Wave Scattering at an Impedance Discontinuity

A change in wave impedance causes lossless signal scattering:

- A traveling wave impinging on an impedance discontinuity will partially reflect from and partially transmit through the discontinuity
- Pressure will be continuous everywhere
- Velocity in = velocity out (junction has no state)
- Signal power (energy) is conserved

Outline

- Plane-Wave Scattering
- The Loaded N-Way Junction
- Lossless Scattering
  - Normalized Scattering Junctions
  - Transformer-Normalized Scattering Junctions
- Junction Passivity
- The Digital Waveguide Mesh

Plane-Wave Scattering

Consider a plane-wave $p_1^+$ propagating from wave impedance $R_1$ into a new wave impedance $R_2$:

$$
\begin{array}{c|c}
R_1 & R_2 \\
\hline
p_1^+ & p_2^+ \\
\end{array}
$$

Physical constraints:

- $p_1^+ + p_1^- = p_2^+$ (pressure continuous across junction)
- $v_1^+ + v_1^- = v_2^+$ (velocity in = velocity out)

Ohm’s Law relations:

- $p_1^+ = R_1 v_1^+$
- $p_1^- = - R_1 v_1^-$

Scattering Solution

Let

$$
p_j \triangleq p_1^+ + p_1^- = p_2^+ \quad \text{(pressure at junction)}
\quad v_j \triangleq v_1^+ + v_1^- = v_2^+ \quad \text{(velocity at junction)}
$$

Then we can write

$$
p_1^+ + p_1^- = p_2^+ = p_j
\Rightarrow R_1 v_1^+ - R_1 v_1^- = R_2 v_2^+ = R_2 v_j
\Rightarrow R_1 v_1^+ - R_1(v_j - v_1^+) = R_2 v_j
\Rightarrow 2 R_1 v_1^+ - R_1 v_j = R_2 v_j
\Rightarrow v_j = \frac{2 R_1}{R_1 + R_2} v_1^+
$$

We have solved for the junction velocity $v_j = v_2^+$. The transmitted pressure is then $p_2^+ = R_2 v_2^+ = R_2 v_j$.

Since $v_j = v_1^+ + v_1^-$, the reflected velocity is simply

$$
v_j = v_1^+ - v_1^- = \left[ \frac{2 R_1}{R_1 + R_2} - 1 \right] v_1^+ = \frac{R_1 - R_2}{R_1 + R_2} v_1^+
$$

Thus, we have solved for the transmitted and reflected velocity waves given the incident wave and the two impedances.
Using the Ohm’s law relations, the pressure waves follow:
\[ p_2^+ = R_2 v_2^+ = R_2 v_j = \frac{2 R_2}{R_1 + R_2} p_1^+ \]
\[ p_1^- = -R_1 v_1^- = \frac{R_2 - R_1}{R_1 + R_2} p_1^+ \]
Define
\[ k = \frac{R_2 - R_1}{R_1 + R_2} = \text{Impedance Step} \]
\[ \frac{1}{R} = \text{Impedance Sum} \]
Then we get the following scattering relations in terms of \( k \) for pressure waves:
\[ p_2^+ = (1 + k) p_1^+ \]
\[ p_1^- = k p_1^+ \]

Signal Flow Graph:
\[ p_1^- \quad 1 + k \quad p_2^+ \]
\[ v_1^- \quad 1 - k \quad v_2^- \]
\[ p_1^- \quad k \quad R_2 \]
\[ R_1 \quad \nabla k \quad R_2 \]
\[ v_1^- \quad R_1 \quad -k \quad R_2 \]

Signal power conserved (left-going power negated):
\[ p_1^- v_1^- = p_2^- v_2^- + (-p_1^- v_1^-) \]

**Longitudinal String Waves**

- **Longitudinal** string waves compress and stretch along the string \( x \) axis
- String may have a nonzero diameter = “stiff string” or “rod”
- In solids, force-density waves are called stress waves

**Superposition of Bidirectional Scattering**

- Stepping from \( R_2 \) to \( R_1 \) negates \( k = \frac{R_2 - R_1}{R_2 + R_1} \)
- Transmission is \( 1 + \) reflection in either direction
- “Kelly-Lochbaum” scattering junction

**Special Cases:**
- \( R_2 = \infty \) ⇒ \( k = 1 \) (e.g., rigid wall reflection)
- \( R_2 = 0 \) ⇒ \( k = -1 \) (e.g., open-ended tube)
- \( R_2 = R_1 \) ⇒ \( k = 0 \) (no reflection)

**Longitudinal Scattering in Strings**

a) Physical picture indicating traveling waves in a continuous medium with wave impedance changing from \( R_0 \) to \( R_1 \) to \( R_2 \) along the horizontal axis, resulting in signal scattering (power-conserving transmission and reflection)

b) Digital simulation diagram

Such a rod might be constructed, for example, using three different materials having three different densities
Computing common velocity at junction:

\[ R_i V_j = F_j = \sum_{i=1}^{N} F_i = \sum_{i=1}^{N} (F_i^+ + F_i^-) \]

\[ = \sum_{i=1}^{N} (R_i V_i^+ - R_i V_i^-) = \sum_{i=1}^{N} (2R_i V_i^+ - R_i V_j) \]

\[ \Rightarrow \]

\[ V_j = 2 \left( R_j + \sum_{i=1}^{N} R_i \right)^{-1} \sum_{i=1}^{N} R_i V_i^+ \]

or

\[ V_j(s) = \sum_{i=1}^{N} A_i(s) V_i^+(s) \]

where

\[ A_i(s) \triangleq \frac{2R_i}{R_j(s) + R_1 + \cdots + R_N} \]

(generalized “alpha parameter”, cf. Wave Digital Filters)

Finally, by continuity, \( V_j = V_i^+ + V_i^- \) \( \Rightarrow \)

\[ V_i^-(s) = V_j(s) - V_i^+(s) \]

Solution Properties

We determined that

\[ V_j(s) = \sum_{i=1}^{N} A_i(s) V_i^+(s) \quad \text{(junction velocity)} \]

\[ V_i^-(s) = V_j(s) - V_i^+(s) \quad \text{(outgoing velocity waves)} \]

where

\[ A_i(s) \triangleq \frac{2R_i}{R_j(s) + R_1 + \cdots + R_N} \]

- Lossless only when \( \text{Re}\{R_j(j\omega)\} \equiv 0 \)
- Memoryless only when \( \text{Im}\{R_j(j\omega)\} \equiv 0 \)
- Dynamic load takes all scattering coefficients into Laplace domain
  - Order of each “scattering-filter” equals load order
  - Normally one filter can serve entire junction
- Junction load equivalent to an \( N + 1 \)st waveguide (with no input) having (generalized) wave impedance given by load impedance \( R_j(s) \)
  (consider “perfect termination” of a transmission line using a resistor equal in value to line impedance)
Recall $N$ velocity waveguides meeting at a series junction:

$$V_J(s) = \sum_{i=1}^{N} \mathcal{A}_i(s) V_i^+(s)$$ (junction velocity)

$$V_i^-(s) = V_J(s) - V_i^+(s)$$ (outgoing velocity waves)

where

$$\mathcal{A}_i(s) = \frac{2R_i}{R_J(s) + R_2 + \cdots + R_N}$$

In the lossless (unloaded) case, $R_J(s) = 0$, and so the alpha parameters are real and positive and add up to 2:

$$\alpha_i = \frac{2R_i}{R_1 + \cdots + R_N}$$

I.e.,

$$0 \leq \alpha_i \leq 2$$

and

$$\sum_{i=1}^{N} \alpha_i = 2$$

Also, we no longer have to be in the Laplace domain ($R_J = 0$)

### Reflection-Free Ports

To suppress series-junction reflections on one of the strings, say string 1, then we set its wave impedance to the series combination (i.e., sum) of the other wave impedances meeting at the series junction:

$$R_1 = R_2 + R_3 + \cdots + R_N$$

- This choice of $R_1$ matches the impedance seen from string 1 when entering junction
- Such matching eliminates an impedance step seen by waves traveling from string 1 into all of the other strings

In this case, the first alpha parameter becomes

$$\alpha_1 = \frac{2R_1}{R_1 + R_2 + \cdots + R_N} = \frac{2R_1}{R_1 + R_1} = 1$$

and the remaining alpha parameters can be expressed as

$$\alpha_i = \frac{2R_i}{2R_1} = \frac{R_i}{R_1}, \quad i = 1, 2, \ldots, N$$

and the sum of the $N - 1$ remaining alpha parameters is therefore 1 since there is no junction load.

### Time-Domain Lossless Series Scattering in terms of Alpha Parameters

$$v_J(t) = \sum_{i=1}^{N} \alpha_i v_i^+(t)$$

$$v_i^-(t) = v_J(t) - v_i^+(t)$$

Alpha parameters conveniently parametrize lossless scattering junctions:

- Explicit coefficients of incoming traveling waves for computing junction velocity
- Losslessness assured when alpha parameters are nonnegative and sum to 2
- When alpha parameters sum to less than 2, there is conceptually a “resistive loss” at the junction

In the lossless, equal-impedance case, $R_i = R, \forall i \Rightarrow$

$$\alpha_i = \frac{2}{N}$$

When $N$ is a power of two, no multiplies are needed (multiply-free reverberators, waveguide meshes, etc., are based on this)

### Normalizing by the Wave Impedance at a Reflection-Free Port

Often we only care about the signal scattering and not the specific impedance values. In that case it is convenient to divide all impedances at the junction by $R_1$, defining $\hat{R}_i = R_i / R_1$, so that

$$\hat{R}_1 = 1$$

$$\hat{R}_i \in [0, 1], \quad i = 2, \ldots, N$$

$$\sum_{i=2}^{N} \hat{R}_i = 1$$

$$\alpha_1 = 1$$

$$\alpha_i = \hat{R}_i, \quad i = 2, \ldots, N$$

$$\sum_{i=2}^{N} \alpha_i = 1$$

String 1 is said to intersect with the other strings at a reflection-free port. For $N > 2$, the other strings are attached to the junction at reflecting ports.
The ideal transformer steps wave impedance from

Recall

Better term = "Normalized-wave scattering junction"

There are engineering approximations, however:

No scattering reflections generated

Physical in principle, but not realizable

There are engineering approximations, however:

- Conical acoustic tube
- Horn loudspeakers
- Quarter-wave microwave transformers

The Digital Waveguide Transformer

The normalized scattering junction

- Recall normalized waves:

- Converting scattering to normalized waves \( \tilde{f}^\pm \) gives

- Better term = "Normalized-wave scattering junction"

- Normalized junction is equivalent to a 2D rotation:

\[
\begin{align*}
\tilde{f}_i^+(t) &= \cos(\theta_i) \tilde{f}_{i-1}^+(t - T) - \sin(\theta_i) \tilde{f}_i^-(t) \\
\tilde{f}_{i-1}^+(t + T) &= k_i(t) \tilde{f}_{i-1}^+(t - T) + \sqrt{1 - k_i^2(t)} \tilde{f}_i^-(t)
\end{align*}
\]

Transformer Scattering Formulas

General Two-Port:

Power conservation:

\[ (p_1^+ - p_1^-) \left( \frac{p_1^+ - \tilde{p}_1^-}{R_1} \right) = -(p_2^+ + p_2^-) \left( \frac{p_2^+ - \tilde{p}_2^-}{R_2} \right) \]

Non-reflecting:

\[
\begin{align*}
\tilde{p}_1^- &= g_1 p_2^+ \\
\tilde{p}_2^- &= g_2 p_1^+
\end{align*}
\]

for some constants \( g_1, g_2 \)

Solution:

\[
\begin{align*}
p_1^+ &= \sqrt{\frac{R_1}{R_2}} \tilde{p}_2^+ = \frac{1}{g} p_1^+ \\
\tilde{p}_2^- &= \sqrt{\frac{R_1}{R_2}} p_1^+ = g p_1^+
\end{align*}
\]
where \( g \triangleq \) transformer “turns ratio”

The ideal 2-port transformer

![Wave-flow diagram](image)

Principles of Passive Construction

We can state the following general principles for passive signal processing:

- Confine all nonlinear operations to physically meaningful wave variables
- Signal power = square of physical variable times admittance or impedance
- Passivity assured if all effective gains less than 1
- Passive rounding:
  - Apply to extended-precision intermediate result
  - Magnitude truncation ("rounding toward zero")
  - Error power feedback
- Limit cycles impossible in passive systems
- Overflow oscillations impossible in passive systems
- Energy in ideal implementation = Lyapunov function bounding energy in the finite-precision implementation

Structural Losslessness - Two-Port Case

- One-multiply scattering junctions are structurally lossless: Only one parameter for which all in-range quantizations correspond to lossless scattering
  - Reflection coefficient \( k_i \in [-1, 1] \)
  - Alpha parameter \( \alpha_i \in [0, 2] \)
- Not all normalized scattering junctions are structurally lossless
  - The four-multiply normalized junction has two parameters, \( s_i \triangleq k_i \) and \( c_i \triangleq \sqrt{1 - k_i^2} \), which may not satisfy \( s_i^2 + c_i^2 = 1 \) after quantization
  - The three-multiply normalized junction requires non-amplifying rounding on the product of the quantized transformer coefficients \((g_i) \cdot (1/g_i) \leq 1\)
Net Signal Power at a Two-Port Scattering Junction

A junction is passive if the power flowing away from it does not exceed the power flowing into it:

\[
\frac{|f^+_i(t)|^2}{R_i(t)} + \frac{|f^-_{i-1}(t+T)|^2}{R_{i-1}(t)} \leq \frac{|f^+_{i-1}(t-T)|^2}{R_{i-1}(t)} + \frac{|f^-_i(t)|^2}{R_i(t)}
\]

Let \( \hat{f} \) denote the finite-precision version of \( f \). Then a sufficient condition for junction passivity is

\[
\left| \hat{f}^+_i(t) \right| \leq |f^+_i(t)|, \quad \left| \hat{f}^-_{i-1}(t+T) \right| \leq |f^-_{i-1}(t+T)|
\]

Digital Waveguide Mesh

2D mesh
- Rectilinear: 4-port junctions (no multiplies)
- Hexagonal (“chicken wire”): 3-port junctions
- Triangular: 6-port junctions (staggered rect. w. diagonals)

3D mesh
- Rectilinear: 6-port junctions
- Diamond crystal lattice (tetrahedral mesh): 4-port junctions (no multiplies)

2D Rectangular Mesh

At each four-port scattering junction:

\[
V_J = \frac{in_1 + in_2 + in_3 + in_4}{2}
\]

\[
\text{out}_k = V_J - i_n_k, \quad k = 1, 2, 3, 4
\]
2D Mesh and the Wave Equation

\[ v_{l,m} = v_{l-1,m}(n) = \frac{1}{2} \left[ v_{l,m+1}(n) + v_{l,m-1}(n) \right] \]

Junction Velocity

Junction velocity \( v_{lm} \) at time \( n \):

\[ v_{lm}(n) = \frac{1}{2} \left[ v_{lm}^{+N}(n) + v_{lm}^{+E}(n) + v_{lm}^{+S}(n) + v_{lm}^{+W}(n) \right] \]

or, assuming \( X = Y \) ("square hole" case),

\[ v_{lm}(n+1) - 2v_{lm}(n) + v_{lm}(n-1) = X^2 \frac{1}{2T^2} \left[ \frac{v_{lm+1}(n) - 2v_{lm}(n) + v_{lm-1}(n)}{Y^2} + \frac{v_{l+1,m}(n) - 2v_{lm}(n) + v_{l-1,m}(n)}{X^2} \right] \]

In the limit,

\[ \frac{\partial^2 v(x, y, t)}{\partial t^2} = \frac{X^2}{2T^2} \left[ \frac{\partial^2 v(x, y, t)}{\partial x^2} + \frac{\partial^2 v(x, y, t)}{\partial y^2} \right] \]

i.e., the ideal 2D wave equation

\[ \frac{\partial^2 v}{\partial t^2} = c^2 \left[ \frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} \right] \]

where \( \nabla^2 \) denotes the Laplacian, and

\[ c = \frac{1}{\sqrt{2T}} \]

Traveling Waves on the 2D Square-Holed Mesh

We found that the 2D digital waveguide mesh satisfies a finite difference scheme which converges to the ideal 2D wave equation with wave propagation speed

\[ c = \frac{X}{\sqrt{2T}} \]

- Every two time steps \( (2T \text{ sec}) \) corresponds to a spatial step of \( \sqrt{2X} \) meters — This is the distance from one diagonal to the next on the square-holed mesh
- Diagonal plane-wave propagation is exact
- Consider Huygens’ principle along a mesh diagonal
- The \( x \) and \( y \) directions are highly dispersive:
  - High frequencies travel slower than low frequencies
  - Dispersion depends on frequency and direction
- The triangular mesh is much closer to isotropic:
  - Dispersion more nearly the same in all directions
- Frequency-dependent dispersion can be addressed using frequency warping
- By construction, there is no attenuation at any frequency in any direction