**Closed-Form Coefficient Functions**

Example: Two-Pole Resonator

\[ H(z) = \frac{g}{1 - 2R \cos(\theta) \cdot z^{-1} + R^2 \cdot z^{-2}} \]

where

\[ \theta = 2\pi f_r T = \text{pole angle (} f_r = \text{resonance frequency)} \]
\[ R = e^{\pi BT} = \text{pole radius (} B = \text{resonant bandwidth)} \]

Generalization:

- Write filter frequency response in terms of the coefficients
- Find formulas for coefficients in terms of useful spectral parameters such as Bandwidth, Peak Gain, Q, and so on
- If the equations can be inverted, then we have formulas for the filter coefficients in terms of the desired spectral parameters

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**Multiple Static Designs/Coefficient Interpolation**

- In this method, we sample the desired multidimensional frequency response, design filters at the sample points, then interpolate between the design locations to get coefficients when running
- Multidimensional Frequency Responses represent the desired frequency response as a function of the desired parameters \( \Psi_i \):
  \[ H_{\text{des}}(\omega, \Psi_1, \ldots, \Psi_n) \]
- The sampled response (sampling across \( \Psi_i \)) can be viewed as a multidimensional matrix:
  \[ H[k, m_1, \ldots, m_n] = H_{\text{des}}(\omega_k, m_1\Delta_1, \ldots, m_n\Delta_n) \]
  where, e.g., \( n_i\Delta_i = \Psi_i^{\text{max}} \).

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**Multidimensional Frequency Response Example 1**

Varying \( \Psi_{\text{corner}} \) in \( H_{\text{des}}(\omega, \Psi_{\text{corner}}, \Psi_{\text{resonance}}) \):
Multidimensional Frequency Response Example 2

Varying $\psi_{\text{resonance}}$ in $H_{\text{des}}(\omega, \psi_{\text{corner}}, \psi_{\text{resonance}})$:

Multidimensional Frequency Response

**FIR Filter Design**

- The filter at each parameter setting is designed using some optimization method (pick your favorite)
- When designing a variable FIR filter, this method works well when the parameters $\psi$ are sampled sufficiently densely
- The resulting FIR coefficients often interpolated well from one $\psi$ samples to the next

When designing variable IIR filters, this method has some drawbacks:

Multidimensional Frequency Response

**IIR Filter Design**

- Since the IIR filter-design problem is normally non-convex, designs for filters at neighboring parameter samples may end up at relatively different local minima
- ‘Small’ changes in parameters may not correspond to ‘small’ changes in filter coefficients
- Filter coefficients can thereby become discontinuous functions of parameters and therefore not interpolate well
- The interpolated filter coefficients may give a filter completely different from either of the statically designed filters it is interpolating between
- Design methods can add constraints to keep coefficients continuous in the parameters.
  - One method used the coefficients from a neighboring, already-designed point as the starting guesses for the design, which reduces the chances of ending up at a distant minimum.

- Importantly, there is no guarantee that the IIR filter will remain stable using interpolated coefficients, even if all the designed filters are stable
- Design methods that design to some stability margins can reduce the problem, at the cost of restricted possible designs
- Desired responses with nearly unstable (“high-Q”) regions will be more prone to this problem (example: virtual analog VCF)

The datasets for multidimensional filter-design method can become very large, especially as the number of control parameters gets large.
Outer Product Expansion

This method performs a principal-components analysis of the multidimensional frequency response, and implements the variable filter as a weighted sum of static filters. With a single control parameter $\Psi_1$, we have the expansion

$$H_{\Psi_1} = \sigma_1 \left[ \begin{array}{c} F_1 \\ \vdots \\ F_n \end{array} \right]$$

Since

$$H_{i,\omega} = \sum_{k=1}^{n} \sigma_k G_k F_k(\omega),$$

we see that any point in control-frequency space $H_{i,\omega}$ is a weighted sum of samples from frequency-space functions $F_k$ (i.e., filters) and control-space functions $G_k$.

We can add more control parameters:

$$H_{i_1, \ldots, i_m, \omega} = \sum_{k=1}^{n} [\sigma_k G_{i_1}(i_1) \cdots G_{i_m}(i_m) F_k(\omega)]$$

Interpolated Outer Product Expansion Filters

- When implemented as shown, we must worry about the filters’ phase responses:
  - The design usually ignores the phase of the desired response, to keep $G_i$ real in the expansion, and to keep them from cancelling each other
  - The paper introducing this idea restricted the phases to be equal, and used linear-phase FIR filters

Interpolated Outer Product Expansion

In the multi-controller outer-product expansion

$$H_{i_1, \ldots, i_m, \omega} = \sum_{k=1}^{n} [\sigma_k G_{i_1}(i_1) \cdots G_{i_m}(i_m) F_k(\omega)]$$

we may interpolate the control space:

$$H_{\Psi_1, \ldots, \Psi_m, \omega} = \sum_{k=1}^{n} [\sigma_k g_{i_1}(\Psi_1) \cdots g_{i_m}(\Psi_m) F_k(\omega)]$$

where the $g_i$ are interpolating functions of the $G_i$.

- The $F_i$ are designed to the frequency-axis components of the outer-product expansion of the desired response
- The $g_{i,j}$ are interpolating functions derived from the control-axes components of the outer-product expansion
- This method overcomes the stability problem of the previous method, since all the filters are static, and can be designed to be stable

where

- $F_i$ = ‘principal-component’ filters
- $g_{i,j}$ = interpolated control functions
Spectrum Warping

This method uses the fact that we can use the bilinear transform to map the unit circle onto itself warped. In particular, we can map $z = 1$ to $\tilde{z} = 1$, $z = -1$ to $\tilde{z} = -1$, and have one more degree of freedom left. Such a map is

$$z^{-1} \leftarrow \frac{z^{-1} - \beta}{1 - \beta z^{-1}}$$

which, when the delays in a filter are replaced by the given allpass filter, preserves the magnitude response, but warps the frequency axis according to $\beta$.

Thus a prototype filter (low-pass, for example) can be warped to place its corner frequency anywhere in the frequency range, so that $\beta$ becomes a simple and efficient tuning sweep control.

This method has a problem when applied to IIR prototype filters: the allpass filter has a delay-free path, which makes the feedback paths of the IIR filter unimplementable. Various methods exist to fix this problem, involving placing a delay in the loop, and fixing up the feedback coefficients. Unfortunately, all of these methods require that the fixed-up coefs be recomputed each time $\beta$ is changed, though some are much more efficient than others.

Another spectrum warping method, which preserves linear-phase in FIR filters, was proposed in “Variable Cutoff Linear Phase Digital Filters”, Oppenheim, Mecklenbräuker, & Mersereau, IEEE Trans. Circuits and Systems, v23 n4, April 1976.

Heterodyne Filters

- Use heterodyning to implement a sweepable filter using a fixed lowpass filter and heterodyning.
- Heterodyning is multiplication by a complex sinusoid $e^{-j\omega_c t}$ tuned to the “sweep frequency” $\omega_c$.
- The fixed filter’s bandwidth is a bandwidth control.
- Requires analytic input signal (i.e., first filter out negative frequencies).

General Advice

- Concentrate on finding a filter structure that makes the transformation from control parameters to coefficients trivially simple and/or efficient.
- Control separability $[a_i(\Psi_1, \ldots, \Psi_n) = \prod_{k=1}^n a_{ik}(\Psi_k)]$ is desired because it makes the $a_{ik}$ much simpler.
- Try to use the Closed-Form Coefficient Function method, with simple coefficient functions.
- Alternatively, a sparsely-sampled version of the Static-Design method is fine, as simple (smooth) coefficient functions should correspond to simple interpolations.
- Most of the previously mentioned methods use direct-form or other standard forms, though most could be applied to nonstandard structures, like the Moog structure, if we have design methods for those structures (which is likely).
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