Outline

• Moving String Termination
• Wave Impedance
• Displacement, Velocity, Acceleration Waves
• Force Waves
• Root-Power Waves

Interactive Animation

String Driven by Moving Termination

Uniformly moving rigid termination for an ideal string (tension $K$, mass density $\epsilon$) at time $0 < t_0 < L/c$.

Driving-Point Impedance $F_0/V_0$:

\[ y'(t_0, 0) = -\frac{v_0 f_0}{ct_0} = -\frac{v_0}{c} = -\frac{v_0}{\sqrt{K/\epsilon}} \]

\[ \Rightarrow f_0 = -K \sin(\theta) \approx -K y'(t, 0) = \sqrt{K/\epsilon} v_0 \frac{\Delta}{R} = v_0 \]

• If the left endpoint moves with constant velocity $v_0$ then the external applied force is $f_0 = R v_0$
• $R = \sqrt{K/\epsilon}$ is wave impedance (for transverse waves)
• Equivalent circuit is a resistor (dashpot) $R > 0$
• We have the simple relation $f_0 = R v_0$ only in the absence of return waves, i.e., until time $t_0 = 2L/c$.

Waveguide “Equivalent Circuits” for the Uniformly Moving Rigid String Termination

a) Velocity waves

b) Force waves

• String moves with speed $v_0$ or $0$ only
• String is always one or two straight segments
• “Helmholtz corner” (slope discontinuity) shuttles back and forth at speed $c$
• String slope increases without bound
• Applied force at termination steps up to infinity
  - Physical string force is labeled $f(n)$
  - $f_0 = R v_0 = \text{incremental force per period}$

http://phet.colorado.edu/sims/sims.php?sim=Wave_on_a_String
Overview of Wave Variable Choices

We have thus far considered only transverse displacement waves. We can also choose:

- **Transverse velocity** \( v \triangleq \dot{y} \)
- **Transverse acceleration** \( a \triangleq \ddot{y} \)
- **Slope waves** \( \dot{y} \)
- **Curvature waves** \( y'' = c^2 \ddot{y} \) (for ideal string)
- Any number of derivatives or integrals of displacement \( y \) with respect to time or position
- Conversion between time derivatives carried out by integrators and differentiators

\[ \ldots \rightarrow \int \rightarrow \dot{y} \rightarrow \frac{d}{dt} \rightarrow \frac{d}{dt} \rightarrow \frac{d}{dt} \rightarrow \frac{d}{dt} \rightarrow \ldots \]

String State, Cont’d

*Velocity waves* are a good overall choice for strings because:

- It is less noisy numerically to integrate for displacement than to differentiate for velocity
- Force (slope) waves = scaling of velocity waves (as we will show shortly)
- Analogous to volume velocity in *acoustic tubes*

Specifying String State

The complete state of the string is given at time \( n \) by:

- \( \{ y(t_n, x_m), \dot{y}(t_n, x_m) \}_{m=0}^{N-1} \) (typical in acoustics)
- \( \{ y(t_n, x_m), \dot{y}(t_{n-1}, x_m) \}_{m=0}^{N-1} \) (typical in acoustic simulations)
- \( \{ y^+(n - m), y^-(n + m) \}_{m=0}^{N-1} \) (what we did)
- \( \{ y^+(n - m), y^-(n + m) \}_{m=0}^{N-1} \) (today)
- \( \{ v^+(n - m), v^-(n + m) \}_{m=0}^{N-1} \) (today)
- Any two linearly independent variables (either physical variables or wave variables)

- All traveling-wave variables can be computed from any others, as long as string state is specified
- Wave variable conversions requiring differentiation or integration are relatively expensive since a large-order digital filter is necessary to do it right

First-Order Discrete-Time Wave-Variable Conversion Filters

**a) First-Order Difference**

\[ \hat{v}(n) = y(n) - y(n-1) \]

**b) First-Order “Leaky” Integrator**

\[ \hat{y}(n) = v(n) + g \hat{y}(n-1), \quad g < 1, g \approx 1 \]

(loss factor \( g \) avoids DC build-up)
Filter Design Approach

• Ideal Digital Differentiator:
  \[ H(e^{j\omega T}) \approx j\omega, \quad \omega \in [-\pi/T, \pi/T] \]

• Ideal Digital Integrator
  \[ H(e^{j\omega T}) \approx \frac{1}{j\omega}, \quad \omega \in [-\pi/T, \pi/T] \]

• Exact match is not possible in finite order

• Minimize
  \[ \| H(e^{j\omega T}) - \hat{H}(e^{j\omega T}) \| \]
  where \( \hat{H} \) is the digital filter frequency response

Spatial Derivatives

Slope waves are simply related to velocity waves. By the chain rule,
\[
y'(t, x) = \frac{\partial}{\partial x} y(t, x) \\
y'_r(t - x/c) + \frac{1}{c} \ddot{y}_r(t + x/c) \\
y'_l(t + x/c) + \frac{1}{c} \ddot{y}_l(t - x/c) \\
\Rightarrow \\
y'^+ = -\frac{1}{c} v^+ \\
y'^- = \frac{1}{c} v^- \\
v'^+ = -cy'^+ \\
v'^- = cy'^- \\
\]

• Physical string slope = (lower rail - upper rail)/c in a velocity-wave simulation

• \( v^-(0 + m) = v^+(0 - m) \forall m \) on a struck string

Ideal Differentiator Frequency Response

• Discontinuity at \( z = -1 \) ensures no exact finite-order solution

• Need oversampling factor, as in interpolator design (e.g., 20 kHz to 22.05 kHz)
  Response is unconstrained between bandlimit and \( f_s/2 \)

• As before, a small increment in oversampling factor yields a much larger decrease in required filter order to meet a given spec

Wave Impedance

We just showed
\[
y'^+ = -\frac{1}{c} v^+ \\
y'^- = \frac{1}{c} v^- \\
\]

Define new wave variables in terms of slope waves as
\[
f^+ = -Ky'^+ \\
f^- = -Ky'^- \\
\]

Note that \( f^\pm \) are in physical units of force.

We have
\[
f^+ = \frac{K}{c} v^+ \\
f^- = -\frac{K}{c} v^- \\
\]

Recall
\[
c = \sqrt{\frac{K}{\epsilon}} \\
\Rightarrow \quad \frac{K}{c} = \sqrt{K\epsilon} \triangleq R \\
\]

which is the wave impedance of the ideal string (force/velocity for traveling waves). Thus,
\[
\begin{align*}
f^+ &= R v^+ \\
f^- &= -R v^- \\
\end{align*}
\]
Ohm’s Law for Traveling Waves

We just derived Ohm’s Law for Traveling Waves on an Ideal String

\[
\begin{align*}
  f^+(n) &= R v^+(n) \\
  f^-(n) &= -R v^-(n)
\end{align*}
\]

where the velocity waves are defined in terms of transverse string displacement by

\[
\begin{align*}
  v^+(n) &\triangleq \dot{y}^+(n) \\
  v^-(n) &\triangleq -\dot{y}^-(n),
\end{align*}
\]

\(f^+\) and \(f^-\) are corresponding force waves, and \(R \triangleq \sqrt{K\epsilon}\) is the wave impedance of the string:

\[
R \triangleq \sqrt{K\epsilon} = \frac{K}{c} = \epsilon c
\]

To unify vibrating strings with acoustic tubes, we choose the force which acts to the right as our force wave variable:

\[
\begin{align*}
  f(t, x) &\triangleq f_r(t, x) = -K\dot{y}(t, x)
\end{align*}
\]

• Analogous to longitudinal pressure in acoustic tubes

We have

\[
f(t, x) = \frac{K}{c} [\dot{y}_r(t - x/c) - \dot{y}_l(t + x/c)]
\]

• Force waves are thus proportional to velocity waves

• Proportionality constant is called the wave impedance (or characteristic impedance) of the string:

\[
R \triangleq \sqrt{K\epsilon} = \frac{K}{c} = \epsilon c
\]

• Wave impedance = geometric mean of spring stiffness and inertial mass

• Traveling force-wave components:

\[
\begin{align*}
  f^+(n) &= R v^+(n) \\
  f^-(n) &= -R v^-(n)
\end{align*}
\]

For acoustic tubes, we have

\[
\begin{align*}
  p^+(n) &= R u^+(n) \\
  p^-(n) &= -R u^-(n)
\end{align*}
\]

where

• \(p^+(n)\) = right-going longitudinal pressure
• \(p^-(n)\) = left-going longitudinal pressure
• \(u^+(n)\) = left and right-going volume-velocity waves
• wave impedance is

\[
R = \frac{\rho c}{A} \quad \text{(Acoustic Tubes)}
\]

where

\[
\begin{align*}
  -\rho &= \text{mass per unit volume of air} \\
  -c &= \text{sound speed in air} \\
  -A &= \text{cross-sectional area of tube}
\end{align*}
\]

• For particle velocity, wave impedance = \(R_0 = \rho c\)

• Particle velocity is appropriate in open air, while volume velocity is used for acoustic tubes
Power Waves

Physically,

\[
\text{Power} = \text{Work/Time} = \text{Force} \times \text{Distance/Time} = \text{Force} \times \text{Velocity}
\]

Traveling power waves:

\[
\mathcal{P}^+(n) \triangleq f^+(n)v^+(n) \\
\mathcal{P}^-(n) \triangleq -f^-(n)v^-(n)
\]

From "Ohm's law" \( f^+ = Ru^+ \) and \( f^- = -Ru^- \), we have

\[
\mathcal{P}^+(n) = R[v^+(n)]^2 = \frac{[f^+(n)]^2}{R} \\
\mathcal{P}^-(n) = R[v^-(n)]^2 = \frac{[f^-(n)]^2}{R}
\]

Note that both \( \mathcal{P}^+ \) and \( \mathcal{P}^- \) are nonnegative

Summing traveling powers gives total power:

\[
\mathcal{P}(t_n, x_m) \triangleq \mathcal{P}^+(n - m) + \mathcal{P}^-(n + m)
\]

If we had instead defined \( \mathcal{P}^-(n) \triangleq f^-(n)v^-(n) \) (no minus sign in front), then summing the traveling powers would give net power flow.

\[
\text{Energy Density Waves}
\]

Energy density = potential + kinetic energy densities:

\[
W(t, x) = \frac{1}{2}Ky^2(t, x) + \frac{1}{2}\epsilon\dot{y}^2(t, x)
\]

Sampled wave energy density can be expressed as

\[
W(t_n, x_m) \triangleq W^+(n - m) + W^-(n + m)
\]

where

\[
W^+(n) = \frac{\mathcal{P}^+(n)}{c} = \frac{f^+(n)v^+(n)}{c} = \epsilon [v^+(n)]^2 = \frac{[f^+(n)]^2}{K} \\
W^-(n) = \frac{\mathcal{P}^-(n)}{c} = -\frac{f^-(n)v^-(n)}{c} = \epsilon [v^-(n)]^2 = \frac{[f^-(n)]^2}{K}
\]

Total wave energy in string of length \( L \):

\[
\mathcal{E}(t) = \int_{x=0}^{L} W(t, x)dx \approx \sum_{m=0}^{L/X-1} W(t, x_m)X
\]

Root-Power Waves

Wave variables normalized to square root of power carried:

\[
\hat{f}^+ \triangleq \frac{f^+}{\sqrt{R}} \quad \hat{f}^- \triangleq \frac{f^-}{\sqrt{R}} \\
\hat{v}^+ \triangleq \frac{v^+}{\sqrt{R}} \quad \hat{v}^- \triangleq \frac{v^-}{\sqrt{R}}
\]

\[
\Rightarrow \mathcal{P}^+ = -f^+v^+ = \hat{f}^+\hat{v}^+ = R(v^+)^2 = (\hat{v}^+)^2 = (f^+)^2 \\
\text{and} \quad \mathcal{P}^- = -f^-v^- = -\hat{f}^+\hat{v}^- = R(v^-)^2 = (\hat{v}^-)^2 = (f^-)^2
\]

- Normalized wave variables \( \hat{f}^\pm \) and \( \hat{v}^\pm \) behave physically like force and velocity waves
- Either can be squared to obtain signal power
- Dynamic range is normalized in \( L_2 \) sense
- Driving a normalized waveguide network with unit variance white noise gives signal power equal to 1 throughout the network
- Time-varying wave impedances do not cause "parametric amplification"