MUS420 Lecture
Digitizing Traveling Waves in Vibrating Strings

Julius O. Smith III (jos@ccrma.stanford.edu)
Center for Computer Research in Music and Acoustics (CCRMA)
Department of Music, Stanford University
Stanford, California 94305

June 27, 2020

Outline

- Ideal vibrating string
- Traveling-wave solution
- Sampled traveling waves

Ideal Vibrating String Model

We know already how to model a string as a *bidirectional delay line* with

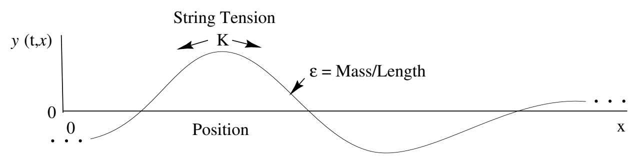
- inverting reflecting terminations (for displacement)
- filters for loss and dispersion
- outputs as sums of traveling-wave components

This model is based on *traveling waves* and the *superposition* of traveling waves as *experimental fact*. In such a model, sound-speed must be measured experimentally.

We now take our string model to the next level based on the *physics* of ideal strings:

- Sound speed becomes a *predicted* quantity
- The very useful concept of *wave impedance* is derived

Ideal String Physics



Wave Equation

$$\boxed{Ky'' = \epsilon \ddot{y}}$$

$$\begin{array}{rcl} K & \stackrel{\Delta}{=} & \text{string tension} \\ \epsilon & \stackrel{\Delta}{=} & \text{linear mass density} \\ y & \stackrel{\Delta}{=} & \text{string displacement} \end{array} \qquad \begin{array}{rcl} y & \stackrel{\Delta}{=} & y(t,x) \\ \dot{y} & \stackrel{\Delta}{=} & \frac{\partial}{\partial t} y(t,x) \\ y' & \stackrel{\Delta}{=} & \frac{\partial}{\partial x} y(t,x) \end{array}$$

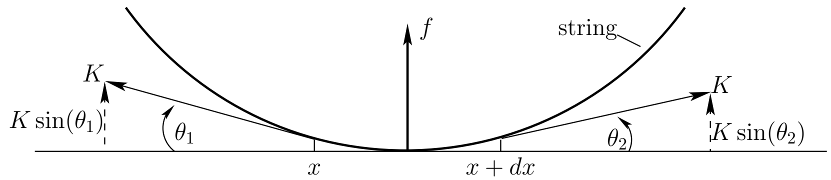
Newton's second law

$$\boxed{\text{Force} = \text{Mass} \times \text{Acceleration}}$$

Assumptions

- Lossless
- Linear
- Flexible (no "Stiffness")
- Slope $y'(t,x) \ll 1$

String Wave Equation Derivation



Force diagram for length dx string element

Total upward force on length dx string element:

$$\begin{aligned} f(x+dx/2) &= K\sin(\theta_1) + K\sin(\theta_2) \\ &\approx K\left[\tan(\theta_1) + \tan(\theta_2)\right] \\ &= K\left[-y'(x) + y'(x+dx)\right] \\ &\approx K\left[-y'(x) + y'(x) + y''(x)dx\right] \\ &= Ky''(x)dx \end{aligned}$$

Mass of length dx string segment: $m = \epsilon\, dx$.

By Newton's law, $f = ma = m\ddot{y}$, we have

$$Ky''(t,x)dx = (\epsilon\, dx)\ddot{y}(t,x)$$

or

$$\boxed{Ky''(t,x) = \epsilon \ddot{y}(t,x)}$$

Traveling-Wave Solution

One-dimensional lossless wave equation:

$$\boxed{Ky'' = \epsilon \ddot{y}}$$

Plug in *traveling wave to the right*:

$$\begin{aligned} y(t,x) &= y_r(t - x/c) \\ \Rightarrow \quad y'(t,x) &= -\frac{1}{c}\dot{y}(t,x) \\ y''(t,x) &= \frac{1}{c^2}\ddot{y}(t,x) \end{aligned}$$

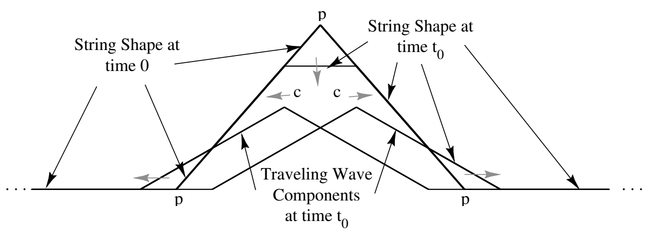
- Given $c \stackrel{\Delta}{=} \sqrt{K/\epsilon}$, the wave equation is satisfied for *any shape traveling to the right at speed* c (but remember slope $\ll 1$)
- Similarly, any *left-going* traveling wave at speed c, $y_l(t + x/c)$, satisfies the wave equation (show)

- General solution to lossless, 1D, second-order wave equation:

$$y(t,x) = y_r(t - x/c) + y_l(t + x/c)$$

- $y_l(\cdot)$ and $y_r(\cdot)$ are arbitrary twice-differentiable functions (slope $\ll 1$)
- **Important point:** Function of two variables $y(t,x)$ is replaced by two functions of a single (time) variable $\Rightarrow$ *reduced computational complexity*.
- Published by d'Alembert in 1747 (wave equation itself introduced in same paper)

Infinitely long string plucked simultaneously at three points marked 'p'



- Initial displacement = sum of two identical triangular pulses
- At time t_0, traveling waves centers are separated by $2ct_0$ meters
- String is not moving where the traveling waves overlap at same slope.
- Nelson Lee's Animation[1]
- Travis Skare's Interactive Animation[2]

[1] http://ccrma.stanford.edu/~jos/rsadmin/TravellingWaveApp.swf
[2] https://ccrma.stanford.edu/~travissk/dwgdemo/

Sampled Traveling Waves in a String

For discrete-time simulation, we must *sample* the traveling waves

- Sampling interval $\stackrel{\Delta}{=} T$ seconds
- Sampling rate $\stackrel{\Delta}{=} f_s$ Hz $= 1/T$
- Spatial sampling interval $\stackrel{\Delta}{=} X$ m/s $\stackrel{\Delta}{=} cT$
  $\Rightarrow$ *systolic grid*

For a vibrating string with length L and fundamental frequency f_0,

$$c = f_0 \cdot 2L \qquad \left(\frac{\text{meters}}{\text{sec}} = \frac{\text{periods}}{\text{sec}} \cdot \frac{\text{meters}}{\text{period}}\right)$$

so that

$$X = cT = (f_0 2L)/f_s = L[f_0/(f_s/2)]$$

Thus, the number of *spatial samples* along the string is

$$\boxed{L/X = (f_s/2)/f_0}$$

or

$$\boxed{\text{Number of spatial samples} = \text{Number of string harmonics}}$$

Examples:

- Spatial sampling interval for CD-quality digital model of Les Paul electric guitar (strings ≈ 26 inches)
  - $X = Lf_0/(f_s/2) = L82.4/22050 \approx 2.5$ mm for low E string
  - $X \approx 10$ mm for high E string (two octaves higher and the same length)
  - Low E string: $(f_s/2)/f_0 = 22050/82.4 = 268$ harmonics (spatial samples)
  - High E string: 67 harmonics (spatial samples)
- Number of harmonics = number of oscillators required in *additive synthesis*
- Number of harmonics = number of two-pole filters required in *subtractive, modal*, or *source-filter decomposition synthesis*
- Digital waveguide model needs only *one delay line* (length $2L$)

Examples (continued):

- Sound propagation in *air*:
  - Speed of sound $c \approx 331$ meters per second
  - $X = 331/44100 = 7.5$ mm
  - Spatial sampling rate $= \nu_s = 1/X = 133$ samples/m
  - Sound speed in air is *comparable* to that of transverse waves on a guitar string (faster than some strings, slower than others)
  - Sound travels much faster in most solids than in air
  - Longitudinal waves in strings travel faster than transverse waves
    - typically an order of magnitude faster

Sampled Traveling Waves in any Digital Waveguide

$$\begin{aligned} x &\to x_m &= mX \\ t &\to t_n &= nT \end{aligned}$$

$\Rightarrow$

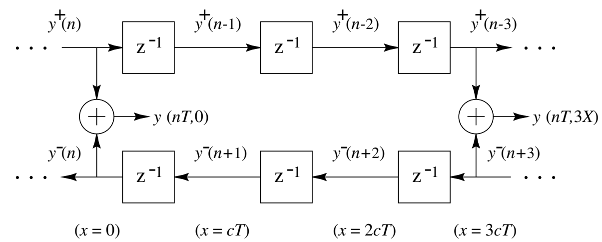
$$\begin{aligned} y(t_n, x_m) &= y_r(t_n - x_m/c) + y_l(t_n + x_m/c) \\ &= y_r(nT - mX/c) + y_l(nT + mX/c) \\ &= y_r\left[(n-m)T\right] + y_l\left[(n+m)T\right] \\ &= y^+(n-m) + y^-(n+m) \end{aligned}$$

when $X = cT$, where we defined

$$y^+(n) \triangleq y_r(nT) \qquad\qquad y^-(n) \triangleq y_l(nT)$$

- "$+$" superscript $\Longrightarrow$ *right-going*
- "$-$" superscript $\Longrightarrow$ *left-going*
- $y_r\left[(n-m)T\right] = y^+(n-m) =$ output of m-sample delay line with input $y^+(n)$
- $y_l\left[(n+m)T\right] \triangleq y^-(n+m) =$ *input* to an m-sample delay line whose *output* is $y^-(n)$

Lossless digital waveguide with observation points at $x = 0$ and $x = 3X = 3cT$



- Recall:

$$\begin{aligned} y(t,x) &= y^+\left(\frac{t - x/c}{T}\right) + y^-\left(\frac{t + x/c}{T}\right) \\ &\downarrow \\ y(nT, mX) &= y^+(n-m) + y^-(n+m) \end{aligned}$$

- *Position* $x_m = mX = mcT$ is *eliminated* from the simulation
- Position x_m remains laid out from left to right
- Left- and right-going traveling waves must be *summed* to produce a *physical* output

$$y(t_n, x_m) = y^+(n-m) + y^-(n+m)$$

- Similar to *ladder* and *lattice digital filters*

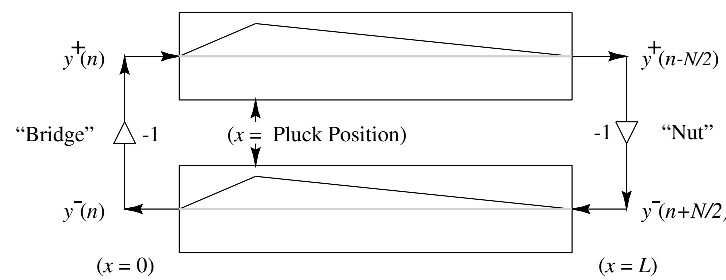
Important point: Discrete time simulation is *exact* at the sampling instants, to within the numerical precision of the samples themselves.

To avoid *aliasing* associated with sampling:

- Require all initial waveshapes be *bandlimited* to $(-f_s/2, f_s/2)$
- Require all external driving signals be similarly bandlimited
- Avoid nonlinearities or keep them "weak"
- Avoid time variation or keep it slow
- Use plenty of oversampling and lowpass filtering with rapid high-frequency roll-off in severely nonlinear and/or time-varying cases
- Prefer "feed-forward" over "feed-back" around nonlinearities and/or modulations when possible

Interactive simulation of a vibrating string:
`http://www.colorado.edu/physics/phet/simulations/-stringwave/stringWave.swf`

Digital Waveguide Plucked-String Model Using Initial Conditions



Initial conditions for the ideal plucked string.

- Amplitude of each traveling-wave $= 1/2$ initial string displacement.
- Sum of the upper and lower delay lines $=$ initial string displacement.

Other Wave Variables (Wave-Impedance Preview)

Transverse Velocity Waves:

$$v^+(n) \stackrel{\Delta}{=} \dot{y}^+(n)$$
$$v^-(n) \stackrel{\Delta}{=} \dot{y}^-(n)$$

Wave Impedance (we'll derive later):

$$\boxed{R = \sqrt{K\epsilon} = \frac{K}{c} = \epsilon c}$$

Force Waves:

$$f^+(n) \stackrel{\Delta}{=} R\,v^+(n)$$
$$f^-(n) \stackrel{\Delta}{=} -R\,v^-(n)$$

Ohm's Law for Traveling Waves:

$$\boxed{\begin{aligned} f^+(n) &= R\,v^+(n) \\ f^-(n) &= -\,R\,v^-(n) \end{aligned}}$$

Acoustic Plane Waves

Pressure Plane Waves:

$$p^+(n) \stackrel{\Delta}{=} R_a\,u^+(n)$$
$$p^-(n) \stackrel{\Delta}{=} -R_a\,u^-(n)$$

where u^+, u^- are
Longitudinal Particle-Velocity Waves

Ohm's Law for Traveling Acoustic Plane Waves:

$$\boxed{\begin{aligned} p^+(n) &= R_a u^+(n) \\ p^-(n) &= -\,R_a u^-(n) \end{aligned}}$$

where

$$\boxed{R_a = \rho c}$$

is the wave impedance of air in terms of mass density ρ (kg/m^3) and sound speed c.

Acoustic Tubes

In acoustic tubes, we again work with
Pressure Plane Waves:

$$\begin{aligned} p^+(n) &\stackrel{\Delta}{=} R_{\tau} U^+(n) \\ p^-(n) &\stackrel{\Delta}{=} -R_{\tau} U^-(n) \end{aligned}$$

However, now U^+, U^- are
Longitudinal Volume-Velocity Waves:

$$\begin{aligned} U^+(n) &\stackrel{\Delta}{=} A\, u^+(n) \\ U^-(n) &\stackrel{\Delta}{=} A\, u^-(n) \end{aligned}$$

where A is the cross-sectional area of the tube. In an acoustic tube, it is volume velocity that is conserved from one tube section to the next.

Ohm's Law for Traveling Plane Waves in an Acoustic Tube:

$$\boxed{\begin{aligned} p^+(n) &= \ R_{\tau} U^+(n) \\ p^-(n) &= -\ R_{\tau} U^-(n) \end{aligned}}$$

where

$$\boxed{R_{\tau} = \frac{\rho c}{A}}$$

is the wave impedance of air in terms of mass density ρ, sound speed c, and tube cross-section area A.