Outline

- Plane Wave at an Angle
- Plane-Wave Scattering at an Impedance Discontinuity
- Reflection and Refraction
- Evanescent Field due to Total Internal Reflection
- Plane-Wave Scattering at an Angle
- Imaginary wavenumbers

Plane Wave at an Angle

\[
\begin{align*}
y & \uparrow \\
\lambda & = \frac{2\pi}{k} \\
x & \longrightarrow \\
\end{align*}
\]

wave crests of the sinusoidal traveling plane wave
\[
p(t, x) = \cos(\omega t - k^T x)
\]

Planar pressure-wave traveling in an arbitrary direction:
\[
p(t, x) = \cos\left(\omega t - k^T x\right), \quad x \in \mathbb{R}^3
\]

where \( k = \text{vector wavenumber} \):

\[
k = \begin{bmatrix} k_x \\ k_y \\ k_z \end{bmatrix} = k \begin{bmatrix} k_x/k \\ k_y/k \\ k_z/k \end{bmatrix} \triangleq k \begin{bmatrix} \cos \alpha \\ \cos \beta \\ \cos \gamma \end{bmatrix} \triangleq k \mathbf{u},
\]

where
• \( \underline{u} \) = (unit) vector of direction cosines
• \( k = \frac{2\pi}{\lambda} \) = (scalar) wavenumber along travel direction

Thus, the vector wavenumber \( \underline{k} = k \underline{u} \) contains
• wavenumber in its magnitude \( k = \| \underline{k} \| \)
• travel direction in its orientation \( \underline{u} = \underline{k} / k \)

Note: wavenumber units are radians per meter (spatial radian frequency)

By continuity, waves must agree on boundary plane:

\[
\langle k_1^+, r \rangle = \langle k_1^-, r \rangle = \langle k_2^+, r \rangle
\]

where \( r = (0, y, z) \) denotes any vector in the boundary plane. Thus, at \( x = 0 \) we have

\[
k_1^+ y + k_1^+ z = k_1^- y + k_1^- z = k_2^+ y + k_2^+ z
\]

If the incident wave is constant along \( z \), then \( k_1^+ = 0 \), requiring \( k_1^- = k_2^- = 0 \), leaving

\[
k_1^+ y = k_1^- y = k_2^+ y
\]

or

\[
\begin{align*}
\sin(\theta_1^+) &= \sin(\theta_1^-) = \sin(\theta_2^+)
\end{align*}
\]
Reflection and Refraction

Above we derived
\[ k_1 \sin(\theta_1^+) = k_1 \sin(\theta_1^-) = k_2 \sin(\theta_2^+) \]
The first equality implies
\[ \theta_1^+ = \theta_1^- \]
(Angle of incidence equals angle of reflection)

Let \( c_i \) denote the phase velocity in wave impedance \( R_i \):
\[ c_i = \frac{\omega}{k_i}, \quad i = 1, 2 \]

In impedance \( R_2 \), we have in particular
\[ \omega^2 = c_2^2 k_2^2 = c_2^2 \left[ (k_{2x}^+)^2 + (k_{2y}^+)^2 \right] \]

Solving for \( k_{2x}^+ \) gives
\[ k_{2x}^+ = \sqrt{\frac{\omega^2}{c_2^2} - (k_{2y}^+)^2} = \sqrt{\frac{\omega^2}{c_2^2} - k_2^2 \sin^2(\theta_2^+)} \]

Since \( k_1 \sin(\theta_1^+) = k_2 \sin(\theta_2^+) \) from above,
\[ k_{2x}^+ = \sqrt{\frac{\omega^2}{c_2^2} - k_1^2 \sin^2(\theta_1^+)} = \sqrt{\frac{\omega^2}{c_2^2} - \frac{\omega^2}{c_1^2} \sin^2(\theta_1^+)} \]

We have derived
\[ k_{2x}^+ = \frac{\omega}{c_2} \sqrt{1 - \frac{c_2^2}{c_1^2} \sin^2(\theta_1^+)} \]

We earlier established \( k_{2y}^+ = k_{1y}^+ \).

- This describes the \textit{refraction} of the plane wave as it passes through the impedance-change boundary.
- Refraction angle depends on ratio of phase velocities \( c_2/c_1 \).
- This ratio is often called the \textit{index of refraction}:
\[ n \triangleq \frac{c_2}{c_1} \]
and the relation \( k_1 \sin(\theta_1^+) = k_2 \sin(\theta_2^+) \) is called \textit{Snell’s Law} (of refraction).
Evanescent Field due to Total Internal Reflection

Note that if \( c_1 < c_2 | \sin(\theta_1^+) | \), the horizontal component of the wavenumber in medium 2 becomes imaginary:

- Acoustic field in medium 2 is “evanescent”
- Wave in medium 1 undergoes “total internal reflection”
- No power travels from medium 1 to medium 2
- Evanescent field decays exponentially to the right
- “Tunneling” possible given medium 3 in which wave propagation resumes

What does it mean to have an imaginary wavenumber?

\[
p(t, x) = \cos(\omega t - k^T x) = \text{re}\left\{ e^{j(\omega t - k_x x - k_y y)} \right\} \\
= \text{re}\left\{ e^{j(\omega t - k_y y)} e^{-j k_x x} \right\}, \quad (\text{let } k_x \overset{\Delta}{=} -j \kappa_x) \\
\overset{\Delta}{=} \text{re}\left\{ e^{j(\omega t - k_y y)} e^{-k_x x} \right\} \\
= e^{-k_x x} \cos(\omega t - k_y y)
\]

- An imaginary wavenumber corresponds to an exponential decay
- Sign of \( \kappa_x \) is chosen to match boundary conditions at the plane
- Time dependence applies to all points to the right of the boundary (no “propagation”)