MUS420 Lecture
Acoustic Scattering at Impedance Changes

Julius O. Smith III (jos@ccrma.stanford.edu)
Center for Computer Research in Music and Acoustics (CCRMA)
Department of Music, Stanford University
Stanford, California 94305

June 27, 2020

Outline

- Plane-Wave Scattering
- The Loaded N-Way Junction
- Lossless Scattering
  - Normalized Scattering Junctions
  - Transformer-Normalized Scattering Junctions
- Junction Passivity
- The Digital Waveguide Mesh

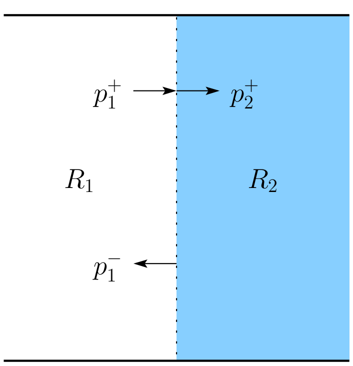
Wave Scattering at an Impedance Discontinuity

A *change in wave impedance* causes lossless signal *scattering*:

- A traveling wave impinging on an *impedance discontinuity* will partially *reflect* from and partially *transmit* through the discontinuity
- *Pressure* will be *continuous* everywhere
- *Velocity in* = *velocity out* (junction has no state)
- *Signal power* (energy) is conserved

Plane-Wave Scattering

Consider a plane-wave p_1^+ propagating from wave impedance R_1 into a new wave impedance R_2:



Physical constraints:

$$\begin{aligned} p_1^+ + p_1^- &= p_2^+ \quad \text{(pressure continuous across junction)} \\ v_1^+ + v_1^- &= v_2^+ \quad \text{(velocity in = velocity out)} \end{aligned}$$

Ohm's Law relations:

$$\begin{aligned} p_i^+ &= \quad R_i v_i^+ \\ p_i^- &= - \ R_i v_i^- \end{aligned}$$

Scattering Solution

Let

$$\begin{aligned} p_j &\stackrel{\Delta}{=} p_1^+ + p_1^- = p_2^+ \quad \text{(pressure at junction)} \\ v_j &\stackrel{\Delta}{=} v_1^+ + v_1^- = v_2^+ \quad \text{(velocity at junction)} \end{aligned}$$

Then we can write

$$\begin{aligned} p_1^+ + p_1^- &= p_2^+ \ = \ p_j \\ \Rightarrow \ R_1 v_1^+ - R_1 v_1^- &= R_2 v_2^+ \ = \ R_2 v_j \\ \Rightarrow \ R_1 v_1^+ - R_1 (v_j - v_1^+) &= R_2 v_j \\ \Rightarrow \ 2\,R_1 v_1^+ - R_1 v_j &= R_2 v_j \\ \Rightarrow \ &\boxed{v_j = \frac{2\,R_1}{R_1 + R_2} v_1^+} \end{aligned}$$

We have solved for the junction velocity $v_j = v_2^+$. The transmitted pressure is then $p_2^+ = R_2 v_2^+ = R_2 v_j$.

Since $v_j = v_1^+ + v_1^-$, the reflected velocity is simply

$$v_1^- = v_j - v_1^+ = \left[\frac{2\,R_1}{R_1 + R_2} - 1\right] v_1^+ = \frac{R_1 - R_2}{R_1 + R_2} v_1^+$$

Thus, we have solved for the transmitted and reflected velocity waves given the incident wave and the two impedances.

Using the Ohm's law relations, the pressure waves follow:

$$p_2^+ \;=\; R_2 v_2^+ = R_2 v_j = \frac{2\,R_2}{R_1 + R_2} p_1^+$$

$$p_1^- \;=\; -R_1 v_1^- = \frac{R_2 - R_1}{R_1 + R_2} p_1^+$$

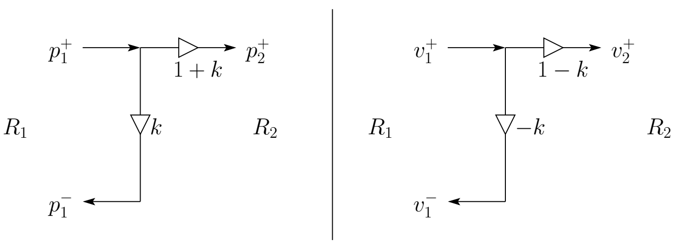
Define

$$\boxed{k = \frac{R_2 - R_1}{R_1 + R_2} = \frac{\text{Impedance Step}}{\text{Impedance Sum}}}$$

Then we get the following scattering relations in terms of k for pressure waves:

$$\boxed{\begin{aligned} p_2^+ \;&=\; (1+k)p_1^+ \\ p_1^- \;&=\; k\,p_1^+ \end{aligned}}$$

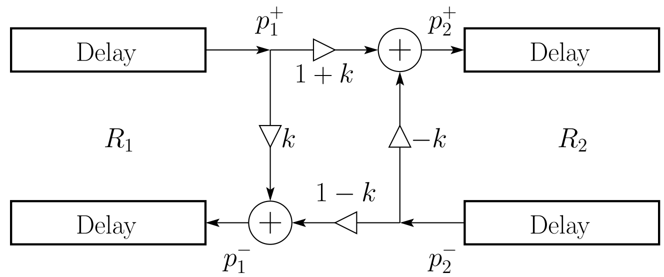
Signal Flow Graph:



Signal power conserved (left-going power negated):

$$p_1^+ v_1^+ = p_2^+ v_2^+ + (-p_1^- v_1^-)$$

Superposition of Bidirectional Scattering



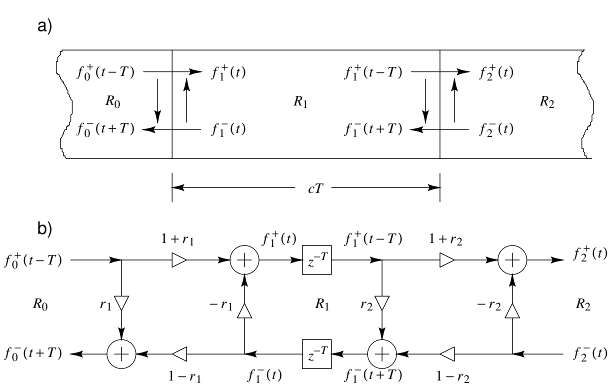
- Stepping from R_2 to R_1 negates $k \triangleq \dfrac{R_2 - R_1}{R_2 + R_1}$
- Transmission is 1 + reflection in either direction
- "Kelly-Lochbaum" scattering junction

Special Cases:

- $R_2 = \infty \;\Rightarrow\; k = 1$ (*e.g.*, rigid wall reflection)
- $R_2 = 0 \;\Rightarrow\; k = -1$ (*e.g.*, open-ended tube)
- $R_2 = R_1 \;\Rightarrow\; k = 0$ (no reflection)

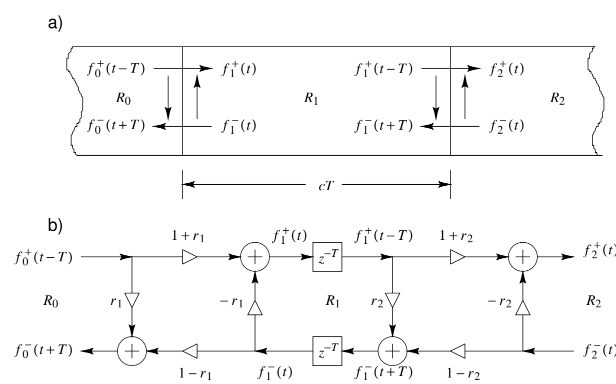
Longitudinal String Waves

- *Longitudinal* string waves compress and stretch along the string x axis
- String may have a nonzero diameter = "stiff string" or "*rod*"
- In solids, *force-density* waves are called *stress* waves



Longitudinal force waves in an ideal rod

Longitudinal Scattering in Strings



A waveguide section between two partial sections

a) Physical picture indicating traveling waves in a continuous medium with wave impedance changing from R_0 to R_1 to R_2 along the horizontal axis, resulting in signal scattering (power-conserving transmission and reflection)

b) Digital simulation diagram

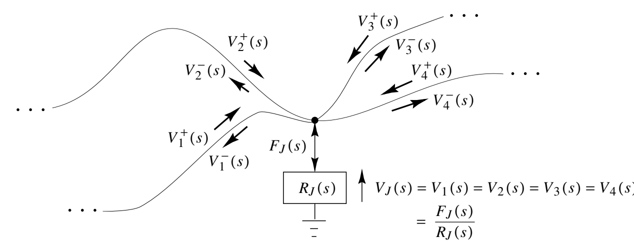
Such a rod might be constructed, for example, using three different materials having three different densities

Longitudinal Scattering in Strings, Notes

- As before, velocity $v_i = v_i^+ + v_i^-$ is defined as positive to the right
- $f_i^+ =$ right-going traveling-wave component of the *stress*, positive when the rod is locally *compressed*
- The stress-wave reflection coefficients are
$$r_i = \frac{R_i - R_{i-1}}{R_i + R_{i-1}}$$
to the right, with corresponding transmission coefficients $1 + r_i$ to the right
- To the left, the impedance step negates, so the reflection coefficients negate for waves propagating to the left
- Wave impedance is now $R_i = \sqrt{E\rho}$ where
$$\begin{aligned} \rho &= \textit{mass density} \\ E &= \textit{Young's modulus} \text{ of the medium} \\ &= \textit{stress} \text{ over } \textit{strain} \end{aligned}$$
- *strain* = *relative* displacement $\delta y / y$
- To minimize the numerical dynamic range, velocity waves may be chosen instead when $R_i > 1$

The Loaded N-Port Scattering Junction

Four Ideal Strings Intersecting at a Load



Series junction ⇔ *common velocity*, *forces sum to 0*:

$$\begin{aligned} V_1(s) = V_2(s) = \cdots = V_N(s) &\triangleq V_J(s) \\ F_1(s) + F_2(s) + \cdots + F_N(s) &= V_J(s)R_J(s) \triangleq F_J(s) \end{aligned}$$

Computing common velocity at junction:

$$\begin{aligned} R_J V_J \;=\; F_J &= \sum_{i=1}^N F_i = \sum_{i=1}^N (F_i^+ + F_i^-) \\ &= \sum_{i=1}^N (R_i V_i^+ - R_i \underbrace{V_i^-}_{V_J - V_i^+}) \\ &= \sum_{i=1}^N (2 R_i V_i^+ - R_i V_J) \end{aligned}$$

⇒

$$V_J = 2\left(R_J + \sum_{i=1}^N R_i\right)^{-1} \sum_{i=1}^N R_i V_i^+$$

or

$$\boxed{V_J(s) = \sum_{i=1}^N \mathcal{A}_i(s) V_i^+(s)}$$

where

$$\boxed{\mathcal{A}_i(s) \triangleq \frac{2R_i}{R_J(s) + R_1 + \cdots + R_N}}$$

(generalized *"alpha parameter"*, cf. Wave Digital Filters)

Finally, by continuity, $V_J = V_i = V_i^+ + V_i^-$ ⇒

$$\boxed{V_i^-(s) = V_J(s) - V_i^+(s)}$$

Solution Properties

We determined that

$$\begin{aligned} V_J(s) &= \sum_{i=1}^N \mathcal{A}_i(s) V_i^+(s) \qquad \text{(junction velocity)} \\ V_i^-(s) &= V_J(s) - V_i^+(s) \qquad \text{(outgoing velocity waves)} \end{aligned}$$

where

$$\mathcal{A}_i(s) \triangleq \frac{2R_i}{R_J(s) + R_1 + \cdots + R_N}$$

- Lossless only when $\text{Re}\{R_J(j\omega)\} \equiv 0$
- Memoryless only when $\text{Im}\{R_J(j\omega)\} \equiv 0$
- Dynamic load takes all scattering coefficients into Laplace domain
  - Order of each "scattering-filter" equals load order
  - Normally *one filter* can serve entire junction
- Junction load equivalent to an $N+1$st waveguide (with no input) having (generalized) wave impedance given by load impedance $R_J(s)$
(consider "perfect termination" of a transmission line using a resistor equal in value to line impedance)

Alpha Parameters

Recall N velocity waveguides meeting at a series junction:

$$V_J(s) \;=\; \sum_{i=1}^{N} \mathcal{A}_i(s) V_i^+(s) \qquad \text{(junction velocity)}$$

$$V_i^-(s) \;=\; V_J(s) - V_i^+(s) \qquad \text{(outgoing velocity waves)}$$

where

$$\mathcal{A}_i(s) \triangleq \frac{2R_i}{R_J(s) + R_1 + \cdots + R_N}$$

In the lossless (unloaded) case, $R_J(s) = 0$, and so the alpha parameters are *real* and *positive* and *add up to 2:*

$$\alpha_i = \frac{2R_i}{R_1 + \cdots + R_N}$$

I.e.,

$$0 \leq \alpha_i \leq 2$$

and

$$\sum_{i=1}^{N} \alpha_i = 2$$

Also, we no longer have to be in the Laplace domain ($R_J = 0$)

Time-Domain Lossless Series Scattering in terms of Alpha Parameters

$$v_J(t) \;=\; \sum_{i=1}^{N} \alpha_i v_i^+(t)$$

$$v_i^-(t) \;=\; v_J(t) - v_i^+(t)$$

Alpha parameters conveniently parametrize lossless scattering junctions:

- Explicit coefficients of incoming traveling waves for computing junction velocity
- Losslessness assured when alpha parameters are nonnegative and sum to 2
- When alpha parameters sum to less than 2, there is conceptually a "resistive loss" at the junction

In the lossless, *equal-impedance* case, $R_i = R, \forall i \;\Rightarrow$

$$\boxed{\alpha_i = \frac{2}{N}}$$

When N is a power of two, *no multiplies* are needed (multiply-free reverberators, waveguide meshes, etc., are based on this)

Reflection-Free Ports

To suppress series-junction *reflections* on one of the strings, say string 1, then we set its wave impedance to the *series combination* (i.e., sum) of the other wave impedances meeting at the series junction:

$$R_1 = R_2 + R_3 + \cdots + R_N$$

- This choice of R_1 *matches* the impedance seen from string 1 when entering junction
- Such matching eliminates an *impedance step* seen by waves traveling from string 1 into all of the other strings

In this case, the first alpha parameter becomes

$$\alpha_1 = \frac{2R_1}{R_1 + R_2 + \cdots + R_N} = \frac{2R_1}{R_1 + R_1} = 1$$

and the remaining alpha parameters can be expressed as

$$\alpha_i = \frac{2R_i}{2R_1} = \frac{R_i}{R_1}, \quad i = 1, 2, \ldots, N$$

and the sum of the $N-1$ remaining alpha parameters is therefore 1 since there is no junction load.

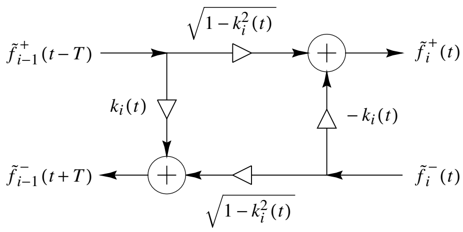
Normalizing by the Wave Impedance at a Reflection-Free Port

Often we only care about the signal scattering and not the specific impedance values. In that case it is convenient to divide all impedances at the junction by R_1, defining $\tilde{R}_i \triangleq R_i/R_1$, so that

$$\begin{aligned} \tilde{R}_1 \;&=\; 1 \\ \tilde{R}_i \;&\in\; [0,1], \quad i = 2, \ldots, N \\ \sum_{i=2}^{N} \tilde{R}_i \;&=\; 1 \\ \alpha_1 \;&=\; 1 \\ \alpha_i \;&=\; \tilde{R}_i \\ \sum_{i=2}^{N} \alpha_i \;&=\; 1 \end{aligned}$$

String 1 is said to intersect with the other strings at a *reflection-free port*. For $N > 2$, the other strings are attached to the junction at reflecting ports.

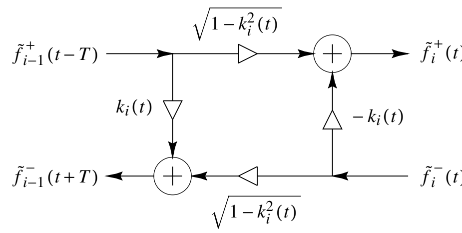
Normalized Scattering Junctions



The normalized scattering junction

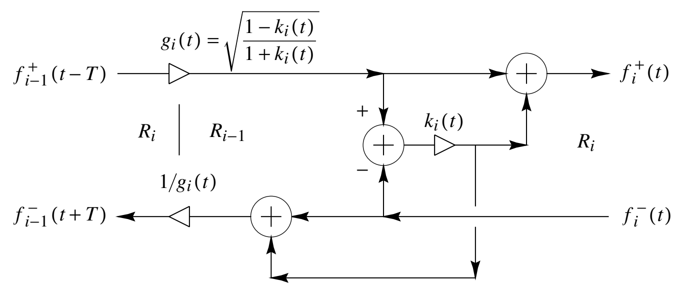
- Recall *normalized waves:*
$$\tilde{f}^+_i \triangleq f^+_i/\sqrt{R_i} \qquad \tilde{v}^+_i \triangleq v^+_i \cdot \sqrt{R_i}$$
- Converting scattering to *normalized waves* $\tilde{f}^{\pm}$ gives
$$\begin{aligned}\tilde{f}^+_i(t) &= \sqrt{1-k_i^2(t)}\,\tilde{f}^+_{i-1}(t-T) - k_i(t)\tilde{f}^-_i(t)\\ \tilde{f}^-_{i-1}(t+T) &= k_i(t)\tilde{f}^+_{i-1}(t-T) + \sqrt{1-k_i^2(t)}\,\tilde{f}^-_i(t)\end{aligned}$$
- Better term = "Normalized-wave scattering junction"
- Normalized junction is equivalent to a *2D rotation*:
$$\begin{aligned}\tilde{f}^+_i(t) &= \cos(\theta_i)\tilde{f}^+_{i-1}(t-T) - \sin(\theta_i)\tilde{f}^-_i(t)\\ \tilde{f}^-_{i-1}(t+T) &= \sin(\theta_i)\tilde{f}^+_{i-1}(t-T) + \cos(\theta_i)\tilde{f}^-_i(t)\end{aligned}$$

Normalized Scattering Junction



The normalized scattering junction

- *Four multiplies* and *two additions* required
- Using *transformer normalization*, we can obtain *three-multiply*, *three-add* variations:



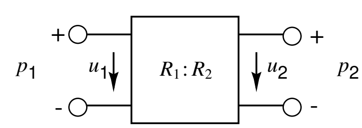
The Digital Waveguide Transformer

The ideal transformer

- scales up pressure and scales down velocity by the same factor
- steps wave impedance from R_1 to R_2 (and vice versa) without reflections
- conserves power
- No scattering reflections generated
- Physical in principle, but not *realizable*
  There are engineering approximations, however:
  - Conical acoustic tube
  - Horn loudspeakers
  - Quarter-wave microwave transformers

Transformer Scattering Formulas

General Two-Port:



The general 2-port.

Power conservation:

$$\begin{aligned} p_1 u_1 &= -p_2 u_2\\ \Leftrightarrow (p_1^+ + p_1^-)\left(\frac{p_1^+ - p_1^-}{R_1}\right) &= -(p_2^+ + p_2^-)\left(\frac{p_2^+ - p_2^-}{R_2}\right)\end{aligned}$$

Non-reflecting:

$$\begin{aligned} p_1^- &= g_1 p_2^+\\ p_2^- &= g_2 p_1^+\end{aligned}$$

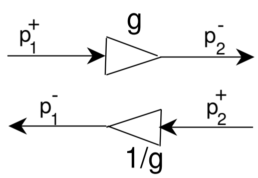
for some constants g_1, g_2

Solution:

$$\begin{aligned} p_1^- &= \sqrt{\frac{R_1}{R_2}}p_2^+ \triangleq \frac{1}{g}p_2^+\\ p_2^- &= \sqrt{\frac{R_2}{R_1}}p_1^+ \triangleq g p_1^+\end{aligned}$$

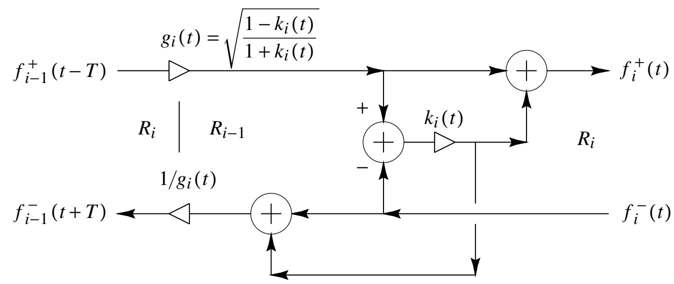
where $g \triangleq$ transformer "turns ratio"

Wave-flow diagram:



The ideal 2-port transformer

Three-Multiply Transformer-Normalized Scattering Junction



- Using transformers, all waveguides are normalized to the same impedance, $R_i \equiv 1$
- g_i and/or $1/g_i$ may have a large dynamic range
- While transformer-normalization trades a multiply for an add, up to 50% more bits needed in junction adders (see text)

Principles of Passive Construction

We can state the following general principles for *passive signal processing*:

- Confine all nonlinear operations to *physically meaningful wave variables*
- Signal power = *square* of physical variable times admittance or impedance
- *Passivity* assured if *all effective gains* less than 1
- Passive rounding:
  - Apply to *extended-precision intermediate result*
  - Magnitude truncation ("rounding toward zero")
  - Error power feedback
- *Limit cycles* impossible in passive systems
- *Overflow oscillations* impossible in passive systems
- *Energy* in ideal implementation = *Lyapunov function* bounding energy in the finite-precision implementation

Structural Losslessness - Two-Port Case

- *One-multiply* scattering junctions are *structurally lossless*: Only *one parameter* for which all *in-range* quantizations correspond to lossless scattering
  - *Reflection coefficient* $k_i \in [-1, 1]$
  - *Alpha parameter* $\alpha_i \in [0, 2]$
- Not all normalized scattering junctions are structurally lossless
  - The four-multiply normalized junction has *two* parameters, $s_i \triangleq k_i$ and $c_i \triangleq \sqrt{1-k_i^2}$, which may not satisfy $s_i^2 + c_i^2 = 1$ after quantization
  - The three-multiply normalized junction requires non-amplifying rounding on the product of the quantized transformer coefficients $(g_i) \cdot (1/g_i) \leq 1$

Net Signal Power at a Two-Port Scattering Junction

A junction is passive if the power flowing away from it does not exceed the power flowing into it

$$\underbrace{\frac{[f_i^+(t)]^2}{R_i(t)} + \frac{[f_{i-1}^-(t+T)]^2}{R_{i-1}(t)}}_{\text{outgoing power}} \le \underbrace{\frac{[f_{i-1}^+(t-T)]^2}{R_{i-1}(t)} + \frac{[f_i^-(t)]^2}{R_i(t)}}_{\text{incoming power}}$$

Let $\hat{f}$ denote the finite-precision version of f. Then a *sufficient* condition for junction passivity is

$$\left|\hat{f}_i^+(t)\right| \le \left|f_i^+(t)\right|$$
$$\left|\hat{f}_{i-1}^-(t+T)\right| \le \left|f_{i-1}^-(t+T)\right|$$

Digital Waveguide Mesh

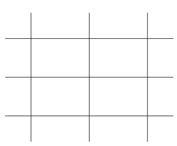
2D mesh

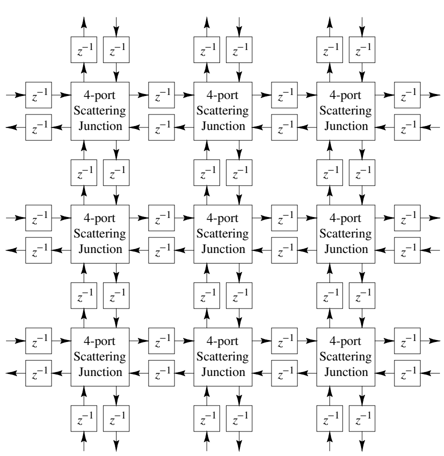
- Rectilinear: 4-port junctions (no multiplies)
- Hexagonal ("chicken wire"): 3-port junctions
- Triangular: 6-port junctions (staggered rect. w. diagonals)

3D mesh

- Rectilinear: 6-port junctions
- Diamond crystal lattice (tetrahedral mesh): 4-port junctions (no multiplies)

2D Rectangular Mesh

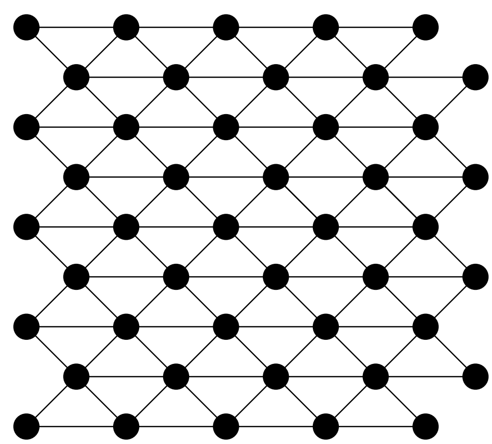




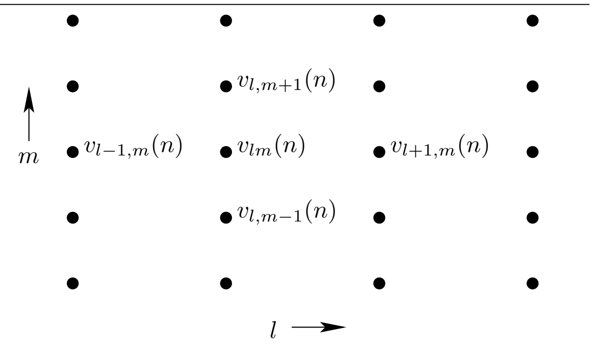
At each four-port scattering junction:

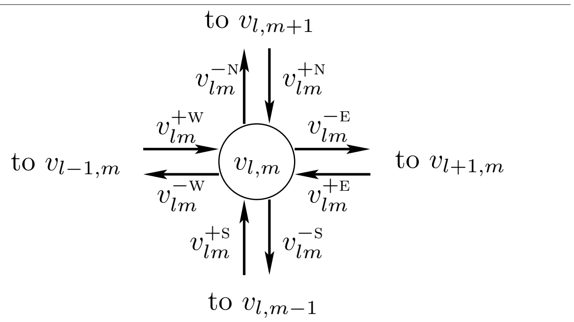
$$V_J = \frac{\mathsf{in}_1 + \mathsf{in}_2 + \mathsf{in}_3 + \mathsf{in}_4}{2}$$
$$\mathsf{out}_k = V_J - \mathsf{in}_k, \qquad k = 1, 2, 3, 4$$

2D Triangular Mesh over Staggered Grid



2D Mesh and the Wave Equation





Junction velocity v_{lm} at time n:

$$v_{lm}(n) = \frac{1}{2}\left[v_{lm}^{+\text{N}}(n) + v_{lm}^{+\text{E}}(n) + v_{lm}^{+\text{S}}(n) + v_{lm}^{+\text{W}}(n)\right]$$

Junction Velocity

Junction velocity v_{lm} at time n:

$$v_{lm}(n) = \frac{1}{2}\left[v_{lm}^{+\text{N}}(n) + v_{lm}^{+\text{E}}(n) + v_{lm}^{+\text{S}}(n) + v_{lm}^{+\text{W}}(n)\right]$$

Equivalent Finite Difference Scheme

We have

$$v_{lm}(n{+}1) = \frac{1}{2}\left[v_{l,m+1}^{-\text{S}}(n) + v_{l+1,m}^{-\text{W}}(n) + v_{l,m-1}^{-\text{N}}(n) + v_{l-1,m}^{-\text{E}}(n)\right]$$

$$v_{lm}(n{-}1) = \frac{1}{2}\left[v_{l,m+1}^{+\text{S}}(n) + v_{l+1,m}^{+\text{W}}(n) + v_{l,m-1}^{+\text{N}}(n) + v_{l-1,m}^{+\text{E}}(n)\right]$$

Adding gives a *finite difference equation* satisfied by the mesh

$$\boxed{v_{lm}(n+1) + v_{lm}(n-1) = \frac{v_{l,m+1} + v_{l+1,m} + v_{l,m-1} + v_{l-1,m}}{2}}$$

- *Physical variables only* (no traveling-wave components)
- Omitted time arguments are all '(n)'

Subtracting $2v_{lm}(n)$ from both sides yields

$$\begin{aligned} &v_{lm}(n+1) - 2v_{lm}(n) + v_{lm}(n-1) \\ &= \frac{1}{2}\{[v_{l,m+1}(n) - 2v_{lm}(n) + v_{l,m-1}(n)] \\ &\quad + [v_{l+1,m}(n) - 2v_{lm}(n) + v_{l-1,m}(n)]\} \end{aligned}$$

or, assuming $X = Y$ ("square hole" case),

$$\begin{aligned} &\frac{v_{lm}(n+1) - 2v_{lm}(n) + v_{lm}(n-1)}{T^2} \\ &= \frac{X^2}{2T^2}\left[\frac{v_{l,m+1}(n) - 2v_{lm}(n) + v_{l,m-1}(n)}{Y^2} \right. \\ &\qquad \left. + \frac{v_{l+1,m}(n) - 2v_{lm}(n) + v_{l-1,m}(n)}{X^2}\right]. \end{aligned}$$

In the limit,

$$\frac{\partial^2 v(x,y,t)}{\partial t^2} = \frac{X^2}{2T^2}\left[\frac{\partial^2 v(x,y,t)}{\partial x^2} + \frac{\partial^2 v(x,y,t)}{\partial y^2}\right]$$

i.e., the ideal 2D wave equation

$$\frac{\partial^2 v}{\partial t^2} = c^2\left[\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2}\right] \triangleq c^2 \nabla^2 v$$

where ∇^2 denotes the Laplacian, and

$$c = \frac{1}{\sqrt{2}}\frac{X}{T}$$

Traveling Waves on the 2D Square-Holed Mesh

We found that the 2D digital waveguide mesh satisfies a finite difference scheme which converges to the ideal 2D wave equation with wave propagation speed

$$c = \frac{1}{\sqrt{2}}\frac{X}{T} = \frac{\sqrt{2}X}{2T}$$

- Every two time steps ($2T$ sec) corresponds to a spatial step of $\sqrt{2}X$ meters — This is the distance from one diagonal to the next on the square-holed mesh
- Diagonal plane-wave propagation is *exact*
- Consider Huygens' principle along a mesh *diagonal*
- The x and y directions are highly *dispersive*:
  - High frequencies travel *slower* than low frequencies
  - Dispersion depends on *frequency* and *direction*
- The *triangular mesh* is much closer to *isotropic*:
  - Dispersion more nearly the same in all directions
- Frequency-dependent dispersion can be addressed using *frequency warping*
- By construction, there is *no attenuation* at any frequency in any direction