

MUS420 Lecture
Elementary Digital Waveguide Models for Vibrating Strings

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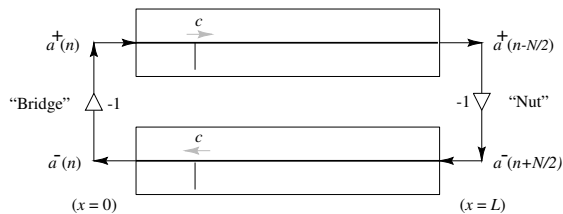
February 5, 2019

Outline

- Terminated string
- Plucked and struck string
- Damping and dispersion
- String Loop Identification
- Nonlinear “overdrive” distortion

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Acceleration-Wave Simulation



Initial conditions for the ideal plucked string: acceleration or curvature waves.

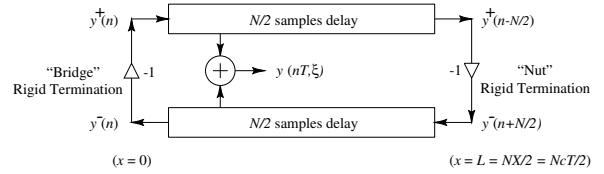
Recall:

$$y'' = \frac{1}{c^2} \ddot{y}$$

Acceleration waves are proportional to “curvature” waves.

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Rigidly Terminated Ideal String



- Reflection *inverts* for displacement, velocity, or acceleration waves (proof below)
- Reflection *non-inverting* for slope or force waves

Boundary conditions:

$$y(t, 0) \equiv 0 \quad y(t, L) \equiv 0 \quad (L = \text{string length})$$

Expand into Traveling-Wave Components:

$$y(t, 0) = y_r(t) + y_l(t) = y^+(t/T) + y^-(t/T)$$

$$y(t, L) = y_r(t - L/c) + y_l(t + L/c)$$

Solving for outgoing waves gives

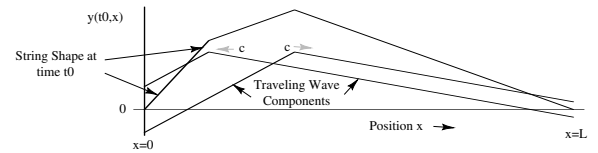
$$y^+(n) = -y^-(n)$$

$$y^-(n + N/2) = -y^+(n - N/2)$$

$N \triangleq 2L/X = \text{round-trip propagation time in samples}$

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Doubly Terminated Ideal Plucked String

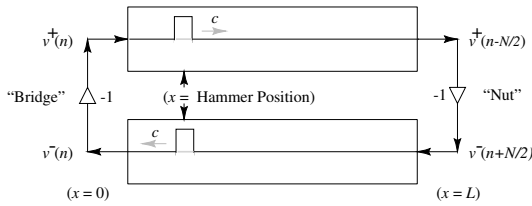


A doubly terminated string, “plucked” at 1/4 its length.

- Shown short time after pluck event.
- Traveling-wave components and physical string-shape shown.
- Note traveling-wave components sum to zero at terminations. (Use image method.)

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Ideal Struck-String Velocity-Wave Simulation



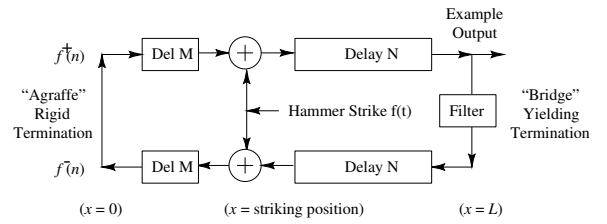
Initial conditions for the ideal struck string in a *velocity wave* simulation.

Hammer strike = *momentum transfer* = velocity step:

$$m_h v_h(0-) = (m_h + m_s) v_s(0+)$$

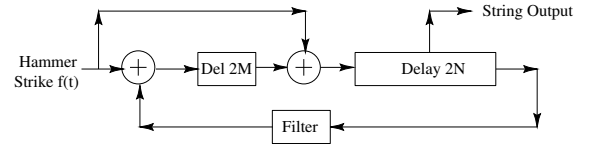
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External String Excitation at a Point



"Waveguide Canonical Form"

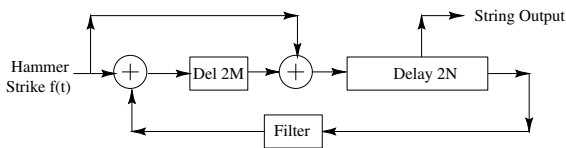
Equivalent System: Delay Consolidation



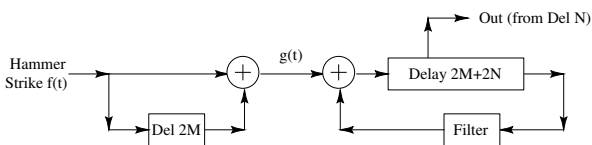
Finally, we "pull out" the comb-filter component:

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Delay Consolidated System (Repeated):



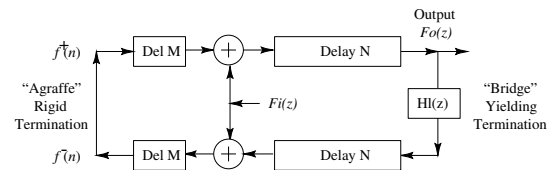
Equivalent System: FFCF Factored Out:



- Extra memory needed.
- Output "tap" can be moved to delay-line output.

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Algebraic Derivation

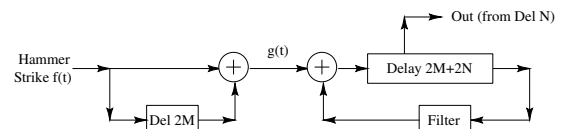


By inspection:

$$F_o(z) = z^{-N} \{ F_i(z) + z^{-2M} [F_i(z) + z^{-N} H_l(z) F_o(z)] \}$$

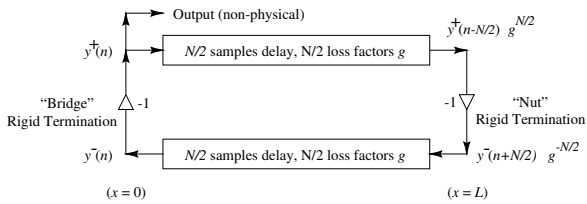
$$\Rightarrow H(z) \triangleq \frac{F_o(z)}{F_i(z)} = z^{-N} \frac{1 + z^{-2M}}{1 - z^{-(2M+2N)} H_l(z)}$$

$$= (1 + z^{-2M}) \frac{z^{-N}}{1 - z^{-(2M+2N)} H_l(z)}$$



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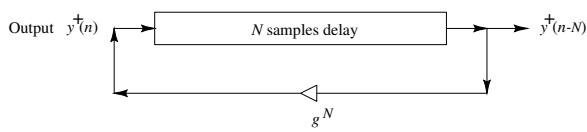
Damped Plucked String



Rigidly terminated string with distributed resistive losses.

- N loss factors g are embedded between the delay-line elements.

Equivalent System: Gain Elements Commuted



All N loss factors g have been “pushed” through delay elements and combined at a *single* point.

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Computational Savings

- $f_s = 50\text{kHz}, f_1 = 100\text{Hz} \Rightarrow \text{delay} = 500$
- Multiplies reduced by two orders of magnitude
- Input-output transfer function unchanged
- Round-off errors reduced

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Frequency-Dependent Damping

- Loss factors g should really be digital filters
- Gains in nature typically decrease with frequency
- Loop gain may not exceed 1 (for stability)
- Gain filters commute with delay elements (LTI)
- Typically only *one* gain filter used per loop

Simplest Frequency-Dependent Loop Filter

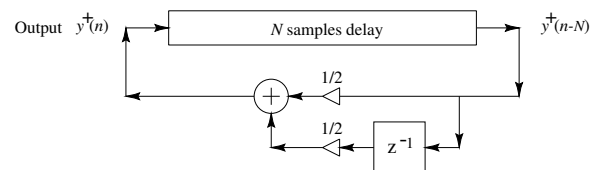
$$H_l(z) = b_0 + b_1 z^{-1}$$

- Linear phase $\Rightarrow b_0 = b_1$ ($\Rightarrow \text{delay} = 1/2$ sample)
- Zero damping at dc $\Rightarrow b_0 + b_1 = 1$
 $\Rightarrow b_0 = b_1 = 1/2$
 \Rightarrow

$$H_l(e^{j\omega T}) = \cos(\omega T/2), \quad |\omega| \leq \pi f_s$$

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Karplus-Strong Algorithm



- To play a note, the delay line is initialized with random numbers (“white noise”)

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KS Physical Interpretation

- Rigidly terminated ideal string with the simplest damping filter
- Damping consolidated at one point and replaced by a one-zero filter approximation
- String *shape* = *sum* of upper and lower delay lines
- The *difference* of upper and lower delay lines corresponds to a nonzero initial string *velocity*. To show this, recall that $f \triangleq -Ky'$ so that

$$y' = -\frac{1}{K}(f^+ + f^-) = -\frac{R}{K}(v^+ - v^-) = \frac{1}{c}(v^- - v^+)$$

implying

$$\boxed{v^+ = -c(y^+)' \quad v^- = c(y^-)'}$$

- Karplus-Strong string is both “plucked” and “struck” by random amounts along entire length of string!
- A “splucked” string?

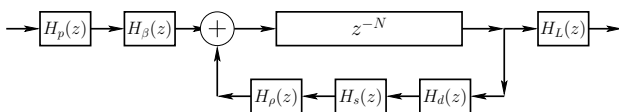
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KS Sound Examples

- “Vintage” 8-bit sound examples:
 - Original Plucked String: (AIFF) (MP3)
 - Drum: (AIFF) (MP3)
 - Stretched Drum: (AIFF) (MP3)
- STK Plucked String: (WAV) (MP3)
- Extended KS (EKS) Scale: (WAV) (MP3)

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Extended Karplus-Strong (EKS) Algorithm



N = pitch period ($2 \times$ string length) in samples

$$H_p(z) = \frac{1-p}{1-pz^{-1}} = \text{pick-direction lowpass filter}$$

$$H_\beta(z) = 1 - z^{-\beta N} = \text{pick-position comb filter, } \beta \in (0, 1)$$

$$H_d(z) = \text{string-damping filter (one/two poles/zeros typical)}$$

$$H_s(z) = \text{string-stiffness allpass filter (several poles and zeros)}$$

$$H_\rho(z) = \frac{\rho(N) - z^{-1}}{1 - \rho(N)z^{-1}} = \text{first-order string-tuning allpass filter}$$

$$H_L(z) = \frac{1 - R_L}{1 - R_L z^{-1}} = \text{dynamic-level lowpass filter}$$

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EKS Sound Example

Bach A-Minor Concerto—Orchestra Part: (WAV) (MP3)

- Executes in real time on one Motorola DSP56001 (20 MHz clock, 128K SRAM)
- Developed for the NeXT Computer introduction at Davies Symphony Hall, San Francisco, 1989
- Solo violin part was played live by Dan Kobiacka of the San Francisco Symphony

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Simplest Frequency-Dependent Loss

Recall that the two-point average used in the Karplus-Strong algorithm can be interpreted as the simplest possible frequency-dependent loss filter for the otherwise ideal vibrating string:

$$H_l(z) = \frac{1 + z^{-1}}{2}$$

Next Simplest Case: Length 3 FIR Loop Filter

$$H_l(z) = b_0 + b_1 z^{-1} + b_2 z^{-2}$$

- Linear phase $\Rightarrow b_0 = b_2$ (\Rightarrow delay = 1 sample)
- Unity dc gain $\Rightarrow b_0 + b_1 + b_2 = 2b_0 + b_1 = 1 \Rightarrow$
 $H_l(e^{j\omega T}) = e^{-j\omega T} [(1 - 2b_0) + 2b_0 \cos(\omega T)]$
- Remaining degree of freedom = *damping control*

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Loop Filter Identification

For loop-filter design, we wish to minimize the error in

- partial decay time (set by amplitude response)
- partial overtone tuning (set by phase response)

Simple and effective method (MUS421 style):

- Estimate pitch (elaborated next page)
- Set Hamming FFT-window length to four periods
- Compute the short-time Fourier transform (STFT)
- Detect peaks in each spectral frame
- Connect peaks through time (amplitude envelopes)
- Amplitude envelopes must decay *exponentially*
- On a dB scale, exponential decay is a *straight line*
- Slope of straight line determines decay time-constant
- Can use `1st-order polyfit` in Matlab or Octave
- For beating decay, connect amplitude envelope peaks
- Decay rates determine ideal amplitude response
- Partial tuning determines ideal phase response

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Length 3 FIR Loop Filter with Variable DC Gain

Relaxing the unity-dc-gain restriction, but keeping linear phase, we have

$$H_l(z) = b_0 + b_1 z^{-1} + b_0 z^{-2} \quad (\text{linear phase})$$

We can use the remaining two degrees of freedom for *brightness B* & *sustain S*:

$$\begin{aligned} g_0 &\triangleq e^{-6.91P/S} \\ b_0 &= g_0(1 - B)/4 = b_2 \\ b_1 &= g_0(1 + B)/2 \end{aligned}$$

where

$$\begin{aligned} P &= \text{period in seconds (total loop delay)} \\ S &= \text{desired sustain time in seconds} \\ B &= \text{brightness parameter in the interval } [0, 1] \end{aligned}$$

Sustain time S is defined here as the time to decay 60 dB (or 6.91 time-constants) when brightness B is maximum ($B = 1$). At minimum brightness ($B = 0$), we have

$$|H_l(e^{j\omega T})| = g_0 \frac{1 + \cos(\omega T)}{2} = g_0 \cos^2(\omega T)$$

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Plucked/Struck String Pitch Estimation

- Take FFT of middle third of plucked string tone
- Detect spectral peaks
- Form histogram of peak spacing Δf_i
- Pitch estimate $\hat{f}_0 \triangleq$ most common spacing Δf_i
- Refine \hat{f}_0 with gradient search using harmonic comb:

$$\begin{aligned} \hat{f}_0 &\triangleq \arg \max_{\hat{f}_0} \sum_{k=1}^K \log |X(k\hat{f}_0)| \\ &= \arg \max_{\hat{f}_0} \prod_{k=1}^K |X(k\hat{f}_0)| \end{aligned}$$

where

$$\begin{aligned} K &= \text{number of peaks, and} \\ k &= \text{harmonic number of } k\text{th peak} \\ &(\text{valid method for non-stiff strings}) \end{aligned}$$

Must skip over any missing harmonics, *i.e.*, omit k whenever $|X(k\hat{f}_0)| \approx 0$.

The text provides further details and pointers to recent papers on pitch estimation.

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Nonlinear “Overdrive”

A popular type of distortion, used in *electric guitars*, is *clipping* of the guitar waveform.

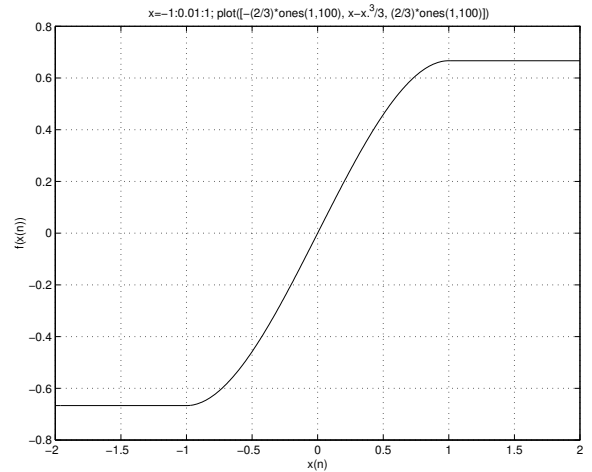
Hard Clipper

$$f(x) = \begin{cases} -1, & x \leq -1 \\ x, & -1 \leq x \leq 1 \\ 1, & x \geq 1 \end{cases}$$

where x denotes the current input sample $x(n)$, and $f(x)$ denotes the output of the nonlinearity.

Soft Clipper

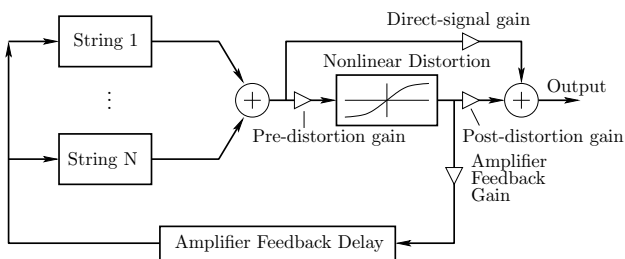
$$f(x) = \begin{cases} -\frac{2}{3}, & x \leq -1 \\ x - \frac{x^3}{3}, & -1 \leq x \leq 1 \\ \frac{2}{3}, & x \geq 1 \end{cases}$$



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Amplifier Distortion + Amplifier Feedback



Distorted Electric Guitar with Amplifier Feedback

- Distortion can be preceded and followed by *EQ*
E.g., integrator “pre” and differentiator “post”
- Distortion output signal often further filtered by an *amplifier cabinet filter*, representing speaker cabinet, driver responses, etc.
- In Class A tube amplifiers, there should be *duty-cycle modulation* as a function of signal level¹
 - 50% at low levels (no duty-cycle modulation)
 - 55-65% duty cycle observed at high levels
⇒ even harmonics come in
 - Example: Distortion input can *offset by a constant* (e.g., input RMS level times some scaling)

¹<http://www.trueaudio.com/at.eetjlm.htm>