

MUS420 Lecture  
Elementary Digital Waveguide Models for Vibrating  
Strings

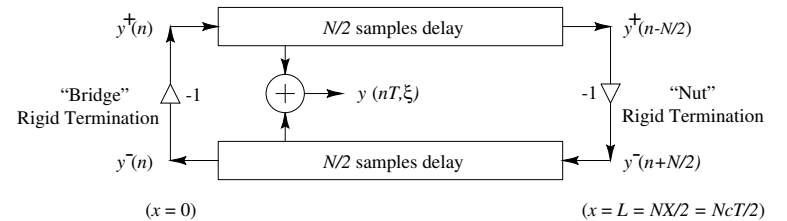
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Outline

- Terminated string
- Plucked and struck string
- Damping and dispersion
- String Loop Identification
- Nonlinear “overdrive” distortion

Rigidly Terminated Ideal String



- Reflection *inverts* for displacement, velocity, or acceleration waves (proof below)
- Reflection *non-inverting* for slope or force waves

Boundary conditions:

$$y(t, 0) \equiv 0 \quad y(t, L) \equiv 0 \quad (L = \text{string length})$$

Expand into *Traveling-Wave Components*:

$$y(t, 0) = y_r(t) + y_l(t) = y^+(t/T) + y^-(t/T)$$

$$y(t, L) = y_r(t - L/c) + y_l(t + L/c)$$

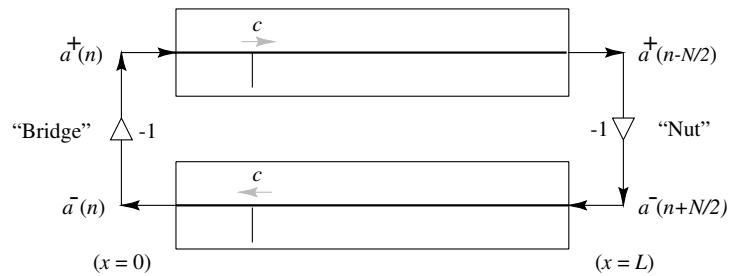
Solving for outgoing waves gives

$$y^+(n) = -y^-(n)$$

$$y^-(n + N/2) = -y^+(n - N/2)$$

$N \triangleq 2L/X = \text{round-trip propagation time in samples}$

## Acceleration-Wave Simulation



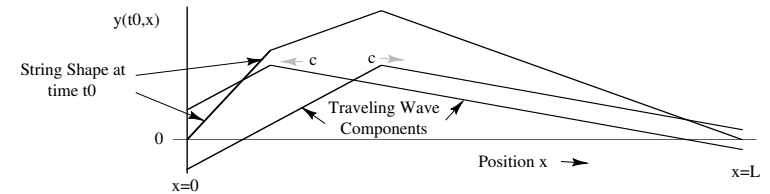
Initial conditions for the ideal plucked string: acceleration or curvature waves.

Recall:

$$y'' = \frac{1}{c^2} \ddot{y}$$

Acceleration waves are proportional to “curvature” waves.

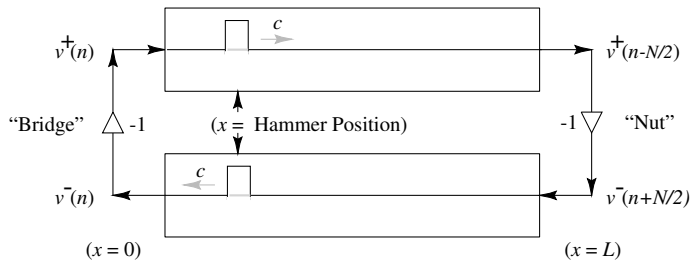
## Doubly Terminated Ideal Plucked String



A doubly terminated string, “plucked” at 1/4 its length.

- Shown short time after pluck event.
- Traveling-wave components and physical string-shape shown.
- Note traveling-wave components sum to zero at terminations. (Use image method.)

## Ideal Struck-String Velocity-Wave Simulation

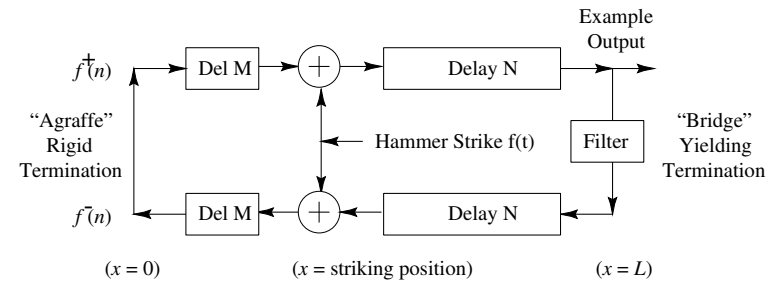


Initial conditions for the ideal struck string in a *velocity wave* simulation.

Hammer strike = *momentum transfer* = velocity step:

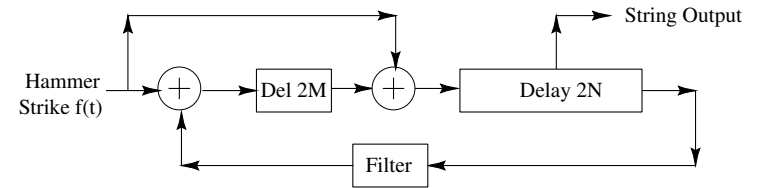
$$m_h v_h(0-) = (m_h + m_s) v_s(0+)$$

## External String Excitation at a Point



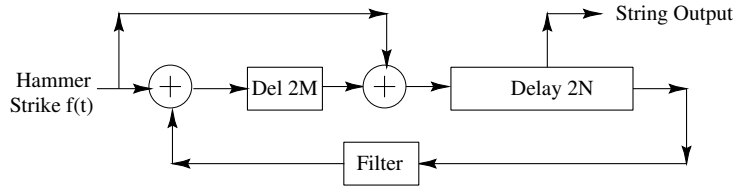
“Waveguide Canonical Form”

## Equivalent System: Delay Consolidation

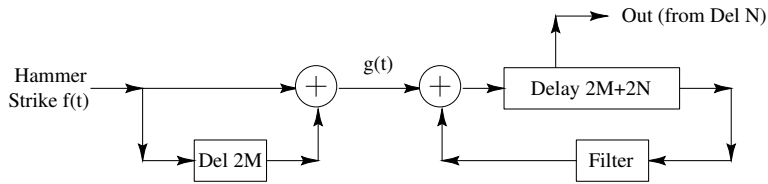


Finally, we “pull out” the comb-filter component:

### Delay Consolidated System (Repeated):

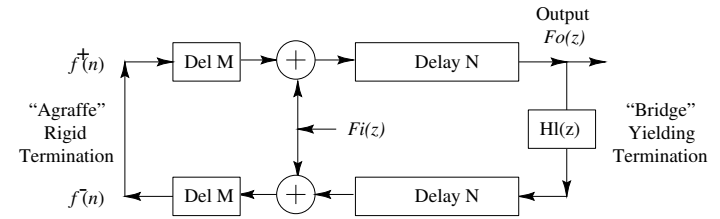


### Equivalent System: FFCF Factored Out:



- Extra memory needed.
- Output “tap” can be moved to delay-line output.

### Algebraic Derivation

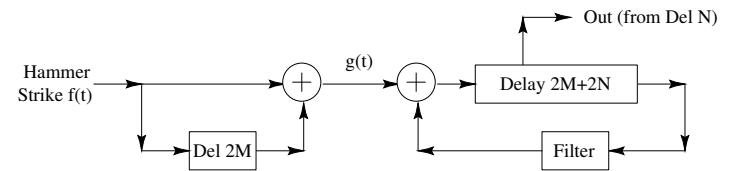


By inspection:

$$F_o(z) = z^{-N} \{ F_i(z) + z^{-2M} [ F_i(z) + z^{-N} H_l(z) F_o(z) ] \}$$

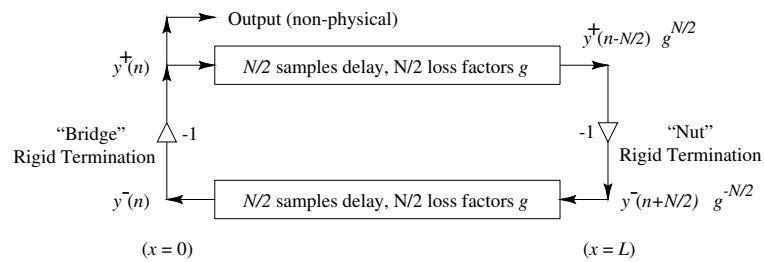
$$\Rightarrow H(z) \triangleq \frac{F_o(z)}{F_i(z)} = z^{-N} \frac{1 + z^{-2M}}{1 - z^{-(2M+2N)} H_l(z)}$$

$$= (1 + z^{-2M}) \frac{z^{-N}}{1 - z^{-(2M+2N)} H_l(z)}$$



## Damped Plucked String

## Computational Savings

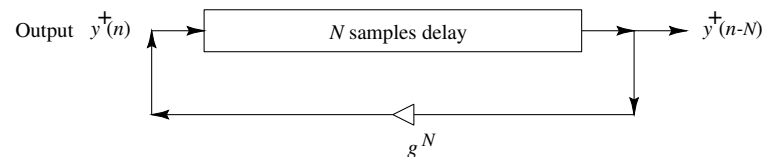


- $f_s = 50\text{kHz}, f_1 = 100\text{Hz} \Rightarrow \text{delay} = 500$
- Multiplies reduced by two orders of magnitude
- Input-output transfer function unchanged
- Round-off errors reduced

Rigidly terminated string with distributed resistive losses.

- $N$  loss factors  $g$  are embedded between the delay-line elements.

### Equivalent System: Gain Elements Commuted



All  $N$  loss factors  $g$  have been "pushed" through delay elements and combined at a *single* point.

## Frequency-Dependent Damping

- Loss factors  $g$  should really be digital filters
- Gains in nature typically decrease with frequency
- Loop gain may not exceed 1 (for stability)
- Gain filters commute with delay elements (LTI)
- Typically only *one* gain filter used per loop

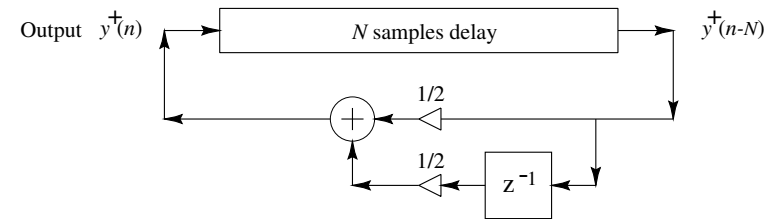
### Simplest Frequency-Dependent Loop Filter

$$H_l(z) = b_0 + b_1 z^{-1}$$

- Linear phase  $\Rightarrow b_0 = b_1$  ( $\Rightarrow$  delay = 1/2 sample)
- Zero damping at dc  $\Rightarrow b_0 + b_1 = 1$   
 $\Rightarrow b_0 = b_1 = 1/2$   
 $\Rightarrow$

$$H_l(e^{j\omega T}) = \cos(\omega T/2), \quad |\omega| \leq \pi f_s$$

## Karplus-Strong Algorithm



- To play a note, the delay line is initialized with random numbers (“white noise”)

## KS Physical Interpretation

- Rigidly terminated ideal string with the simplest damping filter
- Damping consolidated at one point and replaced by a one-zero filter approximation
- String *shape* = *sum* of upper and lower delay lines
- The *difference* of upper and lower delay lines corresponds to a nonzero initial string *velocity*. To show this, recall that  $f \triangleq -Ky'$  so that

$$y' = -\frac{1}{K}(f^+ + f^-) = -\frac{R}{K}(v^+ - v^-) = \frac{1}{c}(v^- - v^+)$$

implying

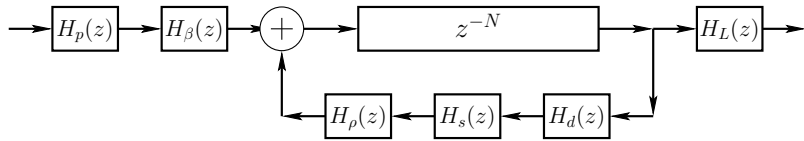
$$\boxed{v^+ = -c(y^+)' \quad v^- = c(y^-)'}$$

- Karplus-Strong string is both “plucked” and “struck” by random amounts along entire length of string!
- A “splucked” string?

## KS Sound Examples

- “Vintage” 8-bit sound examples:
  - Original Plucked String: (AIFF) (MP3)
  - Drum: (AIFF) (MP3)
  - Stretched Drum: (AIFF) (MP3)
- STK Plucked String: (WAV) (MP3)
- Extended KS (EKS) Scale: (WAV) (MP3)

## Extended Karplus-Strong (EKS) Algorithm



$N$  = pitch period ( $2 \times$  string length) in samples

$$H_p(z) = \frac{1 - p}{1 - p z^{-1}} = \text{pick-direction lowpass filter}$$

$$H_\beta(z) = 1 - z^{-\beta N} = \text{pick-position comb filter, } \beta \in (0, 1)$$

$$H_d(z) = \text{string-damping filter (one/two poles/zeros typical)}$$

$$H_s(z) = \text{string-stiffness allpass filter (several poles and zeros)}$$

$$H_\rho(z) = \frac{\rho(N) - z^{-1}}{1 - \rho(N) z^{-1}} = \text{first-order string-tuning allpass filter}$$

$$H_L(z) = \frac{1 - R_L}{1 - R_L z^{-1}} = \text{dynamic-level lowpass filter}$$

## EKS Sound Example

Bach A-Minor Concerto—Orchestra Part: (WAV) (MP3)

- Executes in real time on one Motorola DSP56001 (20 MHz clock, 128K SRAM)
- Developed for the NeXT Computer introduction at Davies Symphony Hall, San Francisco, 1989
- Solo violin part was played live by Dan Kobialka of the San Francisco Symphony



## Simplest Frequency-Dependent Loss

Recall that the two-point average used in the Karplus-Strong algorithm can be interpreted as the simplest possible frequency-dependent loss filter for the otherwise ideal vibrating string:

$$H_l(z) = \frac{1 + z^{-1}}{2}$$

## Next Simplest Case: Length 3 FIR Loop Filter

$$H_l(z) = b_0 + b_1 z^{-1} + b_2 z^{-2}$$

- Linear phase  $\Rightarrow b_0 = b_2$  ( $\Rightarrow$  delay = 1 sample)
- Unity dc gain  $\Rightarrow b_0 + b_1 + b_2 = 2b_0 + b_1 = 1 \Rightarrow$

$$H_l(e^{j\omega T}) = e^{-j\omega T} [(1 - 2b_0) + 2b_0 \cos(\omega T)]$$

- Remaining degree of freedom = *damping control*

## Length 3 FIR Loop Filter with Variable DC Gain

Relaxing the unity-dc-gain restriction, but keeping linear phase, we have

$$H_l(z) = b_0 + b_1 z^{-1} + b_0 z^{-2} \quad (\text{linear phase})$$

We can use the remaining two degrees of freedom for *brightness*  $B$  & *sustain*  $S$ :

$$g_0 \triangleq e^{-6.91P/S}$$

$$b_0 = g_0(1 - B)/4 = b_2$$

$$b_1 = g_0(1 + B)/2$$

where

$P$  = period in seconds (total loop delay)

$S$  = desired sustain time in seconds

$B$  = brightness parameter in the interval  $[0, 1]$

Sustain time  $S$  is defined here as the time to decay 60 dB (or 6.91 time-constants) when brightness  $B$  is maximum ( $B = 1$ ). At minimum brightness ( $B = 0$ ), we have

$$|H_l(e^{j\omega T})| = g_0 \frac{1 + \cos(\omega T)}{2} = g_0 \cos^2(\omega T)$$

## Loop Filter Identification

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For loop-filter design, we wish to minimize the error in

- partial decay time (set by amplitude response)
- partial overtone tuning (set by phase response)

Simple and effective method (MUS421 style):

- Estimate pitch (elaborated next page)
- Set Hamming FFT-window length to four periods
- Compute the short-time Fourier transform (STFT)
- Detect peaks in each spectral frame
- Connect peaks through time (amplitude envelopes)
- Amplitude envelopes must decay *exponentially*
- On a dB scale, exponential decay is a *straight line*
- Slope of straight line determines decay time-constant
- Can use 1st-order polyfit in Matlab or Octave
- For beating decay, connect amplitude envelope peaks
- Decay rates determine ideal amplitude response
- Partial tuning determines ideal phase response

## Plucked/Struck String Pitch Estimation

- Take FFT of middle third of plucked string tone
- Detect spectral peaks
- Form histogram of peak spacing  $\Delta f_i$
- Pitch estimate  $\hat{f}_0 \triangleq$  most common spacing  $\Delta f_i$
- Refine  $\hat{f}_0$  with gradient search using harmonic comb:

$$\begin{aligned}\hat{f}_0 &\triangleq \arg \max_{\hat{f}_0} \sum_{k=1}^K \log |X(k\hat{f}_0)| \\ &= \arg \max_{\hat{f}_0} \prod_{k=1}^K |X(k\hat{f}_0)|\end{aligned}$$

where

$K$  = number of peaks, and

$k$  = harmonic number of  $k$ th peak

(valid method for non-stiff strings)

Must skip over any missing harmonics, *i.e.*, omit  $k$  whenever  $|X(k\hat{f}_0)| \approx 0$ .

The text provides further details and pointers to recent papers on pitch estimation.

## Nonlinear “Overdrive”

A popular type of distortion, used in *electric guitars*, is *clipping* of the guitar waveform.

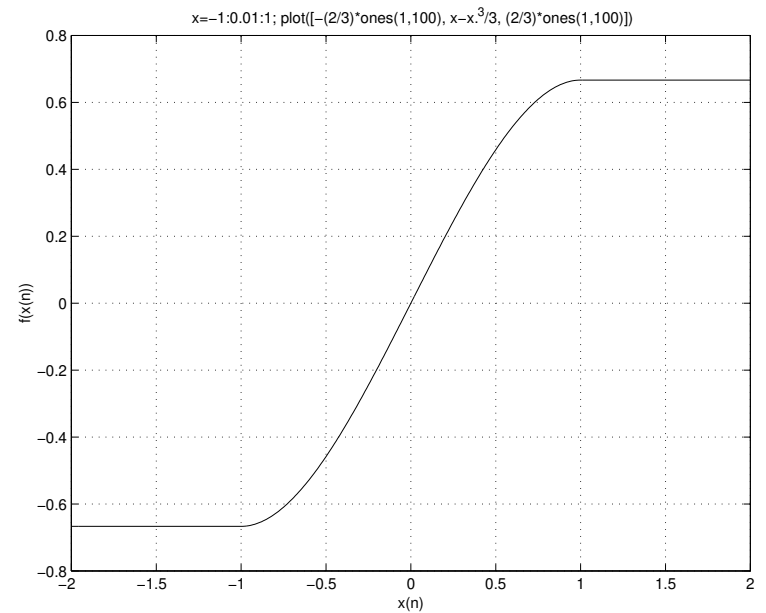
### Hard Clipper

$$f(x) = \begin{cases} -1, & x \leq -1 \\ x, & -1 \leq x \leq 1 \\ 1, & x \geq 1 \end{cases}$$

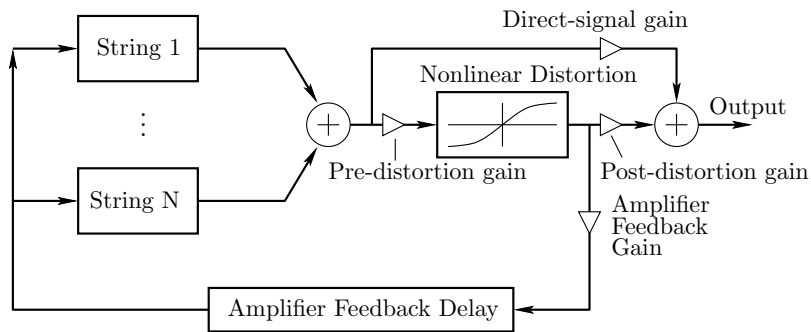
where  $x$  denotes the current input sample  $x(n)$ , and  $f(x)$  denotes the output of the nonlinearity.

## Soft Clipper

$$f(x) = \begin{cases} -\frac{2}{3}, & x \leq -1 \\ x - \frac{x^3}{3}, & -1 \leq x \leq 1 \\ \frac{2}{3}, & x \geq 1 \end{cases}$$



## Amplifier Distortion + Amplifier Feedback



### Distorted Electric Guitar with Amplifier Feedback

- Distortion can be preceded and followed by *EQ*  
E.g., integrator “pre” and differentiator “post”
- Distortion output signal often further filtered by an *amplifier cabinet filter*, representing speaker cabinet, driver responses, etc.
- In Class A tube amplifiers, there should be *duty-cycle modulation* as a function of signal level<sup>1</sup>
  - 50% at low levels (no duty-cycle modulation)
  - 55-65% duty cycle observed at high levels  
⇒ even harmonics come in
  - Example: Distortion input can *offset by a constant* (e.g., input RMS level times some scaling)

<sup>1</sup>[http://www.trueaudio.com/at\\_eetjlm.htm](http://www.trueaudio.com/at_eetjlm.htm)