MUS420 Lecture
Choice of Wave Variables in Digital Waveguide Models

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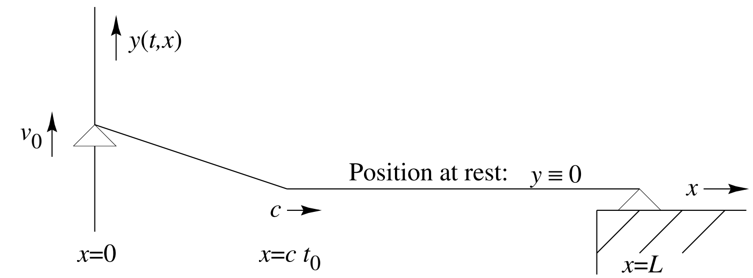
Outline

- Moving String Termination
- Wave Impedance
- Displacement, Velocity, Acceleration Waves
- Force Waves
- Root-Power Waves

Moving Termination: Ideal String



Uniformly moving rigid termination for an ideal string (tension K, mass density ϵ) at time $0 < t_0 < L/c$.

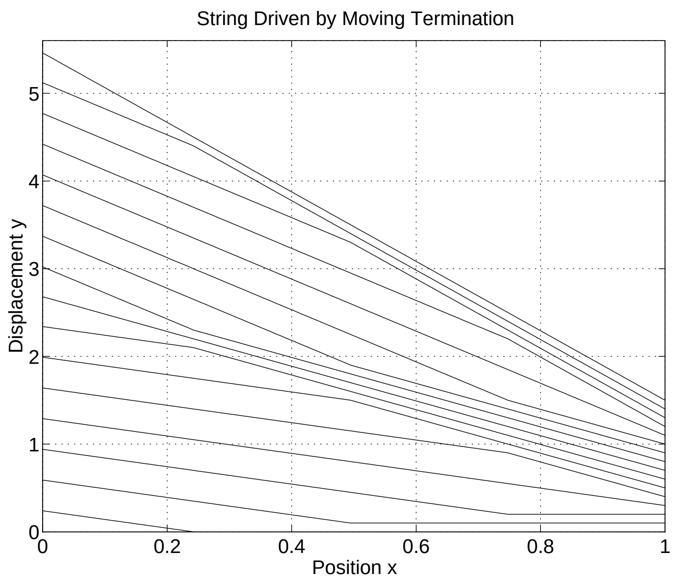
Driving-Point Impedance F_0/V_0:

$$y'(t_0, 0) = -\frac{v_0 t_0}{c t_0} = -\frac{v_0}{c} = -\frac{v_0}{\sqrt{K/\epsilon}}$$

$$\Rightarrow \quad f_0 = -K\sin(\theta) \approx -K y'(t,0) = \sqrt{K\epsilon}\, v_0 \stackrel{\Delta}{=} R\, v_0$$

- If the left endpoint moves with constant velocity v_0 then the external applied force is $f_0 = Rv_0$
- $R \stackrel{\Delta}{=} \sqrt{K\epsilon} \stackrel{\Delta}{=}$ *wave impedance* (for transverse waves)
- Equivalent circuit is a *resistor* (dashpot) $R > 0$
- We have the simple relation $f_0 = Rv_0$ only in the *absence of return waves*, i.e., until time $t_0 = 2L/c$.

- Interactive Animation[1]

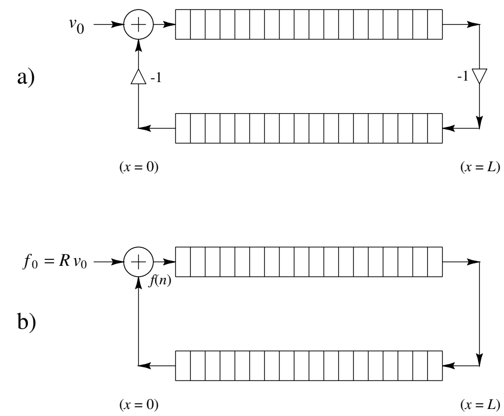


- Successive snapshots of the ideal string with a uniformly moving rigid termination
- Each plot is offset slightly higher for clarity
- GIF89A animation at

  `http://ccrma.stanford.edu/~jos/swgt/movet.html`

[1] http://phet.colorado.edu/simulations/sims.php?sim=Wave_on_a_String

Waveguide "Equivalent Circuits" for the Uniformly Moving Rigid String Termination



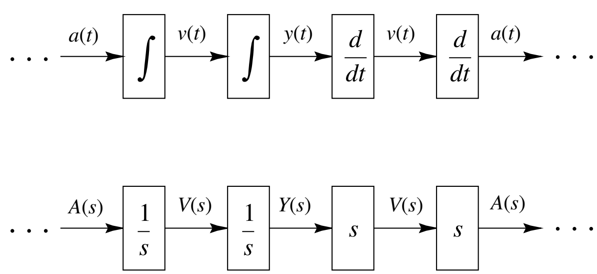
a) Velocity waves b) Force waves

- String moves with speed v_0 or 0 only
- String is always one or two straight segments
- "Helmholtz corner" (slope discontinuity) shuttles back and forth at speed c
- String slope increases without bound
- Applied force at termination steps up to infinity
  - Physical string force is labeled $f(n)$
  - $f_0 = Rv_0 =$ *incremental* force per period

Overview of Wave Variable Choices

We have thus far considered only transverse *displacement* waves. We can also choose

- Transverse *velocity* $v \triangleq \dot{y}$
- Transverse *acceleration* $a \triangleq \ddot{y}$
- *Slope* waves y'
- *Curvature* waves y'' $(= c^2\ddot{y}$ for ideal string)
- Any number of derivatives or integrals of displacement y with respect to time or position
- Conversion between time derivatives carried out by *integrators* and *differentiators*



Specifying String State

The complete *state* of the string is given at time n by

- $\{y(t_n, x_m), \dot{y}(t_n, x_m)\}_{m=0}^{N-1}$ (typical in acoustics)
- $\{y(t_n, x_m), y(t_{n-1}, x_m)\}_{m=0}^{N-1}$ (typical in acoustic simulations)
- $\{y^+(n-m), y^-(n+m)\}_{m=0}^{N-1}$ (what we did)
- $\{y'^+(n-m), y'^-(n+m)\}_{m=0}^{N-1}$ (today)
- $\{v^+(n-m), v^-(n+m)\}_{m=0}^{N-1}$ (today)
- Any *two* linearly independent variables (either *physical* variables or *wave* variables)

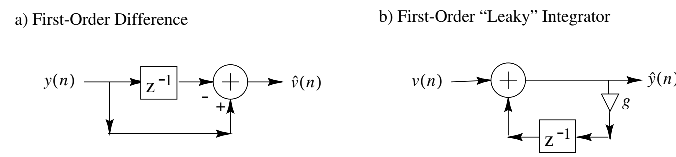
- All traveling-wave variables can be computed from any others, as long as string state is specified
- Wave variable conversions requiring differentiation or integration are relatively expensive since a large-order digital filter is necessary to do it right

String State, Cont'd

Velocity waves are a good overall choice for strings because

- It is less noisy numerically to integrate for displacement than to differentiate for velocity
- Force (slope) waves = scaling of velocity waves (as we will show shortly)
- Analogous to volume velocity in *acoustic tubes*

First-Order Discrete-Time Wave-Variable Conversion Filters



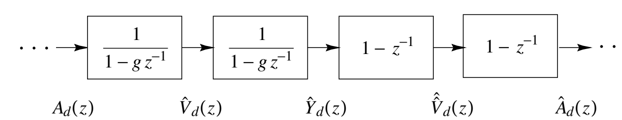
- First-order difference:

$$\hat{v}(n) = y(n) - y(n-1)$$

- First-order "leaky" integrator:

$$\hat{y}(n) = v(n) + g\hat{y}(n-1), \quad g < 1, g \approx 1$$

(loss factor g avoids DC build-up)



Filter Design Approach

- *Ideal Digital Differentiator:*

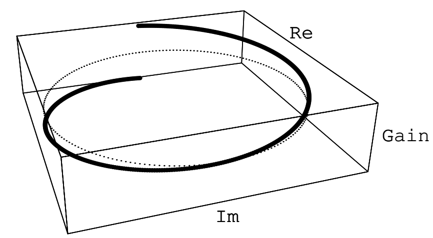
$$H(e^{j\omega T}) \approx j\omega, \quad \omega \in [-\pi/T, \pi/T]$$

- *Ideal Digital Integrator*

$$H(e^{j\omega T}) \approx \frac{1}{j\omega}, \quad \omega \in [-\pi/T, \pi/T]$$

- Exact match is *not possible* in *finite order*
- Minimize $\left\| H(e^{j\omega T}) - \hat{H}(e^{j\omega T}) \right\|$ where $\hat{H}$ is the digital filter frequency response

Ideal Differentiator Frequency Response



- Discontinuity at $z = -1$ ensures no exact finite-order solution
- Need *oversampling factor*, as in interpolator design (e.g., 20 kHz to 22.05 kHz)
  Response is unconstrained between bandlimit and $f_s/2$
- As before, a small increment in oversampling factor yields a much larger decrease in required filter order to meet a given spec

Spatial Derivatives

Slope waves are simply related to velocity waves.
By the chain rule,

$$\begin{aligned} y'(t,x) &\stackrel{\Delta}{=} \frac{\partial}{\partial x} y(t,x) \\ &= y'_r(t-x/c) + y'_l(t+x/c) \\ &= -\frac{1}{c}\dot{y}_r(t-x/c) + \frac{1}{c}\dot{y}_l(t+x/c) \\ &\to -\frac{1}{c}v^+(n-m) + \frac{1}{c}v^-(n+m) \end{aligned}$$

$\Rightarrow$

$$\begin{aligned} y'^+ &= -\ \tfrac{1}{c}v^+ \\ y'^- &= \quad \tfrac{1}{c}v^- \end{aligned}$$

or

$$\begin{aligned} v^+ &= -\ cy'^+ \\ v^- &= \quad cy'^- \end{aligned}$$

- Physical string slope = (lower rail - upper rail)/c in a velocity-wave simulation
- $\Rightarrow v^-(0+m) = v^+(0-m)\,\forall m$ on a struck string

Wave Impedance

We just showed

$$\begin{aligned} y'^+ &= -\ \tfrac{1}{c}v^+ \\ y'^- &= \quad \tfrac{1}{c}v^- \end{aligned}$$

Define new wave variables in terms of slope waves as

$$\begin{aligned} f^+ &\stackrel{\Delta}{=} -Ky'^+ \\ f^- &\stackrel{\Delta}{=} -Ky'^- \end{aligned}$$

Note that $f^{\pm}$ are in physical units of *force*.
We have

$$\begin{aligned} f^+ &= \quad \tfrac{K}{c}v^+ \\ f^- &= -\ \tfrac{K}{c}v^- \end{aligned}$$

Recall

$$c = \sqrt{\frac{K}{\epsilon}}$$

$$\Rightarrow \qquad \frac{K}{c} = \sqrt{K\epsilon} \stackrel{\Delta}{=} R$$

which is the *wave impedance* of the ideal string (force/velocity for traveling waves). Thus,

$$\boxed{\begin{aligned} f^+ &= \quad Rv^+ \\ f^- &= -\ Rv^- \end{aligned}}$$

Ohm's Law for Traveling Waves

We just derived *Ohm's Law for Traveling Waves on an Ideal String*

$$\boxed{\begin{array}{rcl} f^+(n) & = & R\,v^+(n) \\ f^-(n) & = - & R\,v^-(n) \end{array}}$$

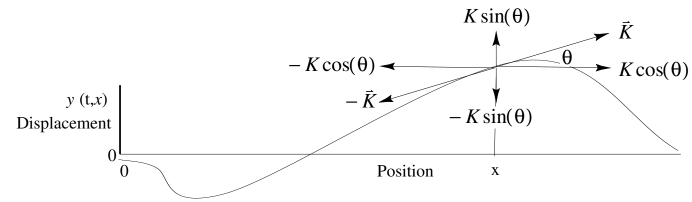
where the *velocity waves* are defined in terms of transverse string displacement by

$$\begin{aligned} v^+(n) &\stackrel{\Delta}{=} \dot{y}^+(n) \\ v^-(n) &\stackrel{\Delta}{=} \dot{y}^-(n), \end{aligned}$$

f^+ and f^- are corresponding *force waves*, and $R \stackrel{\Delta}{=} \sqrt{K\epsilon}$ is the *wave impedance* of the string:

$$\boxed{R \stackrel{\Delta}{=} \sqrt{K\epsilon} = \frac{K}{c} = \epsilon c}$$

Force Waves



- Vertical force acting to the left is

$$f_l(t,x) = K\sin(\theta) \approx K\tan(\theta) = K\,y'(t,x)$$

- Opposing force, acting to the right, is

$$f_r(t,x) = -K\sin(\theta) \approx -K\,y'(t,x)$$

  (Note that a negative slope pulls "up" on the segment to the right)

- These forces must cancel since a nonzero net force on a massless point would produce infinite acceleration

To unify *vibrating strings* with *acoustic tubes*, we choose the force which *acts to the right* as our *force wave variable*:

$$f(t,x) \stackrel{\Delta}{=} f_r(t,x) = \boxed{-Ky'(t,x)}$$

- Analogous to longitudinal pressure in acoustic tubes
- We have
$$f(t,x) = \frac{K}{c}\left[\dot{y}_r(t-x/c) - \dot{y}_l(t+x/c)\right]$$
- Force waves are thus *proportional* to velocity waves
- Proportionality constant is called the *wave impedance* (or *characteristic impedance*) of the string:
$$R \stackrel{\Delta}{=} \sqrt{K\epsilon} = \frac{K}{c} = \epsilon c$$
- Wave impedance = geometric mean of spring stiffness and inertial mass
- Traveling force-wave components:
$$\begin{aligned} f^+(n) &= \quad R\,v^+(n) \\ f^-(n) &= -\; R\,v^-(n) \end{aligned}$$

For *acoustic tubes*, we have

$$\begin{aligned} p^+(n) &= \quad R_{\tau}\,u^+(n) \\ p^-(n) &= -\; R_{\tau}\,u^-(n) \end{aligned}$$

where

- $p^+(n)$ = right-going *longitudinal pressure*
- $p^-(n)$ = left-going longitudinal pressure
- $u^{\pm}(n)$ = left and right-going *volume-velocity waves*
- wave impedance is
$$\boxed{R_{\tau} = \frac{\rho c}{A}} \qquad \text{(Acoustic Tubes)}$$
  where
  - ρ = mass per unit volume of air
  - c = sound speed in air
  - A = cross-sectional area of tube
- For *particle velocity*, wave impedance = $R_0 = \rho c$
- Particle velocity is appropriate in open air, while volume velocity is used for acoustic tubes

Power Waves

Physically,

$$\begin{aligned}\text{Power} &= \text{Work/Time}\\ &= \text{Force} \times \text{Distance/Time}\\ &= \text{Force} \times \text{Velocity}\end{aligned}$$

Traveling power waves:

$$\begin{aligned}\mathcal{P}^+(n) &\stackrel{\Delta}{=} \quad f^+(n)v^+(n)\\ \mathcal{P}^-(n) &\stackrel{\Delta}{=} -\; f^-(n)v^-(n)\end{aligned}$$

From "Ohm's law" $f^+ = Rv^+$ and $f^- = -Rv^-$, we have

$$\begin{aligned}\mathcal{P}^+(n) &= R\,[v^+(n)]^2 = \frac{[f^+(n)]^2}{R}\\ \mathcal{P}^-(n) &= R\,[v^-(n)]^2 = \frac{[f^-(n)]^2}{R}\end{aligned}$$

Note that both $\mathcal{P}^+$ and $\mathcal{P}^-$ are *nonnegative*

Summing traveling powers gives total power:

$$\mathcal{P}(t_n, x_m) \stackrel{\Delta}{=} \mathcal{P}^+(n-m) + \mathcal{P}^-(n+m)$$

If we had instead defined $\mathcal{P}^-(n) \stackrel{\Delta}{=} f^-(n)v^-(n)$ (no minus sign in front), then summing the traveling powers would give *net* power flow.

Energy Density Waves

Energy density = potential + kinetic energy densities:

$$W(t,x) \stackrel{\Delta}{=} \underbrace{\frac{1}{2}Ky'^2(t,x)}_{\text{potential}} + \underbrace{\frac{1}{2}\epsilon\dot{y}^2(t,x)}_{\text{kinetic}}$$

Sampled wave energy density can be expressed as

$$W(t_n, x_m) \stackrel{\Delta}{=} W^+(n-m) + W^-(n+m)$$

where

$$\begin{aligned}W^+(n) &= \frac{\mathcal{P}^+(n)}{c} = \frac{f^+(n)v^+(n)}{c} = \epsilon\left[v^+(n)\right]^2 = \frac{\left[f^+(n)\right]^2}{K}\\ W^-(n) &= \frac{\mathcal{P}^-(n)}{c} = -\frac{f^-(n)v^-(n)}{c} = \epsilon\left[v^-(n)\right]^2 = \frac{\left[f^-(n)\right]^2}{K}\end{aligned}$$

Total wave energy in string of length L:

$$\mathcal{E}(t) = \int_{x=0}^{L} W(t,x)dx \approx \sum_{m=0}^{L/X-1} W(t,x_m)X$$

Root-Power Waves

Wave variables *normalized* to *square root* of power carried:

$$\begin{array}{lll} \tilde{f}^+ \triangleq f^+/\sqrt{R} & \qquad & \tilde{f}^- \triangleq f^-/\sqrt{R} \\ \tilde{v}^+ \triangleq v^+\sqrt{R} & \qquad & \tilde{v}^- \triangleq v^-\sqrt{R} \end{array}$$

$\Rightarrow$

$$\begin{aligned} \mathcal{P}^+ &= \quad f^+v^+ &&= \tilde{f}^+\tilde{v}^+ \\ &= R\,(v^+)^2 &&= (\tilde{v}^+)^2 \\ &= (f^+)^2/R &&= (\tilde{f}^+)^2 \end{aligned}$$

and

$$\begin{aligned} \mathcal{P}^- &= -f^-v^- &&= -\tilde{f}^+\tilde{v}^+ \\ &= R\,(v^-)^2 &&= (\tilde{v}^-)^2 \\ &= (f^-)^2/R &&= (\tilde{f}^-)^2 \end{aligned}$$

- Normalized wave variables $\tilde{f}^\pm$ and $\tilde{v}^\pm$ behave physically like force and velocity waves
- Either can be squared to obtain signal power
- Dynamic range is normalized in L_2 sense
- Driving a normalized waveguide network with unit variance white noise gives signal power equal to 1 throughout the network
- Time-varying wave impedances do not cause "parametric amplification"