Clarinet Tonehole Impedance Parameters

(open-hole shunt impedance) \(R_c^o = R_b(jkt_c + \xi_c)\)
(closed-hole shunt impedance) \(R_c^c = -jR_b\cot(kt_h)\)
(open-hole series impedance) \(R_s^o = -jR_bkt_s^o\)
(closed-hole series impedance) \(R_s^c = -jR_bkt_s^c\)

- \(t_c\) and \(t_s\) are the equivalent series lengths of the open and closed tonehole, respectively:
- \(t_c^o = \frac{0.47b(b/a)^4}{\tanh(1.84t_h/b) + 0.62(b/a)^2 + 0.64(b/a)}\)
- \(t_c^c = \frac{0.47b(b/a)^4}{\coth(1.84t_h/b) + 0.62(b/a)^2 + 0.64(b/a)}\)

where \(a\) is radius of the main bore

- The closed-tonehole height \(V_h/(\pi b^2)\) is \( \approx \)

\[t_h = t_w + \frac{b^2}{8a} \left[ 1 + 0.172 \left( \frac{b}{a} \right)^2 \right] \]

where \(t_w\) is tonehole chimney height at center

Note: The specific resistance of the open tonehole, \(\xi_c\), is the only real impedance and therefore the only source of wave energy loss at the tonehole. It is given by

\[\xi_c = 0.25(kb)^2 + \alpha t_h + (1/4)kd_h\ln(2b/r_c),\]

where \(r_c\) is the radius of curvature of the tonehole, \(d_v\) is the viscous boundary layer thickness which expressible in terms of the shear viscosity \(\eta\) of air as

\[d_v = \sqrt{\frac{2\eta}{\rho\omega}}\]
and $\alpha$ is the real part of the propagation wavenumber (or minus the imaginary part of complex spatial frequency $k$). In the large-tube limit (i.e., when the tube radius is large compared with the viscous boundary layer), $\alpha$ is given by

$$\alpha = \frac{1}{2bc} \left[ \frac{2\pi\omega}{\rho} + (\gamma - 1) \frac{2\kappa\omega}{\rho C_p} \right]$$

where $\gamma = 1.4$ is the adiabatic gas constant for air, $\kappa$ is the thermal conductivity of air, and $C_p$ is the specific heat of air at constant pressure.

For air at $300^\circ$ Kelvin ($26.85^\circ$ C), valid within $\pm 10$ degrees of that temperature:

- $\rho = 1.1769 \times 10^{-3}(1 - 0.00335\Delta T) \text{ g/cm}^3$
- $\eta = 1.846 \times 10^{-4}(1 + 0.0025\Delta T) \text{ g/sec/cm}$
- $\gamma = 1.4017(1 - 0.0002\Delta T)$
- $\nu = \sqrt{\eta C_p/\kappa} = 0.8418(1 - 0.0002\Delta T)$
- $c = 3.4723 \times 10^4(1 + 0.00166\Delta T) \text{ cm/sec}$

$\alpha = \frac{\omega}{c} \left( \frac{1.045}{r_v} + \frac{1.080}{r_v^2} + \frac{0.750}{r_v^3} \right)$ (valid for $r_v > 2$)

- $r_v = b\sqrt{\frac{\nu}{\eta}} = \sqrt{2}\frac{b}{d_v}$ can be interpreted as $\sqrt{2}$ times the ratio of the tonehole radius $b$ to the viscous boundary layer thickness $d_v$.

**Digital Waveguide Tonehole Formulation**

For implementation in a digital waveguide model, the lumped shunt and series impedances must be converted to scattering parameters:

$$[ P_1 \quad U_1 ] = \begin{bmatrix} R_0 & \frac{R_0 R'_0}{R'_0 + R_0} \end{bmatrix} \begin{bmatrix} P_2 \quad U_2 \end{bmatrix}$$

- Substitute $k = \omega/c$ in the impedance values measured by Keefe to convert spatial frequency to temporal frequency
- Substitute
  - $P_i = P^+_i + P^-_i$
  - $U_i = \frac{P^+_i - P^-_i}{R_0}$, $i = 1, 2$
- Finally, solve for the outgoing waves $P^-_1, P^-_2$ in terms of the incoming waves $P^+_1, P^+_2$.

**Derivation**

From Keefe’s transmission-matrix model, we have

$$[ P_1 \quad U_1 ] = \begin{bmatrix} 1 + \frac{R_0}{R'_0} & \frac{R_0}{R'_0} \end{bmatrix} \begin{bmatrix} P_2 \quad U_2 \end{bmatrix}$$
Mathematica code

Clear["t*", "p*", "u*", "r*"]
transmissionMatrix = {{t11, t12}, {t21, t22}};
leftPort = {p2p + p2m}, {(p2p - p2m)/r2};
rightPort = {{p1p + p1m}, (p1p - p1m)/r1};
Format[t11, TeXForm] := "T_{11}"
Format[p1p, TeXForm] := "P_1^+"

P_1^+ = \frac{2P_1^+}{R_1^+R_1T_{11} - P_1^+R_1T_{12} + P_1^+R_1T_{21} + P_1^+R_1T_{22}}

P_1^- = \frac{P_1^-R_1T_{11} - P_1^-R_1T_{12} + P_1^-R_1T_{21} - P_1^-R_1T_{22} + 2P_1^+R_1T_{11}T_{22}}{R_1T_{11} - R_1T_{12} - R_1T_{21} + R_1T_{22}}

Substituting relevant values for Keefe’s tonehole model, we obtain

\begin{align*}
\begin{bmatrix} P_1^+ \\ P_1^- \end{bmatrix} &= \begin{bmatrix} S & T \\ T & S \end{bmatrix} \begin{bmatrix} P_1^+ \\ P_1^- \end{bmatrix} \\
&= \frac{1}{(2R_1 + R_0)(2R_1 + R_1 + 4R_T)} \begin{bmatrix} 4sR_1 + 4R_1^2 - 4R_T^2 & 8sR_1R_0 \\ 8sR_1R_0 & 4R_0 + 2R_1^2 - 4R_T^2 \end{bmatrix} \begin{bmatrix} P_1^+ \\ P_1^- \end{bmatrix}
\end{align*}

Thus,

S(\omega) = \frac{4R_0R_s + R_s^2 - 4R_T^2}{(2R_0 + R_s)(2R_0 + R_s + 4R_s)} \approx -\frac{R_0}{R_0 + 2R_s}

is the reflectance of the tonehole (the same from either direction), and the transmittance is given by

T(\omega) = \frac{8R_0R_s}{(2R_0 + R_s)(2R_0 + R_s + 4R_s)} \approx \frac{2R_s}{R_0 + 2R_s}

- Tonehole reflectance and transmittance are the same in either direction
- The notation "S" for reflectance is chosen because every reflectance is a Schur function (stable and not exceeding unit magnitude on the unit circle in the z plane)
- The approximate forms are obtained by neglecting the negative series inductance \(R_s\) which serves to adjust the effective length of the bore, and which therefore can be implemented elsewhere in the interpolated delay-line calculation as discussed further below

- The open and closed tonehole cases are obtained by substituting \{R_s = R_s^c, R_s = R_s^o\} and \{R_0 = R_0^c, R_0 = R_0^o\}, respectively

Open and Closed Tonehole Reflectance: Measured versus Second-Order Digital Filter Approximation

Impulse Response of Several Toneholes in a Clarinet Bore: Measured versus Digital Waveguide Model
In a manner analogous to converting the four-multiply Kelly-Lochbaum (KL) scattering junction into a one-multiply form, we may pursue a “one-filter” form of the waveguide tonehole model.

- The series inertance gives some initial trouble, since

\[ [1 + S(\omega)] - T(\omega) = \frac{2R_a}{2R_0 + R_a} \Delta L(\omega) \]

instead of zero as in the KL junction.

- In the scattering formulas for the general loaded waveguide junction, the reflectance seen on any branch is always the transmittance from that branch to any other branch minus 1. I.e., if \( \alpha_i \) denotes the transmittance from branch \( i \) to all other branches meeting at the junction, then \( \alpha_i - 1 \) is the reflectance seen on branch \( i \).

- By factoring out the common term \( \alpha_i \), one-multiply two-port scattering junctions are found.

Substituting \( T = 1 + S - L \) into the basic scattering relations, and factoring out \( S \), we obtain, in the frequency domain,

\[
P_1^- (\omega) = S P_1^+ + T P_2^+ \\
= S [P_1^+ + P_2^+] + [1 - L] P_2^+ \\
\Delta = S [P_1^+ + P_2^+] + AP_2^+ 
\]

and, similarly,

\[
P_2^- (\omega) = S [P_1^+ + P_2^+] + AP_1^+ 
\]

The resulting tonehole implementation is shown below:

- Since \( L(\omega) \approx 0 \), it can be neglected to first order, and \( A(\omega) \approx 1 \), reducing both of the above forms to an approximate “one-filter” tonehole implementation.

- Since \( R_a = -j R_b \omega t / c \) is a pure negative reactance, we have

\[
A(\omega) = 1 - L(\omega) = \frac{R_0 - R_a/2}{R_0 + R_a/2} = \frac{p + j\omega}{p - j\omega} = \frac{R_0 c}{R_b t} 
\]

- In this form, it is clear that \( A(\omega) \) is a first-order allpass filter with a single pole-zero pair near infinity.

Unfortunately, the pole is in the right-half-plane and hence unstable. We cannot therefore implement it as shown.
Using elementary manipulations, the unstable allpasses can be moved to the configuration shown below:

![Diagram of the Clarinet Tonehole as a Two-Port Loaded Junction]

Notes:

- **$T(\omega)/A(\omega)$** is stable whenever $T$ is stable.
- The unstable allpasses now operate only on the two incoming wave variables, which implies they can be implemented implicitly by slightly reducing the (interpolated) delay-lines leading to the junction from either side.
- The tonehole requires only one filter $S/A$ or $T/A$.

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### The Clarinet Tonehole as a Two-Port Loaded Junction

It seems reasonable to expect that the tonehole should be representable as a load along a waveguide bore model, thus creating a loaded two-port junction with two identical bore ports on either side of the tonehole. From the general relations for the loaded parallel pressure-wave junction, in the two-port case with $R_1 = R_2 = R_0$, we have

\[
\begin{align*}
P_i(s) &= \alpha P_{i1}^+ + \alpha P_{i2}^- + \alpha = 2\Gamma_0/[G_i(s) + 2\Gamma_0] \\
P_i^+(s) &= P_i(s) - \alpha P_{i2}^+ = (\alpha - 1)P_1^+ + \alpha P_2^+ = \alpha(P_1^+ + P_1^-) - P_1^- \\
P_i^-(s) &= P_i(s) - \alpha P_{i1}^- = (\alpha - 1)P_2^- + \alpha P_1^- = \alpha(P_1^- + P_2^-) - P_2^-
\end{align*}
\]

- The general loaded two-port junction can be implemented in “one-filter shared-transmittance form” as shown above with $A(\omega) = 1$ ($L(\omega) = 0$) and

\[
T(\omega) = \alpha + \frac{2\Gamma_0}{G_i(s) + 2\Gamma_0} \frac{2R_j(s)}{2R_j(s) + R_0}
\]

- The simplified Keefe tonehole model (negative inerntance removed, i.e., $R_o = 0$), is equivalent to a loaded two-port waveguide junction with the two-port load impedance set to the tonehole shunt impedance $R_j = R_a$.

We now see precisely how the negative series inerntance $R_a$ provides a negative, frequency-dependent, length correction for the bore. The phase delay of $A(\omega)$ can be computed as

\[
D_A(\omega) = -\frac{A(\omega)}{\omega} = -2\tan^{-1}(\omega/p) = -2\tan^{-1}(k_t R_b/R_0)
\]

- Negative delay correction goes to zero with any frequency $k = \omega/c$, series tonehole length $t_a$, tonehole radius $R_b$, or main bore admittance $\Gamma_0 = 1/R_0$.
- In practice, it is common to combine all delay corrections into a single “tuning allpass filter” for the whole bore.
- Whenever the desired allpass delay goes negative, we simply add a sample of delay to the desired allpass phase-delay and subtract it from the nearest delay.
- In other words, negative delays have to be “pulled out” of the allpass and used to shorten an adjacent interpolated delay line. Such delay lines are normally available in practical modeling situations.

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- Each series impedance $R_o/2$ in the split-T model of Keefe can be modeled as a series waveguide junction with a load of $R_o/2$.
- Set the transmission matrix parameters to the values $T_{11} = T_{22} = 1$, $T_{12} = R_a/2$, and $T_{21} = 0$ to get

\[
\begin{align*}
P_1^- &= (1 - \alpha)P_1^+ + \alpha P_2^- \\
P_1^+ &= \alpha P_1^+ + (1 - \alpha)P_2^-
\end{align*}
\]

where $\alpha = 2R_o/(2R_0 + R_a/2)$ is the alpha parameter for a series loaded waveguide junction involving two impedance $R_0$ waveguides joined in series with each other and with a load impedance of $R_a/2$.

- Switch to the more general convention in which the “$+” superscript denotes waves traveling into a junction of any number of waveguides. This exchanges “$+$” with “$-$” at port 2 to yield

\[
\begin{align*}
P_1^- &= (1 - \alpha)P_1^+ + \alpha P_2^- \\
P_2^- &= \alpha P_1^+ + (1 - \alpha)P_2^-
\end{align*}
\]

- Convert pressure to velocity using $P_i^+ = R_0 U_i^+$ and $P_i^- = -R_0 U_i^-$ to obtain

\[
\begin{align*}
U_1^- &= (\alpha - 1)U_1^+ - \alpha U_2^+ \\
U_2^- &= -\alpha U_1^+ + (\alpha - 1)U_2^+
\end{align*}
\]
Finally, toggle the reference direction of port 2 (the “current” arrow for $u_2$ on port 2 in Keefe’s split-T model) so that velocity is positive flowing into the junction on both ports (which is the convention typically followed in circuit theory). This amounts to negating $U_2^\pm$, giving

$$U_1^- = U_J - U_1^+$$
$$U_2^- = U_J - U_2^+$$

where $U_J \triangleq (aU_1^+ + aU_2^+)$. This is the canonical form of the two-port series scattering junction.