

# Moving Termination: Ideal String

## MUS420 Lecture Choice of Wave Variables in Digital Waveguide Models

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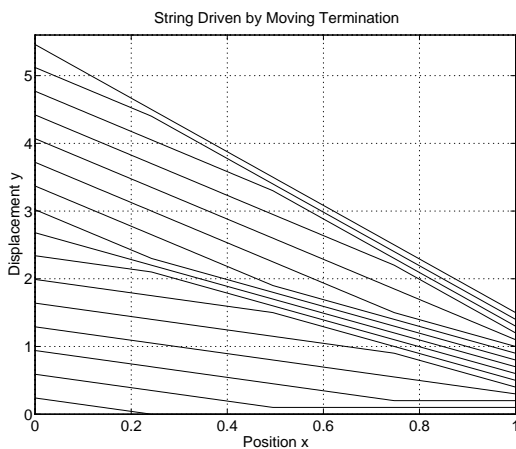
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### Outline

- Moving String Termination
- Wave Impedance
- Displacement, Velocity, Acceleration Waves
- Force Waves
- Root-Power Waves

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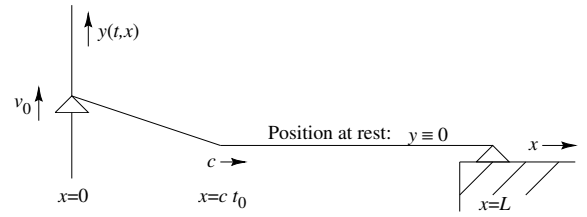
- Interactive Animation<sup>1</sup>



- Successive snapshots of the ideal string with a uniformly moving rigid termination
- Each plot is offset slightly higher for clarity
- GIF89A animation at <http://ccrma.stanford.edu/~jos/swgt/movet.html>

<sup>1</sup>[http://phet.colorado.edu/simulations/sims.php?sim=Wave\\_on\\_a\\_String](http://phet.colorado.edu/simulations/sims.php?sim=Wave_on_a_String)

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Uniformly moving rigid termination for an ideal string (tension  $K$ , mass density  $\epsilon$ ) at time  $0 < t_0 < L/c$ .

### Driving-Point Impedance $F_0/V_0$ :

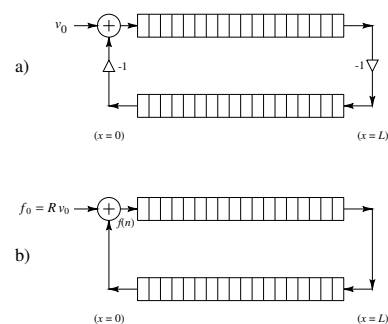
$$y'(t_0, 0) = -\frac{v_0 t_0}{c t_0} = -\frac{v_0}{c} = -\frac{v_0}{\sqrt{K/\epsilon}}$$

$$\Rightarrow f_0 = -K \sin(\theta) \approx -K y'(t, 0) = \sqrt{K\epsilon} v_0 \triangleq R v_0$$

- If the left endpoint moves with constant velocity  $v_0$  then the external applied force is  $f_0 = R v_0$
- $R \triangleq \sqrt{K\epsilon} \triangleq$  wave impedance (for transverse waves)
- Equivalent circuit is a resistor (dashpot)  $R > 0$
- We have the simple relation  $f_0 = R v_0$  only in the absence of return waves, i.e., until time  $t_0 = 2L/c$ .

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### Waveguide “Equivalent Circuits” for the Uniformly Moving Rigid String Termination



a) Velocity waves      b) Force waves

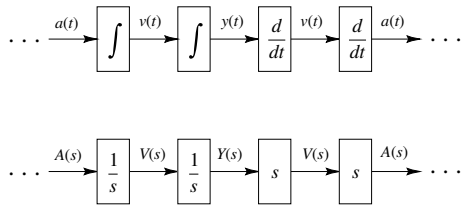
- String moves with speed  $v_0$  or 0 only
- String is always one or two straight segments
- “Helmholtz corner” (slope discontinuity) shuttles back and forth at speed  $c$
- String slope increases without bound
- Applied force at termination steps up to infinity
  - Physical string force is labeled  $f(n)$
  - $f_0 = R v_0 =$  incremental force per period

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## Overview of Wave Variable Choices

We have thus far considered only transverse *displacement* waves. We can also choose

- Transverse *velocity*  $v \triangleq \dot{y}$
- Transverse *acceleration*  $a \triangleq \ddot{y}$
- *Slope waves*  $y'$
- *Curvature waves*  $y''$  ( $= c^2 \ddot{y}$  for ideal string)
- Any number of derivatives or integrals of displacement  $y$  with respect to time or position
- Conversion between time derivatives carried out by *integrators* and *differentiators*



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## String State, Cont'd

*Velocity waves* are a good overall choice for strings because

- It is less noisy numerically to integrate for displacement than to differentiate for velocity
- Force (slope) waves = scaling of velocity waves (as we will show shortly)
- Analogous to volume velocity in *acoustic tubes*

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## Specifying String State

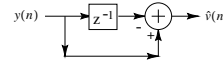
The complete *state* of the string is given at time  $n$  by

- $\{y(t_n, x_m), \dot{y}(t_n, x_m)\}_{m=0}^{N-1}$  (typical in acoustics)
- $\{y(t_n, x_m), y(t_{n-1}, x_m)\}_{m=0}^{N-1}$  (typical in acoustic simulations)
- $\{y^+(n-m), y^-(n+m)\}_{m=0}^{N-1}$  (what we did)
- $\{y'^+(n-m), y'^-(n+m)\}_{m=0}^{N-1}$  (today)
- $\{v^+(n-m), v^-(n+m)\}_{m=0}^{N-1}$  (today)
- Any *two* linearly independent variables (either *physical* variables or *wave variables*)
- All traveling-wave variables can be computed from any others, as long as string state is specified
- Wave variable conversions requiring differentiation or integration are relatively expensive since a large-order digital filter is necessary to do it right

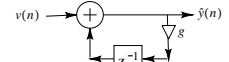
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## First-Order Discrete-Time Wave-Variable Conversion Filters

a) First-Order Difference



b) First-Order "Leaky" Integrator



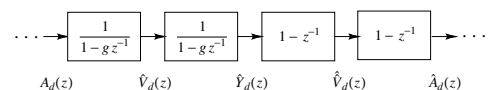
- First-order difference:

$$\hat{v}(n) = y(n) - y(n-1)$$

- First-order "leaky" integrator:

$$\hat{y}(n) = v(n) + g\hat{y}(n-1), \quad g < 1, g \approx 1$$

(loss factor  $g$  avoids DC build-up)



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## Filter Design Approach

- *Ideal Digital Differentiator:*

$$H(e^{j\omega T}) \approx j\omega, \quad \omega \in [-\pi/T, \pi/T]$$

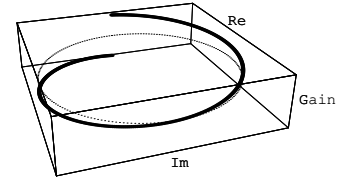
- *Ideal Digital Integrator*

$$H(e^{j\omega T}) \approx \frac{1}{j\omega}, \quad \omega \in [-\pi/T, \pi/T]$$

- Exact match is *not possible* in *finite order*
- Minimize  $\|H(e^{j\omega T}) - \hat{H}(e^{j\omega T})\|$  where  $\hat{H}$  is the digital filter frequency response

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## Ideal Differentiator Frequency Response



- Discontinuity at  $z = -1$  ensures no exact finite-order solution
- Need *oversampling factor*, as in interpolator design (e.g., 20 kHz to 22.05 kHz)  
Response is unconstrained between bandlimit and  $f_s/2$
- As before, a small increment in oversampling factor yields a much larger decrease in required filter order to meet a given spec

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## Spatial Derivatives

*Slope waves* are simply related to velocity waves.  
By the chain rule,

$$\begin{aligned} y'(t, x) &\triangleq \frac{\partial}{\partial x} y(t, x) \\ &= y'_r(t - x/c) + y'_l(t + x/c) \\ &= -\frac{1}{c} \dot{y}_r(t - x/c) + \frac{1}{c} \dot{y}_l(t + x/c) \\ &\rightarrow -\frac{1}{c} v^+(n - m) + \frac{1}{c} v^-(n + m) \end{aligned}$$

$\Rightarrow$

$$\begin{aligned} y'^+ &= -\frac{1}{c} v^+ \\ y'^- &= \frac{1}{c} v^- \end{aligned}$$

or

$$\begin{aligned} v^+ &= -cy'^+ \\ v^- &= cy'^- \end{aligned}$$

- Physical string slope = (lower rail - upper rail)/c in a velocity-wave simulation
- $\Rightarrow v^-(0 + m) = v^+(0 - m) \forall m$  on a struck string

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## Wave Impedance

We just showed

$$\begin{aligned} y'^+ &= -\frac{1}{c} v^+ \\ y'^- &= \frac{1}{c} v^- \end{aligned}$$

Define new wave variables in terms of slope waves as

$$\begin{aligned} f^+ &\triangleq -Ky'^+ \\ f^- &\triangleq -Ky'^- \end{aligned}$$

Note that  $f^\pm$  are in physical units of *force*.

We have

$$\begin{aligned} f^+ &= \frac{K}{c} v^+ \\ f^- &= -\frac{K}{c} v^- \end{aligned}$$

Recall

$$\begin{aligned} c &= \sqrt{\frac{K}{\epsilon}} \\ \Rightarrow \frac{K}{c} &= \sqrt{K\epsilon} \triangleq R \end{aligned}$$

which is the *wave impedance* of the ideal string (force/velocity for traveling waves). Thus,

$$\begin{aligned} f^+ &= Rv^+ \\ f^- &= -Rv^- \end{aligned}$$

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## Ohm's Law for Traveling Waves

We just derived *Ohm's Law for Traveling Waves on an Ideal String*

$$\begin{aligned} f^+(n) &= R v^+(n) \\ f^-(n) &= -R v^-(n) \end{aligned}$$

where the *velocity waves* are defined in terms of transverse string displacement by

$$\begin{aligned} v^+(n) &\triangleq \dot{y}^+(n) \\ v^-(n) &\triangleq \dot{y}^-(n), \end{aligned}$$

$f^+$  and  $f^-$  are corresponding *force waves*, and  $R \triangleq \sqrt{K\epsilon}$  is the *wave impedance* of the string:

$$R \triangleq \sqrt{K\epsilon} = \frac{K}{c} = \epsilon c$$

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To unify *vibrating strings* with *acoustic tubes*, we choose the force which *acts to the right* as our *force wave variable*:

$$f(t, x) \triangleq f_r(t, x) = \boxed{-K y'(t, x)}$$

- Analogous to longitudinal pressure in acoustic tubes
- We have

$$f(t, x) = \frac{K}{c} [\dot{y}_r(t - x/c) - \dot{y}_l(t + x/c)]$$

- Force waves are thus *proportional* to velocity waves
- Proportionality constant is called the *wave impedance* (or *characteristic impedance*) of the string:

$$R \triangleq \sqrt{K\epsilon} = \frac{K}{c} = \epsilon c$$

- Wave impedance = geometric mean of spring stiffness and inertial mass
- Traveling force-wave components:

$$\begin{aligned} f^+(n) &= R v^+(n) \\ f^-(n) &= -R v^-(n) \end{aligned}$$

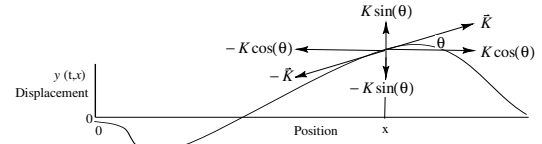
For *acoustic tubes*, we have

$$\begin{aligned} p^+(n) &= R_r u^+(n) \\ p^-(n) &= -R_r u^-(n) \end{aligned}$$

where

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## Force Waves



- Vertical force acting to the left is

$$f_l(t, x) = K \sin(\theta) \approx K \tan(\theta) = K y'(t, x)$$

- Opposing force, acting to the right, is

$$f_r(t, x) = -K \sin(\theta) \approx -K y'(t, x)$$

(Note that a negative slope pulls "up" on the segment to the right)

- These forces must cancel since a nonzero net force on a massless point would produce infinite acceleration

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- $p^+(n)$  = right-going *longitudinal pressure*
- $p^-(n)$  = left-going longitudinal pressure
- $u^\pm(n)$  = left and right-going *volume-velocity waves*
- wave impedance is

$$R_r = \frac{\rho c}{A} \quad (\text{Acoustic Tubes})$$

where

- $\rho$  = mass per unit volume of air
- $c$  = sound speed in air
- $A$  = cross-sectional area of tube

- For *particle velocity*, wave impedance =  $R_0 = \rho c$
- Particle velocity is appropriate in open air, while volume velocity is used for acoustic tubes

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## Power Waves

Physically,

$$\begin{aligned}\text{Power} &= \text{Work/Time} \\ &= \text{Force} \times \text{Distance/Time} \\ &= \text{Force} \times \text{Velocity}\end{aligned}$$

Traveling power waves:

$$\begin{aligned}\mathcal{P}^+(n) &\triangleq f^+(n)v^+(n) \\ \mathcal{P}^-(n) &\triangleq -f^-(n)v^-(n)\end{aligned}$$

From “Ohm’s law”  $f^+ = Rv^+$  and  $f^- = -Rv^-$ , we have

$$\begin{aligned}\mathcal{P}^+(n) &= R[v^+(n)]^2 = \frac{[f^+(n)]^2}{R} \\ \mathcal{P}^-(n) &= R[v^-(n)]^2 = \frac{[f^-(n)]^2}{R}\end{aligned}$$

Note that both  $\mathcal{P}^+$  and  $\mathcal{P}^-$  are *nonnegative*

Summing traveling powers gives total power:

$$\mathcal{P}(t_n, x_m) \triangleq \mathcal{P}^+(n-m) + \mathcal{P}^-(n+m)$$

If we had instead defined  $\mathcal{P}^-(n) \triangleq f^-(n)v^-(n)$  (no minus sign in front), then summing the traveling powers would give *net* power flow.

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## Energy Density Waves

Energy density = potential + kinetic energy densities:

$$W(t, x) \triangleq \underbrace{\frac{1}{2}Ky'^2(t, x)}_{\text{potential}} + \underbrace{\frac{1}{2}\epsilon\dot{y}^2(t, x)}_{\text{kinetic}}$$

Sampled wave energy density can be expressed as

$$W(t_n, x_m) \triangleq W^+(n-m) + W^-(n+m)$$

where

$$\begin{aligned}W^+(n) &= \frac{\mathcal{P}^+(n)}{c} = \frac{f^+(n)v^+(n)}{c} = \epsilon [v^+(n)]^2 = \frac{[f^+(n)]^2}{K} \\ W^-(n) &= \frac{\mathcal{P}^-(n)}{c} = -\frac{f^-(n)v^-(n)}{c} = \epsilon [v^-(n)]^2 = \frac{[f^-(n)]^2}{K}\end{aligned}$$

Total wave energy in string of length  $L$ :

$$\mathcal{E}(t) = \int_{x=0}^L W(t, x) dx \approx \sum_{m=0}^{L/X-1} W(t, x_m) X$$

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## Root-Power Waves

Wave variables *normalized* to square root of power carried:

$$\begin{aligned}\tilde{f}^+ &\triangleq f^+/\sqrt{R} & \tilde{f}^- &\triangleq f^-/\sqrt{R} \\ \tilde{v}^+ &\triangleq v^+\sqrt{R} & \tilde{v}^- &\triangleq v^-\sqrt{R}\end{aligned}$$

$\Rightarrow$

$$\begin{aligned}\mathcal{P}^+ &= f^+v^+ = \tilde{f}^+\tilde{v}^+ \\ &= R(v^+)^2 = (\tilde{v}^+)^2 \\ &= (f^+)^2/R = (\tilde{f}^+)^2\end{aligned}$$

and

$$\begin{aligned}\mathcal{P}^- &= -f^-v^- = -\tilde{f}^-\tilde{v}^- \\ &= R(v^-)^2 = (\tilde{v}^-)^2 \\ &= (f^-)^2/R = (\tilde{f}^-)^2\end{aligned}$$

- Normalized wave variables  $\tilde{f}^\pm$  and  $\tilde{v}^\pm$  behave physically like force and velocity waves
- Either can be squared to obtain signal power
- Dynamic range is normalized in  $L_2$  sense
- Driving a normalized waveguide network with unit variance white noise gives signal power equal to 1 throughout the network
- Time-varying wave impedances do not cause “parametric amplification”

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