MUS420 Supplement
Woodwind Tone-Hole Modeling

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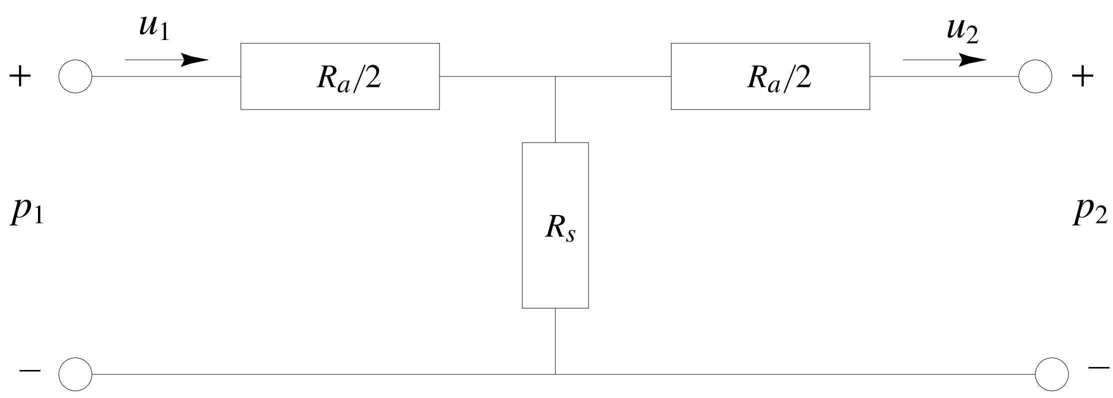
Outline

- Keefe Tonehole Model
- Transmission Matrix Formulation
- Conversion to Digital Waveguide Formulation
- One-Filter Forms
- “Loaded Waveguide Junction” Interpretation

Keefe Tonehole Model



- The clarinet tonehole model developed by Keefe (1990) is parametrized in terms of series and shunt impedances (resistance and reactance), as shown.
- The *transmission matrix* description of this two-port is given by the product of the transmission matrices for the series impedance $R_a/2$, shunt impedance R_s, and series impedance $R_a/2$, respectively:

$$\begin{bmatrix} P_1 \\ U_1 \end{bmatrix} = \begin{bmatrix} 1 & R_a/2 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ R_s^{-1} & 1 \end{bmatrix} \begin{bmatrix} 1 & R_a/2 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} P_2 \\ U_2 \end{bmatrix}$$

$$= \begin{bmatrix} 1 + \frac{R_a}{2R_s} & R_a[1 + \frac{R_a}{4R_s}] \\ \frac{1}{R_s} & 1 + \frac{R_a}{2R_s} \end{bmatrix} \begin{bmatrix} P_2 \\ U_2 \end{bmatrix}$$

where all quantities are written in the frequency domain

Clarinet Tonehole Impedance Parameters

$$
\begin{array}{rrcl}
\text{(open-hole shunt impedance)} & R_s^o & = & R_b(jkt_e + \xi_e) \\
\text{(closed-hole shunt impedance)} & R_s^c & = & -jR_b \cot(kt_h) \\
\text{(open-hole series impedance)} & R_a^o & = & -jR_b k t_a^o \\
\text{(closed-hole series impedance)} & R_a^c & = & -jR_b k t_a^c
\end{array}
$$

- tonehole $=$ acoustic tube of cross-sectional area πb^2
- $R_b = \rho c/(\pi b^2) =$ wave impedance in tonehole
- $\rho =$ density and $c =$ sound speed as usual
- $k = \omega/c = 2\pi/\lambda =$ wavenumber (radian spatial frequency)
- $t_e =$ open-tonehole effective length (greater than physical length due to "air piston" at tonehole exit)
- $\xi_e =$ "specific resistance" of the open tonehole due to air viscosity in and radiation from the hole
- $t_h =$ closed-tonehole height, defined such that its product times the cross-sectional area of the tonehole exactly equals the geometric volume V_h of the closed tonehole

- t_a^o and t_a^c are the equivalent *series* lengths of the open and closed tonehole, respectively:

$$t_a^c = \frac{0.47b(b/a)^4}{\tanh(1.84t_h/b) + 0.62(b/a)^2 + 0.64(b/a)}$$
$$t_a^o = \frac{0.47b(b/a)^4}{\coth(1.84t_h/b) + 0.62(b/a)^2 + 0.64(b/a)}$$

  where $a =$ radius of the main bore

- The closed-tonehole height $V_h/(\pi b^2)$ is $\approx$

$$t_h = t_w + \frac{1}{8}\frac{b^2}{a}\left[1 + 0.172\left(\frac{b}{a}\right)^2\right]$$

  where $t_w =$ tonehole chimney height at center

Note: The specific resistance of the open tonehole, ξ_e, is the only real impedance and therefore the only source of wave energy loss at the tonehole. It is given by

$$\xi_e = 0.25(kb)^2 + \alpha t_h + (1/4)kd_v \ln(2b/r_c),$$

where r_c is the radius of curvature of the tonehole, d_v is the viscous boundary layer thickness which expressible in terms of the shear viscosity η of air as

$$d_v = \sqrt{\frac{2\eta}{\rho\omega}}$$

and α is the real part of the propagation wavenumber (or minus the imaginary part of complex spatial frequency k). In the large-tube limit (i.e., when the tube radius is large compared with the viscous boundary layer), α is given by

$$\alpha = \frac{1}{2bc}\left[\sqrt{\frac{2\eta\omega}{\rho}} + (\gamma - 1)\sqrt{\frac{2\kappa\omega}{\rho C_p}}\right]$$

where $\gamma = 1.4$ is the adiabatic gas constant for air, κ is the thermal conductivity of air, and C_p is the specific heat of air at constant pressure.

For air at 300° Kelvin (26.85° C), valid within ± 10 degrees of that temperature:

$$\begin{aligned}
\rho &= 1.1769 \times 10^{-3}(1 - 0.00335\Delta T)\,\text{g/cm}^3 \\
\eta &= 1.846 \times 10^{-4}(1 + 0.0025\Delta T)\,\text{g/sec/cm} \\
\gamma &= 1.4017(1 - 0.00002\Delta T) \\
\nu &= \sqrt{\eta C_p/\kappa} = 0.8418(1 - 0.0002\Delta T) \\
c &= 3.4723 \times 10^4(1 + 0.00166\Delta T)\,\text{cm/sec} \\
\alpha &= \frac{\omega}{c}\left(\frac{1.045}{r_v} + \frac{1.080}{r_v^2} + \frac{0.750}{r_v^3}\right) \quad \text{(valid for } r_v > 2\text{)}
\end{aligned}$$

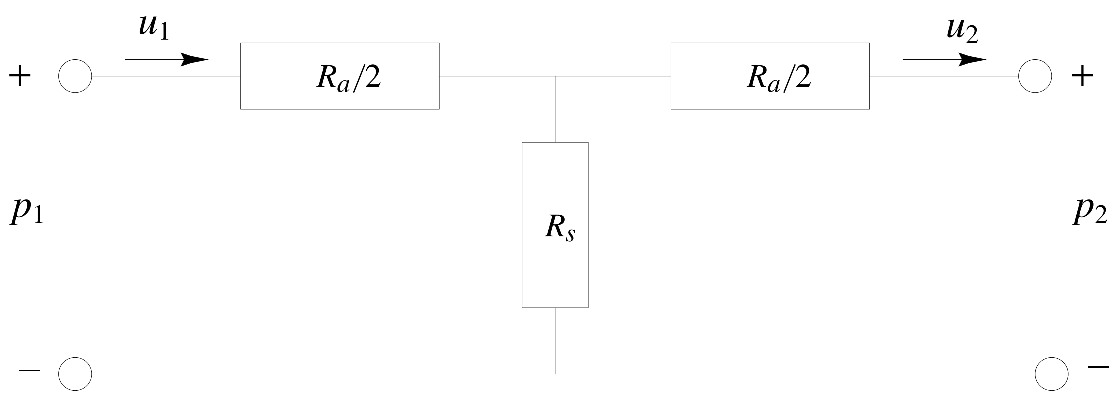
- $r_v = b\sqrt{\frac{\rho\omega}{\eta}} = \sqrt{2}\frac{b}{d_v}$ can be interpreted as $\sqrt{2}$ times the ratio of the tonehole radius b to the viscous boundary layer thickness d_v

- The constant ν^2 is referred to as the Prandtl number,
- η is the shear viscosity coefficient
- r_v is greater than 8 under practical conditions in musical acoustics, $\Rightarrow$ sufficient to keep only the first and second-order terms in the expression above for α.

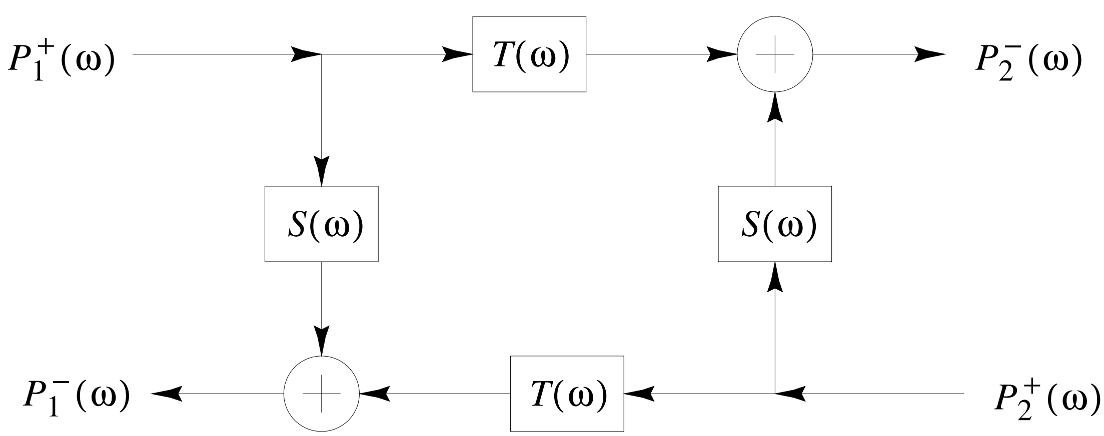
The open-hole effective length t_e, assuming no pad above the hole, is

$$t_e = \frac{(1/k)\tan(kt) + b[1.40 - 0.58(b/a)^2]}{1 - 0.61kb\tan(kt)}$$

Digital Waveguide Tonehole Formulation



For implementation in a digital waveguide model, the lumped shunt and series impedances must be converted to scattering parameters:



Derivation

From Keefe's transmission-matrix model, we have

$$\begin{bmatrix} P_1 \\ U_1 \end{bmatrix} = \begin{bmatrix} 1+\frac{R_a}{2R_s} & R_a[1+\frac{R_a}{4R_s}] \\ \frac{1}{R_s} & 1+\frac{R_a}{2R_s} \end{bmatrix} \begin{bmatrix} P_2 \\ U_2 \end{bmatrix}$$

- Substitute $k = \omega/c$ in the impedance values measured by Keefe to convert spatial frequency to temporal frequency
- Substitute

  $$P_i = P_i^+ + P_i^-, \qquad U_i = \frac{P_i^+ - P_i^-}{R_0}, \qquad i = 1, 2$$

  to convert physical variables to wave variables
- $\left(R_0 = \frac{\rho c}{\pi a^2}\right.$ is the bore wave impedance$\left.\right)$
- Finally, solve for the outgoing waves P_1^-, P_2^- in terms of the incoming waves P_1^+, P_2^+

Mathematica code

```
Clear["t*", "p*", "u*", "r*"]
transmissionMatrix = {{t11, t12}, {t21, t22}};
leftPort = {{p2p+p2m}, {(p2p-p2m)/r2}};
rightPort = {{p1p+p1m}, {(p1p-p1m)/r1}};
Format[t11, TeXForm] := "{T_{11}}"
Format[p1p, TeXForm] := "{P_1^+}"
... (etc. for all variables) ...
TeXForm[Simplify[Solve[leftPort ==
           transmissionMatrix . rightPort,
           {p1m, p2p}]]]
```

$$P_1^- = \frac{2P_2^- R_1 - P_1^+ R_1 T_{11} - P_1^+ T_{12} + P_1^+ R_1 R_2 T_{21} + P_1^+ R_2 T_{22}}{R_1 T_{11} - T_{12} - R_1 R_2 T_{21} + R_2 T_{22},}$$

$$P_2^+ = \frac{P_2^- R_1 T_{11} - P_2^- T_{12} + P_2^- R_1 R_2 T_{21} - 2P_1^+ R_2 T_{12} T_{21} - P_2^- R_2 T_{22} + 2P_1^+ R_2 T_{11} T_{22}}{R_1 T_{11} - T_{12} - R_1 R_2 T_{21} + R_2 T_{22}}$$

Substituting relevant values for Keefe's tonehole model, we obtain

$$\begin{bmatrix} P_1^- \\ P_2^+ \end{bmatrix} = \begin{bmatrix} S & T \\ T & S \end{bmatrix} \begin{bmatrix} P_1^+ \\ P_2^- \end{bmatrix}$$
$$= \frac{1}{(2R_0 + R_a)(2R_0 + R_a + 4R_s)} \begin{bmatrix} 4R_a R_s + R_a^2 - 4R_0^2 & 8R_0 R_s \\ 8R_0 R_s & 4R_a R_s + R_a^2 - 4R_0^2 \end{bmatrix} \begin{bmatrix} P_1^+ \\ P_2^- \end{bmatrix}$$

Thus,

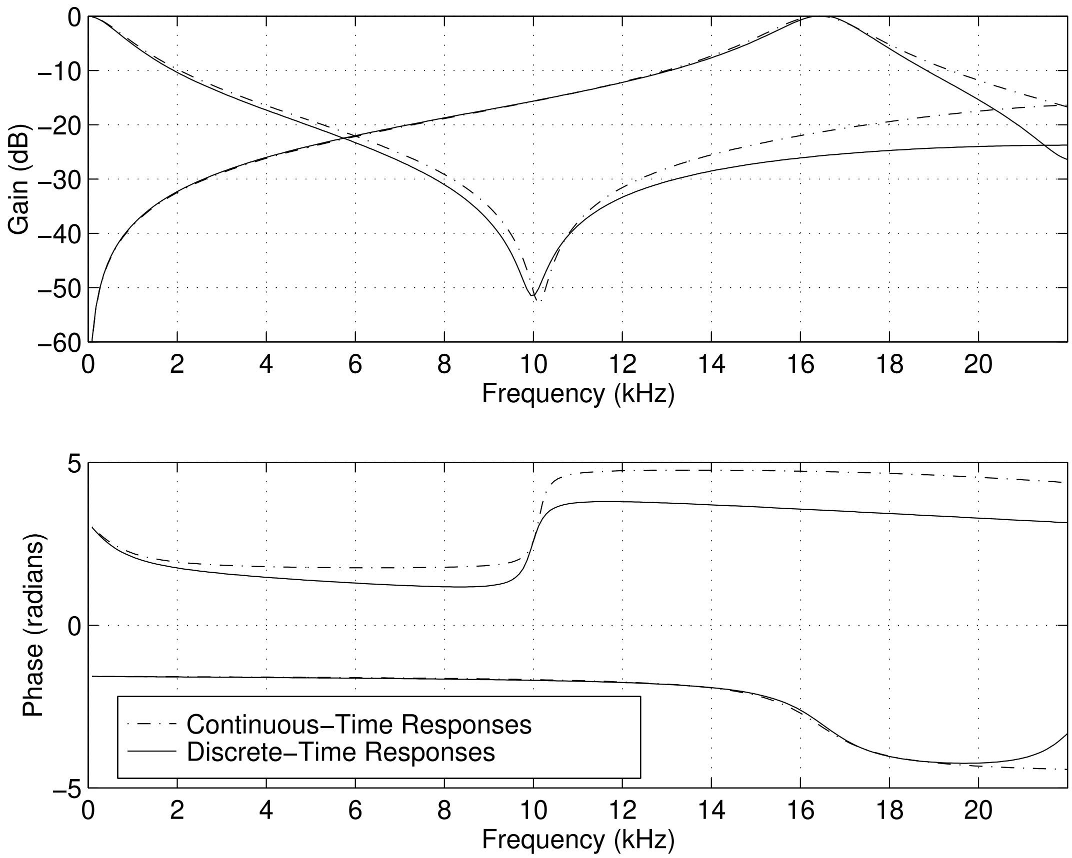
$$S(\omega) = \frac{4R_aR_s + R_a^2 - 4R_0^2}{(2R_0 + R_a)(2R_0 + R_a + 4R_s)} \approx -\frac{R_0}{R_0 + 2R_s}$$

is the *reflectance* of the tonehole (the same from either direction), and the *transmittance* is given by

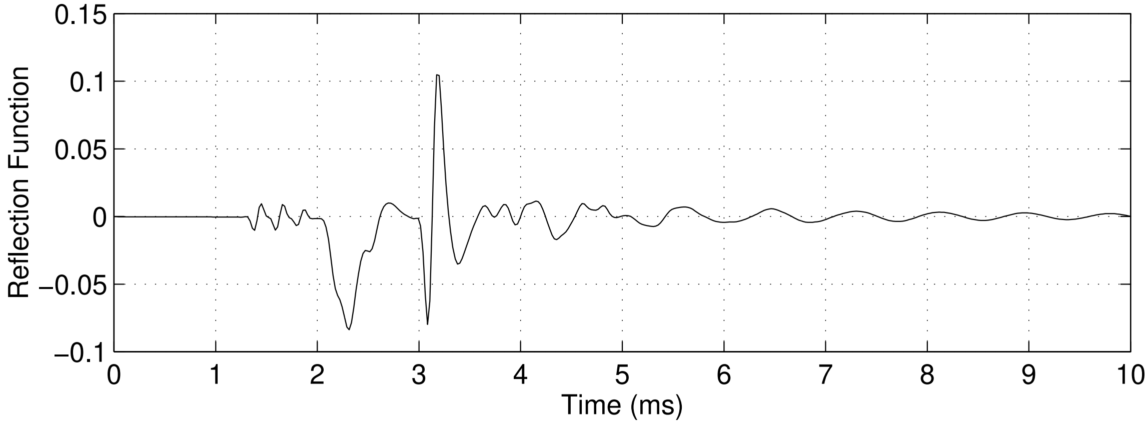
$$T(\omega) = \frac{8R_0R_s}{(2R_0 + R_a)(2R_0 + R_a + 4R_s)} \approx \frac{2R_s}{R_0 + 2R_s}$$

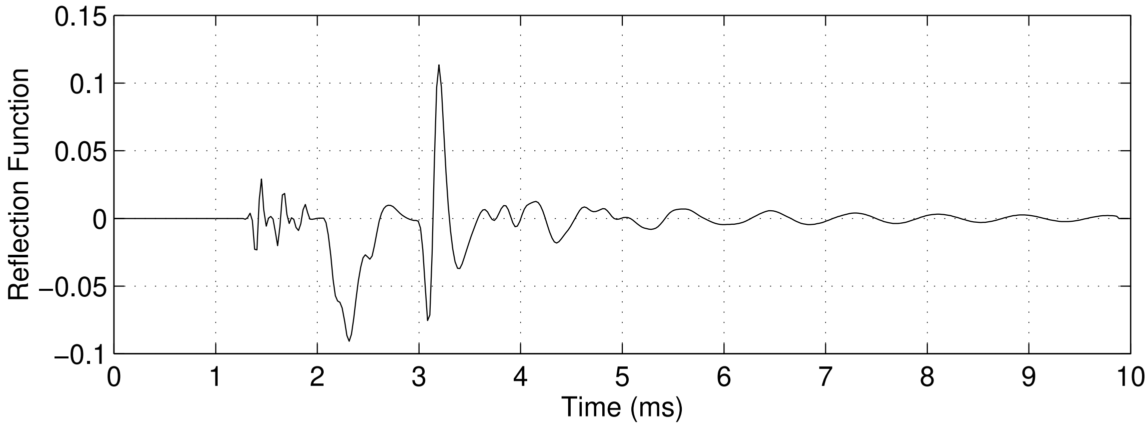
- Tonehole reflectance and transmittance are the same in either direction
- The notation "S" for reflectance is chosen because every reflectance is a *Schur function* (stable and not exceeding unit magnitude on the unit circle in the z plane)
- The approximate forms are obtained by neglecting the negative series inertance R_a which serves to adjust the effective length of the bore, and which therefore can be implemented elsewhere in the interpolated delay-line calculation as discussed further below
- The open and closed tonehole cases are obtained by substituting $\{R_a = R_a^o, R_s = R_s^o\}$ and $\{R_a = R_a^c, R_s = R_s^c\}$, respectively

Open and Closed Tonehole Reflectance: Measured versus Second-Order Digital Filter Approximation



Impulse Response of Several Toneholes in a Clarinet Bore: Measured versus Digital Waveguide Model





One-Filter Forms

In a manner analogous to converting the four-multiply Kelly-Lochbaum (KL) scattering junction into a one-multiply form, we may pursue a "one-filter" form of the waveguide tonehole model.

- The series inertance gives some initial trouble, since
$$[1+S(\omega)]-T(\omega)=\frac{2R_a}{2R_0+R_a}\stackrel{\Delta}{=}L(\omega)$$
instead of zero as in the KL junction.
- In the scattering formulas for the general loaded waveguide junction, the reflectance seen on any branch is always the transmittance from that branch to any other branch minus 1. I.e., if α_i denotes the transmittance from branch i to all other branches meeting at the junction, then $\alpha_i - 1$ is the reflectance seen on branch i.
- By factoring out the common term α_i, one-multiply two-port scattering junctions are found.

Substituting

$$T = 1 + S - L$$

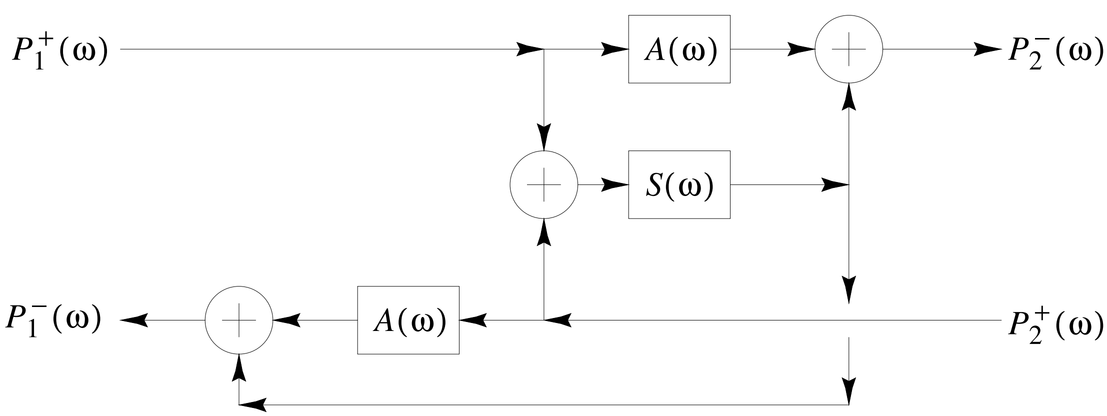
into the basic scattering relations, and factoring out S, we obtain, in the frequency domain,

$$\begin{aligned} P_1^-(\omega) &= SP_1^+ + TP_2^+ \\ &= SP_1^+ + [1 + S - L]P_2^+ \\ &= S[P_1^+ + P_2^+] + [1 - L]P_2^+ \\ &\stackrel{\Delta}{=} S[P_1^+ + P_2^+] + AP_2^+ \end{aligned}$$

and, similarly,

$$P_2^-(\omega) = S[P_1^+ + P_2^+] + AP_1^+$$

The resulting tonehole implementation is shown below:

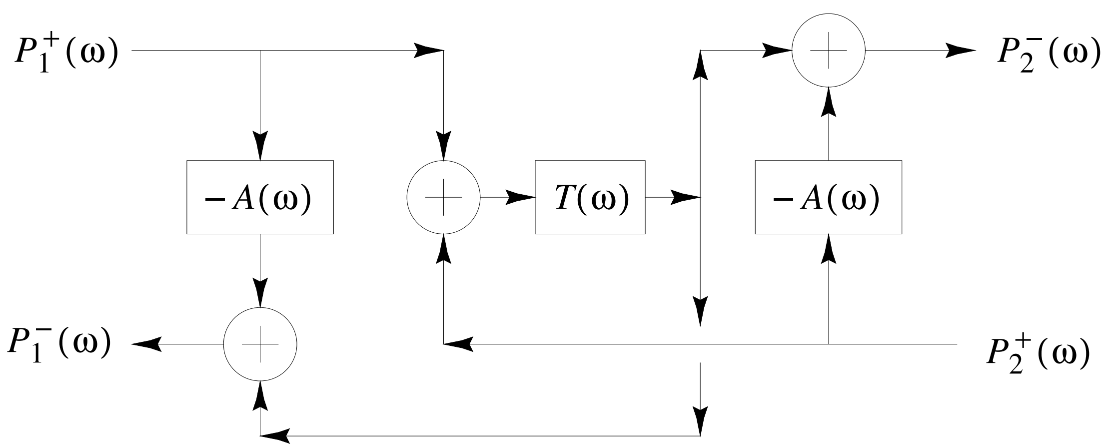


In the same way, an alternate form is obtained from the substitution

$$S = T - 1 + L$$

which yields the “shared transmittance” form:

$$\begin{aligned} P_1^- &= T[P_1^+ + P_2^+] - AP_1^+ \\ P_2^- &= T[P_1^+ + P_2^+] - AP_2^+ \end{aligned}$$



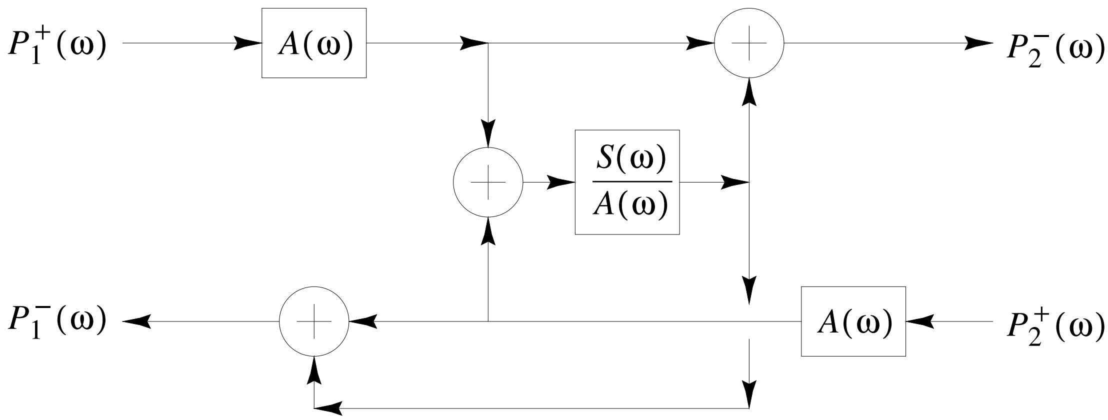
- Since $L(\omega) \approx 0$, it can be neglected to first order, and $A(\omega) \approx 1$, reducing both of the above forms to an approximate "one-filter" tonehole implementation.
- Since $R_a = -jR_b\omega t_a/c$ is a pure negative reactance, we have

$$A(\omega) = 1 - L(\omega) = \frac{R_0 - R_a/2}{R_0 + R_a/2} = \frac{p + j\omega}{p - j\omega}, \quad p = \frac{R_0 c}{R_b t_a}$$

- In this form, it is clear that $A(\omega)$ is a first-order *allpass* filter with a single pole-zero pair near infinity.

Unfortunately, the pole is in the right-half-plane and hence *unstable*. We cannot therefore implement it as shown.

Using elementary manipulations, the unstable allpasses can be moved to the configuration shown below:



Notes:

- $T(\omega)/A(\omega)$ is stable whenever T is stable.
- The unstable allpasses now operate only on the two incoming wave variables, which implies they can be implemented *implicitly* by slightly reducing the (interpolated) delay-lines leading to the junction from either side.
- The tonehole requires only one filter S/A or T/A.

We now see precisely how the negative series inertance R_a provides a *negative, frequency-dependent, length correction* for the bore. The phase delay of $A(\omega)$ can be computed as

$$D_A(\omega) \triangleq -\frac{\angle A(\omega)}{\omega} = -2\tan^{-1}(\omega/p) = -2\tan^{-1}(kt_aR_b/R_0)$$

- Negative delay correction goes to zero with any frequency $k = \omega/c$, series tonehole length t_a, tonehole radius R_b, or main bore admittance $\Gamma_0 = 1/R_0$.
- In practice, it is common to combine all delay corrections into a single "tuning allpass filter" for the whole bore
- Whenever the desired allpass delay goes negative, we simply add a sample of delay to the desired allpass phase-delay and subtract it from the nearest delay.
- In other words, negative delays have to be "pulled out" of the allpass and used to shorten an adjacent interpolated delay line. Such delay lines are normally available in practical modeling situations.

The Clarinet Tonehole as a Two-Port Loaded Junction

It seems reasonable to expect that the tonehole should be representable as a *load* along a waveguide bore model, thus creating a loaded two-port junction with two identical bore ports on either side of the tonehole. From the general relations for the loaded parallel pressure-wave junction, in the two-port case with $R_1 = R_2 = R_0$, we have

$$\begin{aligned} P_J(s) &= \alpha P_1^+ + \alpha P_2^+, \quad \alpha = 2\Gamma_0/[G_J(s) + 2\Gamma_0] \\ P_1^-(s) &= P_J(s) - P_1^+ = (\alpha - 1)P_1^+ + \alpha P_2^+ = \alpha(P_1^+ + P_2^+) - P_1^+ \\ P_2^-(s) &= P_J(s) - P_2^+ = \alpha P_1^+ + (\alpha - 1)P_2^+ = \alpha(P_1^+ + P_2^+) - P_2^+ \end{aligned}$$

- The general loaded two-port junction can be implemented in "one-filter shared-transmittance form" as shown above with $A(\omega) = 1$ $\big(L(\omega) = 0\big)$ and

$$T(\omega) = \alpha = \frac{2\Gamma_0}{2\Gamma_0 + G_J(s)} = \frac{2R_J(s)}{2R_J(s) + R_0}$$

- The *simplified* Keefe tonehole model (negative inertance removed, i.e., $R_a = 0$), is equivalent to a loaded two-port waveguide junction with the two-port load impedance set to the tonehole shunt impedance $R_J = R_s$.

- Each series impedance $R_a/2$ in the split-T model of Keefe can be modeled as a *series* waveguide junction with a load of $R_a/2$:

- Set the transmission matrix parameters to the values $T_{11} = T_{22} = 1$, $T_{12} = R_a/2$, and $T_{21} = 0$ to get

$$\begin{aligned} P_1^- &= (1-\alpha)P_1^+ + \alpha P_2^- \\ P_2^+ &= \alpha P_1^+ + (1-\alpha)P_2^- \end{aligned}$$

  where $\alpha = 2R_0/(2R_0 + R_a/2)$ is the alpha parameter for a series loaded waveguide junction involving two impedance R_0 waveguides joined in series with each other and with a load impedance of $R_a/2$.

- Switch to the more general convention in which the "+" superscript denotes waves traveling *into* a junction of any number of waveguides. This exchanges "+" with "−" at port 2 to yield

$$\begin{aligned} P_1^- &= (1-\alpha)P_1^+ + \alpha P_2^+ \\ P_2^- &= \alpha P_1^+ + (1-\alpha)P_2^+ \end{aligned}$$

- Convert pressure to velocity using $P_i^+ = R_0 U_i^+$ and $P_i^- = -R_0 U_i^-$ to obtain

$$\begin{aligned} U_1^- &= (\alpha-1)U_1^+ - \alpha U_2^+ \\ U_2^- &= -\alpha U_1^+ + (\alpha-1)U_2^+ \end{aligned}$$

- Finally, toggle the reference direction of port 2 (the "current" arrow for u_2 on port 2 in Keefe's split-T model) so that velocity is positive flowing *into* the junction on both ports (which is the convention typically followed in circuit theory). This amounts to negating $U_2^{\pm}$, giving

$$\begin{aligned} U_1^- &= U_J - U_1^+ \\ U_2^- &= U_J - U_2^+ \end{aligned}$$

  where $U_J \stackrel{\Delta}{=} (\alpha U_1^+ + \alpha U_2^+)$.

- This is the canonical form of the two-port series scattering junction.