.

Since $\sum_{i=1}^N \alpha_i = 2$, we have $\boxed{\alpha_N = 1}$ and $\boxed{\sum_{i=1}^{N-1} \alpha_i = 1}$.

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Example

See page 42 of David Yeh's WDF Tutorial¹

Shockley diode equation (“diode law”)

$$I(t) = I_s \cdot \left(e^{\frac{V_d}{nV_T}} - 1 \right)$$

where

I = diode current

I_s = diode reverse leakage current

V_d = voltage across the diode

n = ideality factor (1 for ideal, up to 2 or more otherwise)

V_T = thermal voltage kT/q

k = Boltzmann constant

q = electron charge

T = temperature

¹<https://ccrma.stanford.edu/~dtyeh/papers/wdftutorial.pdf>

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Topology Issues

- Classical WDFs are composed of *parallel* and *series* connections of elements
- A Binary Connection Tree (BCT) can represent any such parallel/series network
- *R*-Nodes
 - Some circuits, such as the “bridged T” circuit, cannot be represented using parallel/series connections of elements
 - These circuits are modeled using more general *scattering matrices*
 - Such circuits are called *R*-Nodes in the overall WDF network graph
 - *R*-Nodes connect naturally to BCT graphs, since all signals are compatible traveling-wave components
 - An open issue is how to minimize the computational complexity of *R*-node scattering matrices

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WDF State Space Interpretation

Digital filters can be expressed in state-space form as

$$\underline{x}(n+1) = A \underline{x}(n) + B \underline{u}(n)$$

by simply enumerating all delay elements as state variables $\underline{x}^T(n) = [x_1(n), x_2(n), \dots, x_N(n)]$, and finding the state transition matrix A by inspection. Any inputs are collected in $\underline{u}(n)$ and determine the B matrix.

- For WDFs, the A matrix is a *scattering matrix*
- The A matrix is *orthogonal* (lossless) for reactive elements (masses, springs)
- The state variables are all *sampled traveling waves*
- Physical state variables (bilinear transformed) are obtainable by *summing* (capacitors, springs) or *subtracting* (inductors, masses) the input and output of the unit delays:

$$y_k(n) = x_k(n) \pm x_k(n-1)$$

- In comparison to other state-space models, WDF state-space form has top numerical properties due to its lossless scattering formulation

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SPQR Decomposition

Every graph can be decomposed into Series (S), Parallel (P), and R (“Rigid”) type subgraphs (Q is the degenerate case consisting of only one graph edge)

- S and P handled by standard WDF methods (BCT)
- R node characterized by its scattering matrix
- Modified Nodal Analysis (MNA) may be used to find the R-node scattering matrix (see Werner et al. reference below)

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Nonlinear Wave Digital Filters

A WDF network tree can have a multiport *instantaneous nonlinearity* at its root:

- A typical instantaneous nonlinearity is a *nonlinear resistor* $R(v)$ (such as a diode) or a *dependent source* (as used in transistor models, etc.)
- Because the resistance of a nonlinear resistor depends on the voltage across it, there is no way to avoid an instantaneous reflection in general (no fixed port-impedance can match it for all input conditions)
- The nonlinearity is placed at the *root* of the BCT. A delay-free path is “computable” only there (we get one per tree)
- Each sample, computations propagate up the tree to the root, reflecting instantaneously, then back down to all the reflection-free ports
- The nonlinear reflectance can be pre-computed and stored for fast interpolated table look-up in real time (no iterations)
- If the nonlinearity cannot be placed at the root of the WDF BCT (e.g., because there are two or more

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nonlinearities in the circuit) the delay-free-path may be solved iteratively using Newton's method et al.

- Alternatively, all nonlinearities can be placed at the root of the WDF tree and connect to the BCT through an R-Node. References:

1. "Wave Digital Filter Adaptors for Arbitrary Topologies and Multiport Linear Elements"²

2. "Resolving Wave Digital Filters with Multiple/Multiport Nonlinearities"³

Kurt Werner et al.

Int. Conf. Digital Audio Effects (DAFx-15)
Trondheim, Norway, 2015

Dynamic Nonlinearities

Nonlinearities can be *instantaneous* or *dynamic* (having *memory*)

- A dynamic nonlinearity can sometimes be converted into an instantaneous nonlinearity:
- Convert to the physical units in which the nonlinearity is instantaneous

²http://www.ntnu.edu/documents/1001201110/1266017954/DAFx-15_submission_53.pdf

³https://www.ntnu.edu/documents/1001201110/1266017954/DAFx-15_submission_54.pdf

Choice of WDF Topology

Summarizing points above,

- Generally try to make a Binary Connection Tree (BCT) using only three-port adaptors
- At the *root* of the tree, include all
 - nonlinearities
 - non-adaptable elements such as switches
- When everything is linear and adaptable, place a time-varying element at the root, to minimize update propagation when that element changes
- When multiple elements are at the root, or when topology is not merely series + parallel connections, there will generally be at least one *R* node

Free WDF Software

Real Time Wave Digital Filter Software (DAFx-2016):

- GitHub: RT-WDF
- DAFx16 Paper

Overview and Demo of Various Wave Digital Filter Software (DAFx-2015, KeyNote 2, Part 2):

- Video (YouTube)
- Slides (PDF)

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 18. Kurt Werner et al., "Resolving Wave Digital Filters with Multiple/Multiport Nonlinearities"⁵, DAFx-15, Trondheim, Norway, 2015. *Int. Conf. Digital Audio Effects (DAFx-15)*

⁴http://www.ntnu.edu/documents/1001201110/1266017954/DAFx-15_submission_53.pdf
⁵https://www.ntnu.edu/documents/1001201110/1266017954/DAFx-15_submission_54.pdf