MUS420 Lecture
Acoustic Scattering at Impedance Changes

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• The Loaded $N$-Way Junction
• Lossless Scattering
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  – Transformer-Normalized Scattering Junctions
• Junction Passivity
• The Digital Waveguide Mesh
Wave Scattering at an Impedance Discontinuity

A change in wave impedance causes lossless signal scattering:

- A traveling wave impinging on an impedance discontinuity will partially reflect from and partially transmit through the discontinuity
- Pressure will be continuous everywhere
- Velocity in = velocity out (junction has no state)
- Signal power (energy) is conserved
Plane-Wave Scattering

Consider a plane-wave $p_1^+$ propagating from wave impedance $R_1$ into a new wave impedance $R_2$:

Physical constraints:

$$p_1^+ + p_1^- = p_2^+ \quad \text{(pressure continuous across junction)}$$

$$v_1^+ + v_1^- = v_2^+ \quad \text{(velocity in = velocity out)}$$

Ohm’s Law relations:

$$p_i^+ = R_i v_i^+$$

$$p_i^- = - R_i v_i^-$$
Scattering Solution

Let
\[ p_j \triangleq p_1^+ + p_1^- = p_2^+ \quad \text{(pressure at junction)} \]
\[ v_j \triangleq v_1^+ + v_1^- = v_2^+ \quad \text{(velocity at junction)} \]

Then we can write
\[ p_1^+ + p_1^- = p_2^+ = p_j \]
\[ \Rightarrow R_1 v_1^+ - R_1 v_1^- = R_2 v_2^+ = R_2 v_j \]
\[ \Rightarrow R_1 v_1^+ - R_1 (v_j - v_1^+) = R_2 v_j \]
\[ \Rightarrow 2 R_1 v_1^+ - R_1 v_j = R_2 v_j \]
\[ \Rightarrow \quad v_j = \frac{2 R_1}{R_1 + R_2} v_1^+ \]

We have solved for the junction velocity \( v_j = v_2^+ \). The transmitted pressure is then \( p_2^+ = R_2 v_2^+ = R_2 v_j \).

Since \( v_j = v_1^+ + v_1^- \), the reflected velocity is simply
\[ v_1^- = v_j - v_1^+ = \left[ \frac{2 R_1}{R_1 + R_2} - 1 \right] v_1^+ = \frac{R_1 - R_2}{R_1 + R_2} v_1^+ \]

Thus, we have solved for the transmitted and reflected velocity waves given the incident wave and the two impedances.
Using the Ohm’s law relations, the pressure waves follow:

\[ p_2^+ = R_2v_2^+ = R_2v_j = \frac{2R_2}{R_1 + R_2}p_1^+ \]

\[ p_1^- = -R_1v_1^- = \frac{R_2 - R_1}{R_1 + R_2}p_1^+ \]

Define

\[ k = \frac{R_2 - R_1}{R_1 + R_2} = \text{Impedance Step} \quad \frac{R_1 + R_2}{R_1 + R_2} = \text{Impedance Sum} \]

Then we get the following scattering relations in terms of \( k \) for pressure waves:

\[ p_2^+ = (1 + k)p_1^+ \]

\[ p_1^- = kp_1^+ \]

Signal Flow Graph:

\[ p_1^+ \quad \downarrow k \quad R_1 \quad \downarrow k \quad R_2 \]

\[ p_1^- \quad 1 + k \quad p_2^+ \]

\[ v_1^+ \quad \downarrow -k \quad R_1 \quad \downarrow -k \quad R_2 \]

\[ v_1^- \quad 1 - k \quad v_2^+ \]

Signal power conserved (left-going power negated):

\[ p_1^+ v_1^+ = p_2^+ v_2^+ + ( -p_1^- v_1^- ) \]
Superposition of Bidirectional Scattering

\[ (1 + k) \]

- Stepping from \( R_2 \) to \( R_1 \) negates \( k = \frac{R_2 - R_1}{R_2 + R_1} \)
- Transmission is \( 1 + \) reflection in either direction
- "Kelly-Lochbaum" scattering junction

Special Cases:

- \( R_2 = \infty \Rightarrow k = 1 \) (e.g., rigid wall reflection)
- \( R_2 = 0 \Rightarrow k = -1 \) (e.g., open-ended tube)
- \( R_2 = R_1 \Rightarrow k = 0 \) (no reflection)
Longitudinal String Waves

- *Longitudinal* string waves compress and stretch along the string *x* axis
- String may have a nonzero diameter = “stiff string” or “rod”
- In solids, *force-density* waves are called *stress* waves

\[
\begin{align*}
  f_0^+ (t-T) & \quad f_1^+ (t) & \quad f_1^- (t) & \quad f_2^+ (t) \\
  R_0 & \quad & \quad & \quad \\
  f_0^- (t+T) & \quad f_1^- (t) & \quad f_1^- (t+T) & \quad f_2^- (t) \\
  & \quad & \quad & \quad
\end{align*}
\]

Longitudinal force waves in an ideal rod
Longitudinal Scattering in Strings

a) A waveguide section between two partial sections

b) Physical picture indicating traveling waves in a continuous medium with wave impedance changing from $R_0$ to $R_1$ to $R_2$ along the horizontal axis, resulting in signal scattering (power-conserving transmission and reflection)

b) Digital simulation diagram

Such a rod might be constructed, for example, using three different materials having three different densities
Longitudinal Scattering in Strings, Notes

- As before, velocity $v_i = v_i^+ + v_i^-$ is defined as positive to the right.

- $f_i^+ = \text{right-going traveling-wave component of the stress}$, positive when the rod is locally compressed.

- The stress-wave reflection coefficients are
  
  $$r_i = \frac{R_i - R_{i-1}}{R_i + R_{i-1}}$$

  to the right, with corresponding transmission coefficients $1 + r_i$ to the right.

- To the left, the impedance step negates, so the reflection coefficients negate for waves propagating to the left.

- Wave impedance is now $R_i = \sqrt{E\rho}$
  
  where

  $\rho = \text{mass density}$

  $E = \text{Young’s modulus}$ of the medium

  $= \text{stress over strain}$

- $\text{strain} = \text{relative displacement} \, \delta y / y$

- To minimize the numerical dynamic range, velocity waves may be chosen instead when $R_i > 1$. 
The Loaded $N$-Port Scattering Junction

Four Ideal Strings Intersecting at a Load

Series junction $\Leftrightarrow$ common velocity, forces sum to 0:

$$V_1(s) = V_2(s) = \cdots = V_N(s) \overset{\Delta}{=} V_J(s)$$

$$F_1(s) + F_2(s) + \cdots + F_N(s) = V_J(s)R_J(s) \overset{\Delta}{=} F_J(s)$$
Computing common velocity at junction:

\[ R_J V_J = F_J = \sum_{i=1}^{N} F_i = \sum_{i=1}^{N} (F_i^+ + F_i^-) \]

\[ = \sum_{i=1}^{N} (R_i V_i^+ - R_i V_{i+}^+) \]

\[ = \sum_{i=1}^{N} (2R_i V_i^+ - R_i V_J) \]

\[ \Rightarrow \]

\[ V_J = 2 \left( R_J + \sum_{i=1}^{N} R_i \right)^{-1} \sum_{i=1}^{N} R_i V_i^+ \]

or

\[ V_J(s) = \sum_{i=1}^{N} A_i(s)V_i^+(s) \]

where

\[ A_i(s) \triangleq \frac{2R_i}{R_J(s) + R_1 + \cdots + R_N} \]

(generalized "alpha parameter", cf. Wave Digital Filters)

Finally, by continuity, \( V_J = V_i = V_i^+ + V_i^- \) \Rightarrow

\[ V_i^-(s) = V_J(s) - V_i^+(s) \]
Solution Properties

We determined that

\[ V_J(s) = \sum_{i=1}^{N} A_i(s)V_i^+(s) \quad \text{(junction velocity)} \]

\[ V_i^-(s) = V_J(s) - V_i^+(s) \quad \text{(outgoing velocity waves)} \]

where

\[ A_i(s) \triangleq \frac{2R_i}{R_J(s) + R_1 + \cdots + R_N} \]

- Lossless only when \( \text{Re}\{R_J(j\omega)\} \equiv 0 \)
- Memoryless only when \( \text{Im}\{R_J(j\omega)\} \equiv 0 \)
- Dynamic load takes all scattering coefficients into Laplace domain
  - Order of each “scattering-filter” equals load order
  - Normally one filter can serve entire junction
- Junction load equivalent to an \( N + 1 \)st waveguide (with no input) having (generalized) wave impedance given by load impedance \( R_J(s) \)
  (consider “perfect termination” of a transmission line using a resistor equal in value to line impedance)
Alpha Parameters

Recall $N$ velocity waveguides meeting at a series junction:

$$V_J(s) = \sum_{i=1}^{N} A_i(s)V_i^+(s) \quad \text{(junction velocity)}$$

$$V_i^-(s) = V_J(s) - V_i^+(s) \quad \text{(outgoing velocity waves)}$$

where

$$A_i(s) \triangleq \frac{2R_i}{R_J(s) + R_1 + \cdots + R_N}$$

In the lossless (unloaded) case, $R_J(s) = 0$, and so the alpha parameters are real and positive and add up to 2:

$$\alpha_i = \frac{2R_i}{R_1 + \cdots + R_N}$$

I.e.,

$$0 \leq \alpha_i \leq 2$$

and

$$\sum_{i=1}^{N} \alpha_i = 2$$

Also, we no longer have to be in the Laplace domain ($R_J = 0$)
Time-Domain Lossless Series Scattering in terms of Alpha Parameters

\[ v_J(t) = \sum_{i=1}^{N} \alpha_i v_i^+(t) \]

\[ v_i^-(t) = v_J(t) - v_i^+(t) \]

Alpha parameters conveniently parametrize lossless scattering junctions:

- Explicit coefficients of incoming traveling waves for computing junction velocity
- Losslessness assured when alpha parameters are nonnegative and sum to 2
- When alpha parameters sum to less than 2, there is conceptually a “resistive loss” at the junction

In the lossless, equal-impedance case, \( R_i = R, \forall i \) \( \Rightarrow \)

\[ \alpha_i = \frac{2}{N} \]

When \( N \) is a power of two, no multiplies are needed (multiply-free reverberators, waveguide meshes, etc., are based on this)
Reflection-Free Ports

To suppress series-junction *reflections* on one of the strings, say string 1, then we set its wave impedance to the *series combination* (i.e., sum) of the other wave impedances meeting at the series junction:

\[ R_1 = R_2 + R_3 + \cdots + R_N \]

- This choice of \( R_1 \) *matches* the impedance seen from string 1 when entering junction
- Such matching eliminates an *impedance step* seen by waves traveling from string 1 into all of the other strings

In this case, the first alpha parameter becomes

\[ \alpha_1 = \frac{2R_1}{R_1 + R_2 + \cdots + R_N} = \frac{2R_1}{R_1 + R_1} = 1 \]

and the remaining alpha parameters can be expressed as

\[ \alpha_i = \frac{2R_i}{2R_1} = \frac{R_i}{R_1}, \quad i = 1, 2, \ldots, N \]

and the sum of the \( N - 1 \) remaining alpha parameters is therefore 1 since there is no junction load.
Normalizing by the Wave Impedance at a Reflection-Free Port

Often we only care about the signal scattering and not the specific impedance values. In that case it is convenient to divide all impedances at the junction by $R_1$, defining $\tilde{R}_i \triangleq R_i/R_1$, so that

$$\tilde{R}_1 = 1$$
$$\tilde{R}_i \in [0, 1], \quad i = 2, \ldots, N$$
$$\sum_{i=2}^{N} \tilde{R}_i = 1$$

$$\alpha_1 = 1$$
$$\alpha_i = \tilde{R}_i$$
$$\sum_{i=2}^{N} \alpha_i = 1$$

String 1 is said to intersect with the other strings at a reflection-free port. For $N > 2$, the other strings are attached to the junction at reflecting ports.
Normalized Scattering Junctions

\[ \tilde{f}_{i-1}^+(t-T) \xrightarrow{k_i(t)} \tilde{f}_{i}^+(t) \]
\[ \tilde{f}_{i-1}^-(t+T) \xleftarrow{-k_i(t)} \tilde{f}_{i}^-(t) \]

The normalized scattering junction

- Recall *normalized waves*:
  \[ \tilde{f}_i^+ \triangleq \frac{f_i^+}{\sqrt{R_i}} \quad \tilde{v}_i^+ \triangleq v_i^+ \cdot \sqrt{R_i} \]
- Converting scattering to *normalized waves* \( \tilde{f}^\pm \) gives
  \[ \tilde{f}_i^+(t) = \sqrt{1 - k_i^2(t)} \tilde{f}_{i-1}^+(t-T) - k_i(t) \tilde{f}_{i}^-(t) \]
  \[ \tilde{f}_{i-1}^-(t+T) = k_i(t) \tilde{f}_{i-1}^+(t-T) + \sqrt{1 - k_i^2(t)} \tilde{f}_{i}^- (t) \]
- Better term = “Normalized-wave scattering junction”
- Normalized junction is equivalent to a 2D rotation:
  \[ \tilde{f}_i^+(t) = \cos(\theta_i) \tilde{f}_{i-1}^+(t-T) - \sin(\theta_i) \tilde{f}_{i}^- (t) \]
  \[ \tilde{f}_{i-1}^-(t+T) = \sin(\theta_i) \tilde{f}_{i-1}^+(t-T) + \cos(\theta_i) \tilde{f}_{i}^- (t) \]
The normalized scattering junction

- **Four multiplies and two additions** required
- **Using transformer normalization**, we can obtain **three-multiply, three-add** variations:

\[
g_i(t) = \sqrt{\frac{1 - k_i(t)}{1 + k_i(t)}}
\]
The Digital Waveguide Transformer

The ideal transformer

- scales up pressure and scales down velocity by the same factor
- steps wave impedance from $R_1$ to $R_2$ (and vice versa) without reflections
- conserves power
- No scattering reflections generated
- Physical in principle, but not realizable

There are engineering approximations, however:

- Conical acoustic tube
- Horn loudspeakers
- Quarter-wave microwave transformers
Transformer Scattering Formulas

General Two-Port:

\[ p_1 u_1 = -p_2 u_2 \]

\[ \Leftrightarrow \left( p_1^+ + p_1^- \right) \left( \frac{p_1^+ - p_1^-}{R_1} \right) = -\left( p_2^+ + p_2^- \right) \left( \frac{p_2^+ - p_2^-}{R_2} \right) \]

Non-reflecting:

\[ p_1^- = g_1 p_2^+ \]
\[ p_2^- = g_2 p_1^+ \]

for some constants \( g_1, g_2 \)

Solution:

\[ p_1^- = \sqrt{\frac{R_1}{R_2}} p_2^+ \Delta \frac{1}{g} p_2^+ \]
\[ p_2^- = \sqrt{\frac{R_2}{R_1}} p_1^+ \Delta g p_1^+ \]
where \( g \triangleq \text{transformer “turns ratio”} \)

**Wave-flow diagram:**

![Wave-flow diagram](image)

The ideal 2-port transformer
Three-Multiply Transformer-Normalized Scattering Junction

Using transformers, all waveguides are normalized to the same impedance, $R_i \equiv 1$

$g_i(t) = \sqrt{\frac{1-k_i(t)}{1+k_i(t)}}$

$1/g_i(t)$

$g_i$ and/or $1/g_i$ may have a large dynamic range

While transformer-normalization trades a multiply for an add, up to 50% more bits needed in junction adders (see text)
Principles of Passive Construction

We can state the following general principles for passive signal processing:

• Confine all nonlinear operations to physically meaningful wave variables
• Signal power = square of physical variable times admittance or impedance
• Passivity assured if all effective gains less than 1
• Passive rounding:
  – Apply to extended-precision intermediate result
  – Magnitude truncation ("rounding toward zero")
  – Error power feedback
• Limit cycles impossible in passive systems
• Overflow oscillations impossible in passive systems
• Energy in ideal implementation = Lyapunov function bounding energy in the finite-precision implementation
Structural Losslessness - Two-Port Case

- **One-multiply** scattering junctions are *structurally lossless*: Only one parameter for which all in-range quantizations correspond to lossless scattering
  - *Reflection coefficient* $k_i \in [-1, 1]$
  - *Alpha parameter* $\alpha_i \in [0, 2]$

- Not all normalized scattering junctions are structurally lossless
  - The four-multiply normalized junction has *two* parameters, $s_i \triangleq k_i$ and $c_i \triangleq \sqrt{1 - k_i^2}$, which may not satisfy $s_i^2 + c_i^2 = 1$ after quantization
  - The three-multiply normalized junction requires non-amplifying rounding on the product of the quantized transformer coefficients $(g_i) \cdot \left(\frac{1}{g_i}\right) \leq 1$
Net Signal Power at a Two-Port Scattering Junction

A junction is passive if the power flowing away from it does not exceed the power flowing into it

\[
\frac{[f_i^+(t)]^2}{R_i(t)} + \frac{[f_{i-1}^-(t + T)]^2}{R_{i-1}(t)} \leq \frac{[f_{i-1}^+(t - T)]^2}{R_{i-1}(t)} + \frac{[f_i^-(t)]^2}{R_i(t)}
\]

Let \( \hat{f} \) denote the finite-precision version of \( f \). Then a sufficient condition for junction passivity is

\[
\left| \hat{f}_i^+(t) \right| \leq \left| f_i^+(t) \right| \\
\left| \hat{f}_{i-1}^-(t + T) \right| \leq \left| f_{i-1}^-(t + T) \right|
\]
Digital Waveguide Mesh

2D mesh

- Rectilinear: 4-port junctions (no multiplies)
- Hexagonal (“chicken wire”): 3-port junctions
- Triangular: 6-port junctions (staggered rect. w. diagonals)

3D mesh

- Rectilinear: 6-port junctions
- Diamond crystal lattice (tetrahedral mesh): 4-port junctions (no multiplies)
At each four-port scattering junction:

\[ V_J = \frac{\text{in}_1 + \text{in}_2 + \text{in}_3 + \text{in}_4}{2} \]

\[ \text{out}_k = V_J - \text{in}_k, \quad k = 1, 2, 3, 4 \]
2D Triangular Mesh over Staggered Grid
Junction velocity $v_{lm}$ at time $n$:

$$v_{lm}(n) = \frac{1}{2} \left[ v_{lm}^+ (n) + v_{lm}^- (n) + v_{lm}^E (n) + v_{lm}^S (n) \right]$$
Junction Velocity

Junction velocity $v_{lm}$ at time $n$:

$$v_{lm}(n) = \frac{1}{2} \left[ v_{lm}^{+N}(n) + v_{lm}^{+E}(n) + v_{lm}^{+S}(n) + v_{lm}^{+W}(n) \right]$$

**Equivalent Finite Difference Scheme**

We have

$$v_{lm}(n+1) = \frac{1}{2} \left[ v_{l,m+1}^{-S}(n) + v_{l+1,m}^{-W}(n) + v_{l,m-1}^{-N}(n) + v_{l-1,m}^{-E}(n) \right]$$

$$v_{lm}(n-1) = \frac{1}{2} \left[ v_{l,m+1}^{+S}(n) + v_{l+1,m}^{+W}(n) + v_{l,m-1}^{+N}(n) + v_{l-1,m}^{+E}(n) \right]$$

Adding gives a *finite difference equation* satisfied by the mesh

$$v_{lm}(n + 1) + v_{lm}(n - 1) = \frac{v_{l,m+1} + v_{l+1,m} + v_{l,m-1} + v_{l-1,m}}{2}$$

- **Physical variables only** (no traveling-wave components)
- Omitted time arguments are all $'(n)'$

Subtracting $2v_{lm}(n)$ from both sides yields

$$v_{lm}(n + 1) - 2v_{lm}(n) + v_{lm}(n - 1) = \frac{1}{2} \left\{ [v_{l,m+1}(n) - 2v_{lm}(n) + v_{l,m-1}(n)] + [v_{l+1,m}(n) - 2v_{lm}(n) + v_{l-1,m}(n)] \right\}$$
or, assuming $X = Y$ ("square hole" case),

$$v_{lm}(n + 1) - 2v_{lm}(n) + v_{lm}(n - 1)$$

$$= \frac{T^2}{X^2} \left[ v_{l,m+1}(n) - 2v_{lm}(n) + v_{l,m-1}(n) \right.\}

$$+ \left. v_{l+1,m}(n) - 2v_{lm}(n) + v_{l-1,m}(n) \right] \right].$$

In the limit,

$$\frac{\partial^2 v(x, y, t)}{\partial t^2} = \frac{X^2}{2T^2} \left[ \frac{\partial^2 v(x, y, t)}{\partial x^2} + \frac{\partial^2 v(x, y, t)}{\partial y^2} \right]$$

i.e., the ideal 2D wave equation

$$\frac{\partial^2 v}{\partial t^2} = c^2 \left[ \frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} \right] \equiv c^2 \nabla^2 v$$

where $\nabla^2$ denotes the Laplacian, and

$$c = \frac{1}{\sqrt{2T}} \frac{X}{T}$$
Traveling Waves on the 2D Square-Holed Mesh

We found that the 2D digital waveguide mesh satisfies a finite difference scheme which converges to the ideal 2D wave equation with wave propagation speed

\[ c = \frac{1}{\sqrt{2}} \frac{X}{T} = \frac{\sqrt{2}X}{2T} \]

• Every two time steps \((2T \text{ sec})\) corresponds to a spatial step of \(\sqrt{2}X\) meters — This is the distance from one diagonal to the next on the square-holed mesh

• Diagonal plane-wave propagation is exact

• Consider Huygens’ principle along a mesh diagonal

• The \(x\) and \(y\) directions are highly dispersive:
  – High frequencies travel \textit{slower} than low frequencies
  – Dispersion depends on \textit{frequency} and \textit{direction}

• The \textit{triangular mesh} is much closer to \textit{isotropic}:
  – Dispersion more nearly the same in all directions

• Frequency-dependent dispersion can be addressed using \textit{frequency warping}

• By construction, there is \textit{no attenuation} at any frequency in any direction