Outline

- Single-Reed Instruments
  - Schematic Physical Model
  - Digital Waveguide Model
- Bernoulli Effect
- Single-Reed Theory
- Single-Reed Computation Models

Single-Reed Instruments

Schematic Model

- Main control variable = air pressure applied to reed
- Secondary control variable = reed embouchure
- Pressure waves = natural choice for simulation
- Bell \( \approx \) power-complementary “cross-over” filter:
  - Low frequencies reflect (inverted)
  - High frequencies transmit
  - Cross-over frequency \( \approx 1500 \text{ Hz} \) for clarinet
    (where wavelength \( \approx \) bore diameter)
- Radiation \( \approx \) “Omni” at LF, more directional at HF
Single-Reed Digital Waveguide Model

- Bore = bidirectional delay line (losses lumped)
- Bore length = 1/4 wavelength in lowest register
  - Bell reflection ≈ -1 at low frequencies
  - Mouthpiece reflection ≈ +1
- Reflection filter depends on first few open toneholes
- In a simple implementation, the bore is “cut to a new length” for each pitch
- Reed = Nonlinear scattering junction
  - Reed mass neglected
  - Reed table interpolated (usually linear)
  - In software, simple “if statement” possible
  - Embouchure can be a simple address offset

Qualitative Description of Single-Reed Oscillation

- Apply pressure at the mouthpiece
- Reed is “biased” in “negative-resistance” region
- High-pressure front travels to open tonehole or bell where it reflects with a sign inversion
- A “canceling wave” travels back toward mouthpiece
- The canceling wave reflects from the mouthpiece with no inversion
- A negative-pressure “wake” is left behind
- The reflected-canceling-wave travels back to the open end where it reflects with inversion
- The negative-pressure throughout bore is canceled by this wave as it travels to the mouthpiece
- Upon reaching mouthpiece, one period is finished
- One period = four trips across bore length
Reed Causes Oscillation Growth

- Wave loses energy propagating in bore (mainly at the open-end reflection)
- The nonlinear reed action must restore this lost energy
- “Power supply” = “dc mouth pressure”
  \[ \Rightarrow \text{Reed converts dc to ac} \]
- Reed action “sharpens” pressure transitions
  - Reed closure increases reflection coefficient in bore
  - As pressure falls in bore, it is amplified by increasing \( \rho \)
  - As pressure falls in bore, it is further amplified by decreasing flow input from the mouth
  - As pressure rises in bore, it is amplified by increasing mouth flow input (although reflection coefficient decreases)
  - Reflection of a positive wave is boosted when the incoming wave is below a certain level and it is attenuated above that level
- When the oscillation reaches a very high amplitude, it is limited on the negative side by the shutting of the reed, and on the positive side by the attenuation described above
**Bernoulli’s Equation**

In an ideal *inviscid, incompressible flow*, we have, by conservation of energy,

\[ p + \frac{1}{2} \rho u^2 + \rho gh = \text{constant} \]

where

- \( p \) = pressure (Newtons/m\(^2\) = kg/(m s\(^2\))
- \( u \) = particle velocity (m/s)
- \( \rho \) = volume density of air (kg/m\(^3\))
- \( g \) = Newton’s gravitational constant (m/s\(^2\))
- \( h \) = Height of flow’s center-of-mass axis (m)

“*Inviscid*” = “Frictionless”, “Lossless”

**Pressure is Proportional to Kinetic Energy**

\[ p = \frac{1}{3} \rho \langle u^2 \rangle \]

- *Average momentum transfer / area / time*
- Caused by collisions of gas molecules with boundary
- Proof: See *Kinetic Theory of Gases*

**Bernoulli Effect**

- \( p_m \) = “mouth pressure”
- Flow inside “mouth” neglected
- Pressure kinetic energy converts to flow kinetic energy within channel
- Jet “carries its own pressure” until it dissipates
- Jet dissipation can go to
  - heat (now allowing “friction” into the model)
  - vortices (angular momentum)
  - radiation (sound waves)
  - pressure recovery:
    (flow kinetic energy \( \rightarrow \) pressure kinetic energy)
Single-Reed Theory

\[ p_m \triangleq \text{mouth pressure (constant)} \]
\[ p_b \triangleq \text{bore pressure (dynamic)} \]
\[ p_{\Delta} \triangleq p_m - p_b \triangleq \text{pressure drop across mouthpiece} \]
\[ u_m \triangleq \text{resulting flow into mouthpiece} \]
\[ R_m(p_{\Delta}) \triangleq \text{reed-aperture impedance (measured)} \]

where

\[ u_b + u_m = 0 \quad \text{(by continuity of volume velocity)} \]
\[ u_m(p_{\Delta}) \triangleq \frac{p_{\Delta}}{R_m(p_{\Delta})} \quad \text{("Ohm’s law" for the reed)} \]
\[ u_b \triangleq u_b^+ + u_b^- = \frac{p_b^+ - p_b^-}{R_b}, \quad R_b \triangleq \text{bore impedance} \]

Toward a Computational Model

Given:

\[ p_m = \text{Mouth pressure} \]
\[ p_b^+ = \text{Incoming traveling bore pressure} \]

Find:

\[ p_b^- = \text{Outgoing traveling bore pressure} \]

such that:

\[ 0 = u_m + u_b = \frac{p_{\Delta}}{R_m(p_{\Delta})} + \frac{p_b^+ - p_b^-}{R_b}, \]
\[ p_{\Delta} \triangleq p_m - p_b = p_m - (p_b^+ + p_b^-) \]

Solving for \( p_b^- \) is not immediate because \( R_m \) depends on \( p_{\Delta} \) which depends on \( p_b^- \).
Graphical Solution Technique

Graphically solve:

\[ G(p_\Delta) = p_\Delta^+ - p_\Delta, \quad p_\Delta^+ = p_m - 2p_b^+ \]

where

\[ G(p_\Delta) \triangleq R_b\mu_m(p_\Delta) = R_b p_\Delta / R_m(p_\Delta) \]

- Introduced by Friedlander and Keller (1953)
- Analogous to finding the “operating point” of a transistor by intersecting its “operating curve” with the “load line” determined by the load resistance.
- Outgoing wave is then \( p_b^- = p_m - p_b^+ - p_\Delta(p_\Delta^+) \)

Scattering Formulation

Solve for \( p_b^- \) to obtain

\[ p_b^- = \frac{1 - r}{1 + r} p_b^+ + \frac{r}{1 + r} p_m \]
\[ = \rho p_b^+ + \frac{1 - \rho}{2} p_m \]
\[ = \frac{p_m}{2} - \rho \frac{p_\Delta^+}{2} \]

where

\[ \rho(p_\Delta) \triangleq \frac{1 - r(p_\Delta)}{1 + r(p_\Delta)}, \quad r(p_\Delta) \triangleq \frac{R_b}{R_m(p_\Delta)} \]

\[ \rho(p_\Delta) = \text{signal-dependent reflection coefficient.} \]

In practice, \( R_m \gg R_b \Rightarrow r \approx 0, \rho \approx 1 \)

(mouthpiece looks largely like a closed end)
For Faster Real-Time Computation

Pre-solve the graphical intersection and store the result in a look-up table.

Let $h$ denote half-pressure $p/2$. Then

$$p_b^+ = -\rho(p_\Delta) \cdot h_\Delta^+$$

Subtracting both sides from $p_b^+$ and solving for $\rho$ gives

$$\rho(p_\Delta) = \frac{p_\Delta}{h_\Delta^+} - 1$$

Now, for each $h_\Delta^+ = -p_b^+$, find $p_\Delta$ graphically, and store the resulting reflection coefficient $\rho(p_\Delta)$ as a function of $h_\Delta^+$:

$$\hat{\rho}(h_\Delta^+) = \rho(p_\Delta(h_\Delta^+))$$

Then the real-time reed computation reduces simply to

$$p_b^- = -\hat{\rho}(h_\Delta^+) \cdot h_\Delta^+$$

This is the form chosen for implementation above.

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Table-Reduced Reed Reflection Coefficient

- Control variable = mouth half-pressure
- $h_\Delta^+ = -p_b^+$ computed from incoming bore pressure by a subtraction
- Table is indexed by $h_\Delta^+$
- Result of lookup is multiplied by $h_\Delta^+$
- Result of multiplication is subtracted from
- Total reed cost = two subtractions, one multiply, and one table lookup per sample
Simple Piecewise-Linear Reed Table

Further Details

- Input mouth pressure is summed with a small amount of white noise corresponding to turbulence, e.g., 0.1% plus more during attacks
- Turbulence level and spectral shape should be computed automatically as a function of pressure drop $p_\Delta$ and reed opening geometry (research topic)
- Simple reflection filter:

$$H(z) = \frac{1 + a_1(t)}{1 + a_1(t)z^{-1}}$$

where $a_1(t) = v(t) - 0.642$, $v(t) = A_v \sin(2\pi f_v t)$, $A_v =$ vibrato amplitude (e.g., 0.03), and $f_v =$ vibrato frequency (e.g., 5 Hz)
- Further loop filtering occurs as a result of using a linearly interpolated delay line for the bore
- Only one (double length) delay line is really used in typical implementations
- To avoid finger-hole models, legato note transitions can be managed using two delay line taps and cross-fading from one to the other during a transition

\[ \hat{\rho}(h_\Delta^+) = \begin{cases} 1 - m(h_c^+ - h_\Delta^+), & -1 \leq h_\Delta^+ < h_c^+ \\ 1, & h_c^+ \leq h_\Delta^+ \leq 1 \end{cases} \]

- Corner point $h_c^+ = $ smallest pressure difference giving reed closure
- In fixed-point, $h_\Delta^+ \overset{\Delta}{=} p_m/2 - p_b^+$ is confined to $[-1, 1)$
- Embouchure and reed-stiffness set by $h_c^+$ and slope $m$ $[m = 1/(h_c^+ + 1)$ in the figure$]$
- Zero at maximum negative pressure $h_\Delta^+ = -1$ is not physical but is practical for inhibiting overflow
- A brighter tone is obtained by increasing the curvature as the reed begins to open [E.g., $\hat{\rho}^k(h_\Delta^+)$, for $k \gg 1$]
Alternative Reed Models

- A direct signal lookup, though requiring much higher resolution, would eliminate the multiply associated with the scattering coefficient
- Coefficient tables can be quantized more heavily in address and word length than direct lookup of a signal value such as $p_{\Delta}(p_{\Delta}^+)$
- Piecewise polynomial approximations are also used

Software

clarinet.cpp in the Synthesis Toolkit (STK):

http://ccrma.stanford.edu/CCRMA/Software/STK/