MUS420/EE367A Lecture
Lumped Elements, One-Ports, and Passive Impedances

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Outline

- Review of Mass-Spring-Dashpot ODEs
- Impedance
- Elementary Impedances
- One-Ports
- Passive Impedances
- Guitar Bridge Modeling
- Matlab for Synthetic Guitar Bridge Admittance

Impedance

We will get a lot of mileage out of *"impedance analysis"*:

- *Laplace-transform analysis* with *zero initial conditions*
- Also called *steady state analysis*
- For mechanical systems,

$$\boxed{\text{Impedance} = \frac{\text{force}}{\text{velocity}}}$$

  defined at some *driving point*

- For LTI systems, impedance may vary with *frequency:*

$$\boxed{R(\omega) = \frac{F(\omega)}{V(\omega)} = \frac{\text{force}(\omega)}{\text{velocity}(\omega)}}$$

Example: Force Driving a Mass

Consider a force $f(t)$ driving a mass m to produce a velocity $v(t) = \dot{x}(t)$ at the driving point. By Newton's 2nd law,

$$f(t) \;=\; m\,\dot{v}(t) \;\longleftrightarrow\; F(s) = ms\,V(s)$$

if $v(0) = 0$, so that

$$\boxed{R_m(s) \;\stackrel{\Delta}{=}\; \frac{F(s)}{V(s)} \;=\; ms}$$

Sinusoidal Driving Force

Let's drive a mass m sinusoidally at radian frequency ω:

$$f(t) = \cos(\omega t) = \frac{1}{2}e^{j\omega t} + \frac{1}{2}e^{-j\omega t}$$

The resulting velocity (using $V(j\omega) = F(j\omega)/R_m(j\omega)$) is

$$\begin{aligned} v(t) \;&=\; \frac{1}{jm\omega}\cdot\frac{1}{2}e^{j\omega t} + \frac{1}{jm(-\omega)}\cdot\frac{1}{2}e^{-j\omega t} \\ &=\; \frac{1}{m\omega}\cdot\frac{1}{2j}e^{j\omega t} - \frac{1}{m\omega}\cdot\frac{1}{2j}e^{-j\omega t} \;=\; \frac{1}{m\omega}\sin(\omega t) \end{aligned}$$

Impedance in Other Physical Systems

- For electrical systems, impedance is voltage divided by current: $R = V/I$
- For transverse traveling waves on a vibrating string, the *wave impedance* is given by

$$R = \frac{F^+(j\omega)}{V^+(j\omega)} = \sqrt{K\epsilon} = \epsilon c$$

  where

  - $F^+ =$ transverse force wave
  - $V^+ =$ transverse velocity wave
  - $\epsilon =$ string density (mass per unit length)
  - $K =$ string tension (stretching force)

- For longitudinal plane-waves in air, the *wave impedance* is given by *pressure* p divided by *particle velocity* u:

$$R = \frac{P^+(j\omega)}{U^+(j\omega)} = \sqrt{\gamma P_0 \rho} = \rho c$$

  where ρ is the density (mass per unit volume) of air, c is the speed of sound propagation, P_0 is ambient air pressure, and $\gamma_c = 1.4$ is the *adiabatic gas constant* for air (ratio of the specific heat of air at constant pressure to that at constant volume)

- For longitudinal plane-wave sections in an *acoustic tube*, the *wave impedance* is given by *pressure* p divided by *volume velocity* v:

$$R = \frac{P(j\omega)}{V(j\omega)} = \frac{\rho c}{A}$$

  where A is the cross-sectional area of the tube section. Note that volume velocity is in units of meters cubed per second.

- Typical physical units used in practice are the Standard International (SI) units:
  - *force* in *Newtons* (kilograms times meters per second squared)
  - *pressure* in *Newtons per meter squared*
  - *velocity* in *meters per second*
  - *mass* in *kilograms*

- Related terms:
  - *admittance* $= \frac{1}{\text{impedance}}$
  - *reactance* = purely imaginary impedance
  - *susceptance* = purely imaginary admittance
  - *immittance* = either impedance or admittance

- For transverse electromagnetic (TEM) waves in a *transmission line*, the *characteristic impedance* (wave impedance) is given by electric potential V in volts divided by electric current I in amperes (coulombs per second):

$$R = \frac{V}{I} = \sqrt{\frac{L}{C}} = Lc$$

  where L and C are the inductance and capacitance, respectively, per unit length along the transmission line

- In a vacuum, the wave impedance for light (also a TEM wave) is

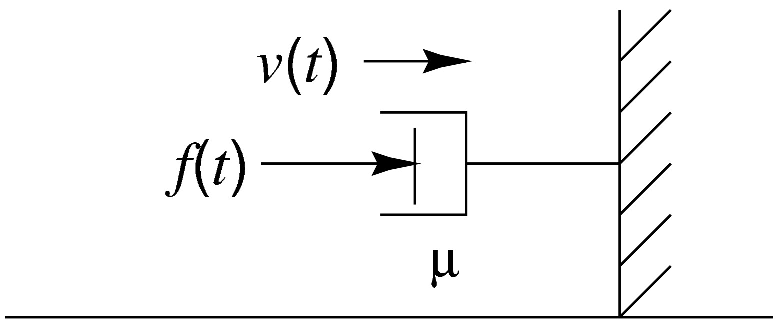
$$R = \sqrt{\frac{\mu_0}{\epsilon_0}} = \mu_0 c$$

  where μ_0 and ϵ_0 are the *permeability* and *permittivity*, respectively, of the vacuum

  It is odd and interesting that waves in the vacuum are subject to the special theory of relativity (speed of light always measured to be the same, irrespective of one's velocity)

Elementary Impedances

Ideal Dashpot



Ideal dashpot characterized by a constant impedance μ

- Dynamic friction law

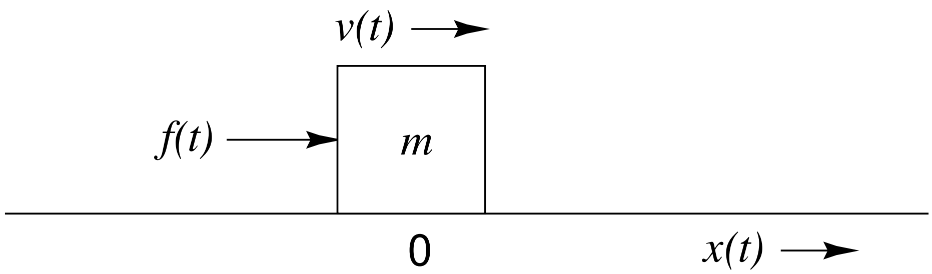
$$f(t) \approx \mu v(t) \qquad \text{"Ohm's Law"}$$
$$(\text{Force} = \text{FrictionCoefficient} \times \text{Velocity})$$

- Impedance

$$\boxed{R_\mu(s) \stackrel{\Delta}{=} \mu \geq 0}$$

- Electrical analogue: *Resistor* $R = \mu$
- More generally, losses due to friction are
  - *frequency dependent*
  - *hysteretic*

Ideal Mass



Ideal mass of m kilograms sliding on a frictionless surface

- Newton's 2nd Law

$$f(t) = ma(t) \stackrel{\Delta}{=} m\dot{v}(t) \stackrel{\Delta}{=} m\ddot{x}(t)$$

$$(\text{Force} = \text{Mass} \times \text{Acceleration})$$

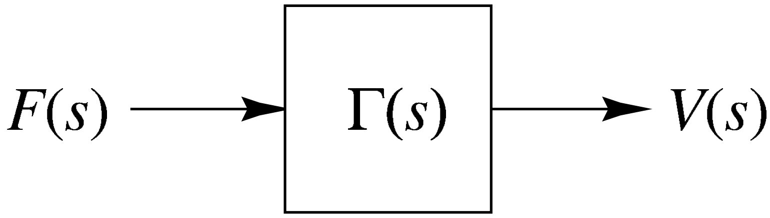
- *Differentiation Theorem* for Laplace Transform

$$F(s) = m[sV(s) - v(0)] = msV(s)$$

when $v(0) = 0$.

- *Impedance*

$$\boxed{R_m(s) \stackrel{\Delta}{=} \frac{F(s)}{V(s)} = ms}$$



"Black Box" Description

- *Admittance* (also called *Mobility*)

$$\Gamma_m(s) \stackrel{\Delta}{=} \frac{1}{R_m(s)} = \frac{1}{ms}$$

- *Impulse Response* (unit-momentum input)

$$\gamma_m(t) \stackrel{\Delta}{=} \mathcal{L}^{-1}\{\Gamma_m(s)\} = \frac{1}{m}u(t)$$

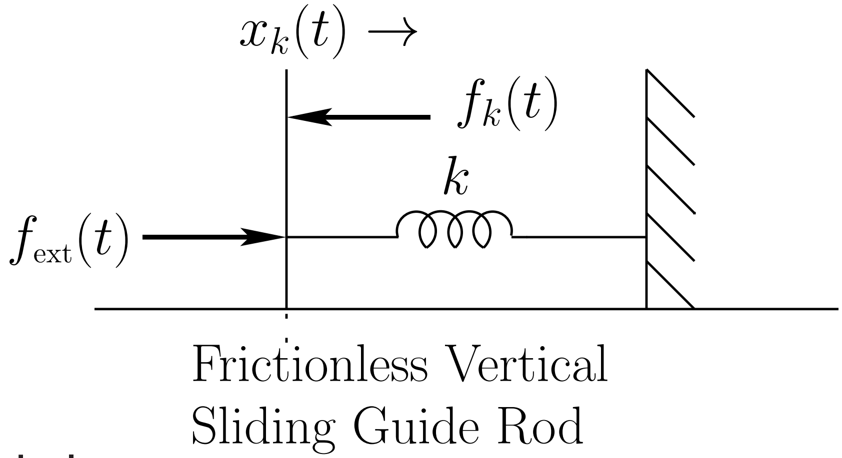
- *Frequency Response*

$$\Gamma_m(j\omega) = \frac{1}{mj\omega}$$

- Mass admittance = *Integrator*
  (for force input, velocity output)

- Electrical analogue: *Inductor* $L = m$.

Ideal Spring



- Hooke's law

$$f(t) = k\,x(t) \stackrel{\Delta}{=} k\int_0^t v(\tau)\,d\tau$$

(Force = Stiffness × Displacement)

- *Impedance*

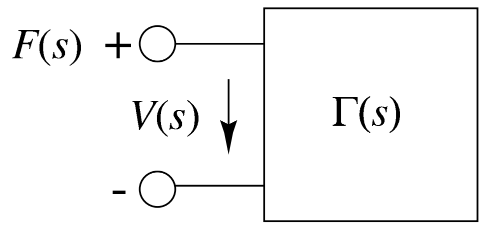
$$\boxed{R_k(s) \stackrel{\Delta}{=} \frac{F(s)}{V(s)} = \frac{k}{s}}$$

- *Frequency Response*

$$\Gamma_k(j\omega) = \frac{j\omega}{k}$$

- Spring = *differentiator* (force input, velocity output)
- Velocity $v(t)$ = "compression velocity"
- Electrical analogue: *Capacitor* $C = 1/k$
  (1/stiffness = "compliance")

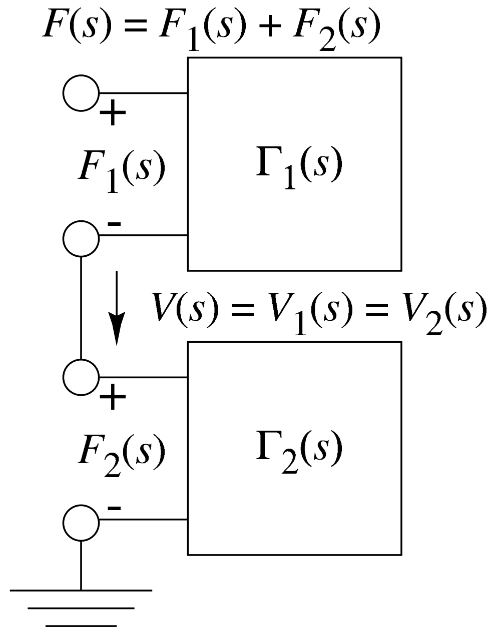
One-Port Driving-Point Impedances



One-Port "Black Box"

- Port *state* $s(t) = \{f(t), v(t)\}$ determined by *force* $f(t)$ and *velocity* $v(t)$ at the port
- Exactly analogous to *network theory* for RLC circuits
- Velocity typically *positive* flowing *into* device
- "Network" characterized by port *impedance* (*admittance*)
- $R(s) \stackrel{\Delta}{=} \frac{F(s)}{V(s)} =$ impedance "seen" at port
- $R(s)$ also called the *driving-point impedance*
- $\Gamma(s) \stackrel{\Delta}{=} \frac{V(s)}{F(s)} =$ *driving-point admittance*

Series Connection of One-Ports



- Series Impedances *Sum*:

$$R(s) = R_1(s) + R_2(s)$$

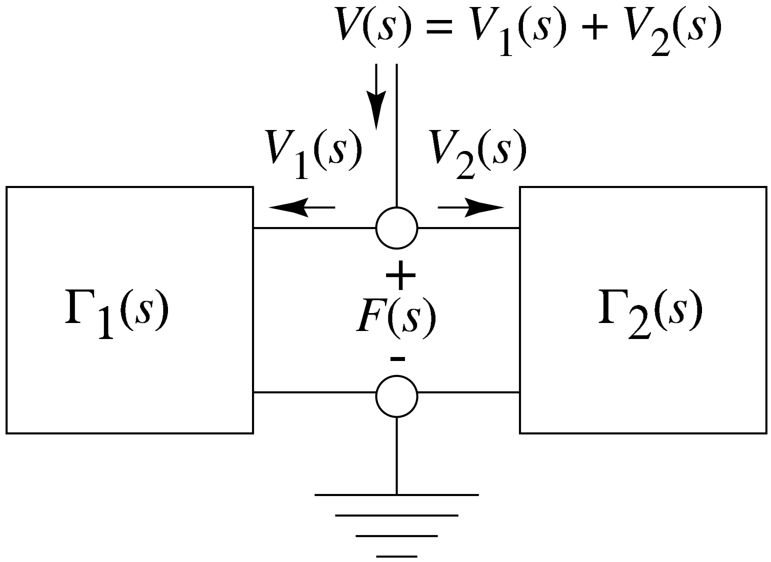
- Admittance:

$$\Gamma(s) = \frac{1}{\frac{1}{\Gamma_1(s)} + \frac{1}{\Gamma_2(s)}} = \frac{\Gamma_1\Gamma_2}{\Gamma_1 + \Gamma_2}$$

Physical Reasoning:

- *Common Velocity* ⇒ *Series* connection
- *Summing Forces* ⇒ *Series* connection

Parallel Combination of One-Ports



$$F(s) = F_1(s) = F_2(s)$$

- Parallel Admittances *Sum*

$$\Gamma(s) \;=\; \Gamma_1(s) + \Gamma_2(s)$$

- Impedance:

$$R(s) \;=\; \frac{1}{\dfrac{1}{R_1(s)} + \dfrac{1}{R_2(s)}} \;=\; \frac{R_1 R_2}{R_1 + R_2}$$

or, for EEs,

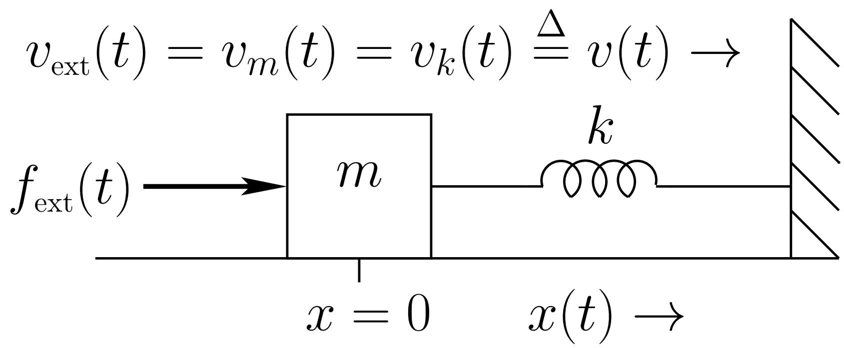
$$R \;=\; R_1 \,\|\, R_2$$

Physical Reasoning:

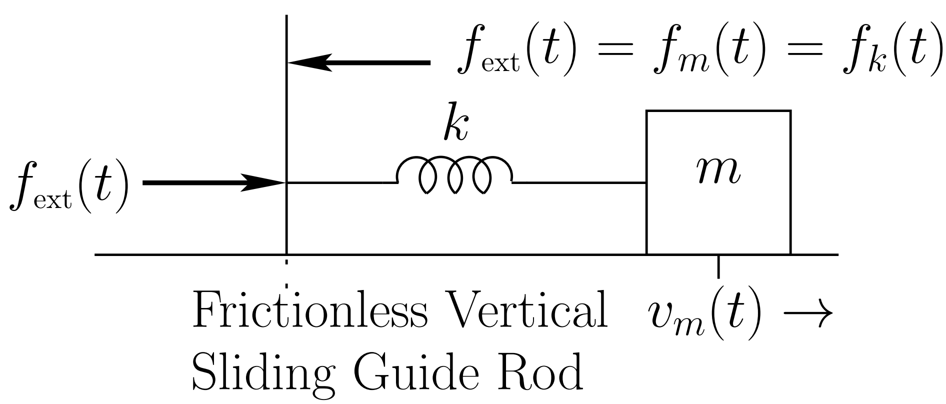
- *Common Force* $\Rightarrow$ *Parallel* connection
- *Summing Velocities* $\Rightarrow$ *Parallel* connection

Examples

- Mass-spring-wall = *series* mass-spring combination:

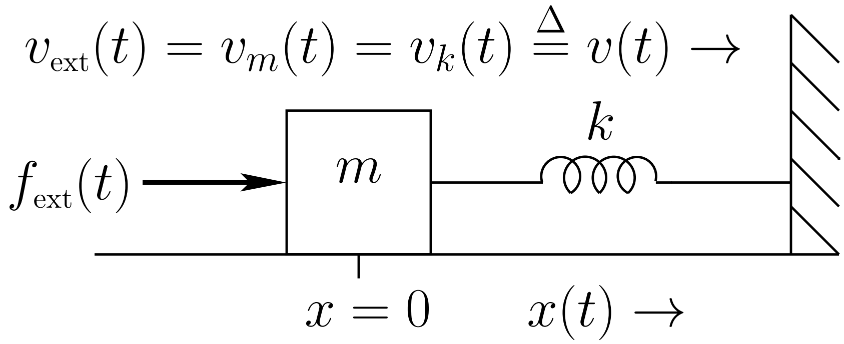


- Spring-mass = *parallel* mass-spring combination

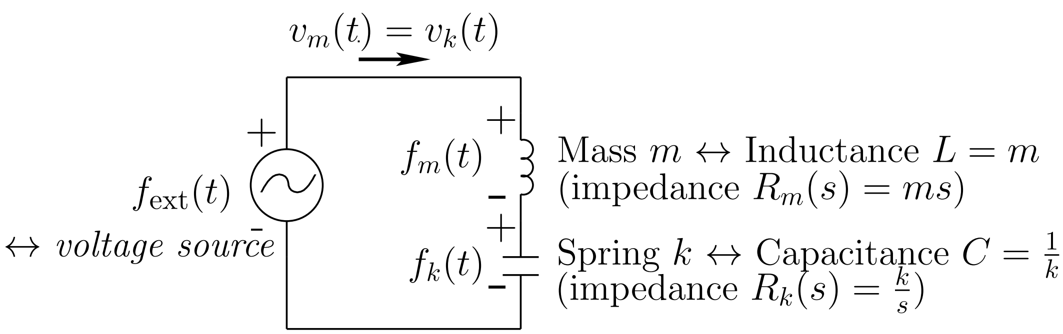


Mass-Spring-Wall (Series)

Physical Diagram:



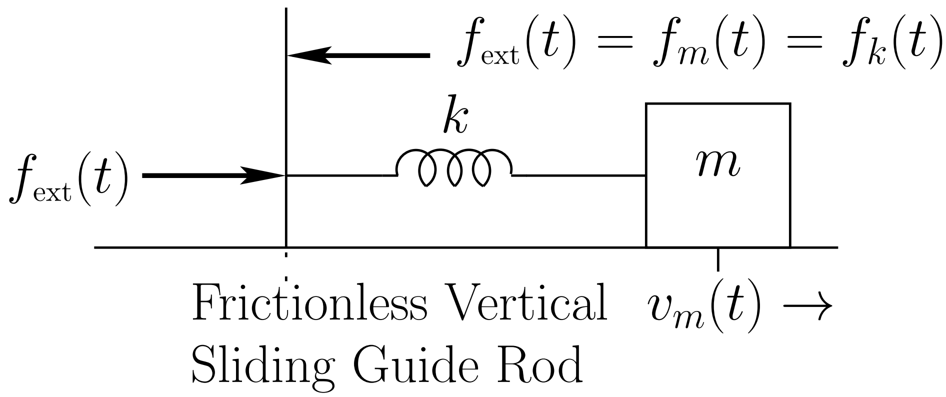
Electrical Equivalent Circuit:



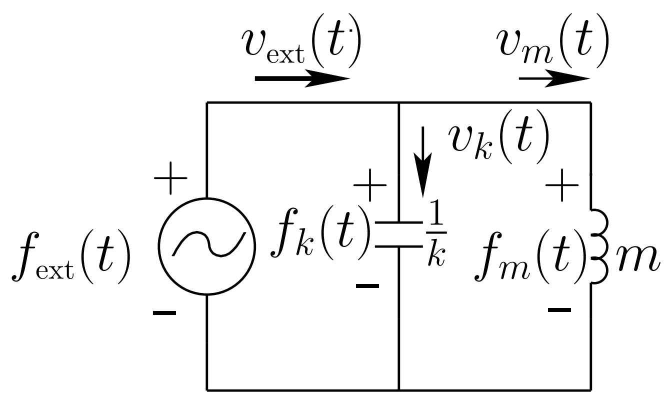
Note sign conventions for *action force* $f_{\text{ext}}(t)$ and *reaction forces* $f_m(t)$, $f_k(t)$

Spring-Mass (Parallel)

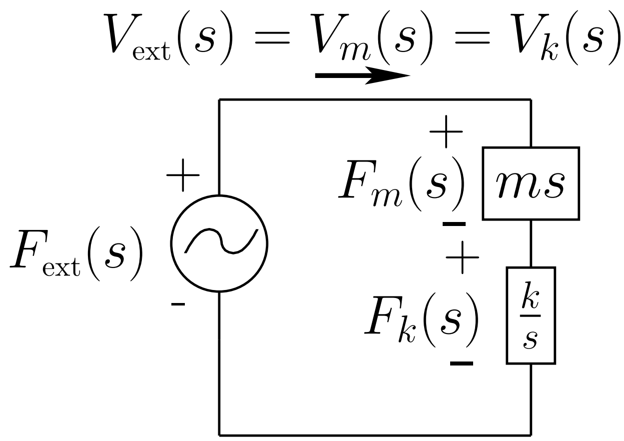
Physical Diagram:



Electrical Equivalent Circuit:



Mass-Spring-Wall (Series) Impedance Diagram



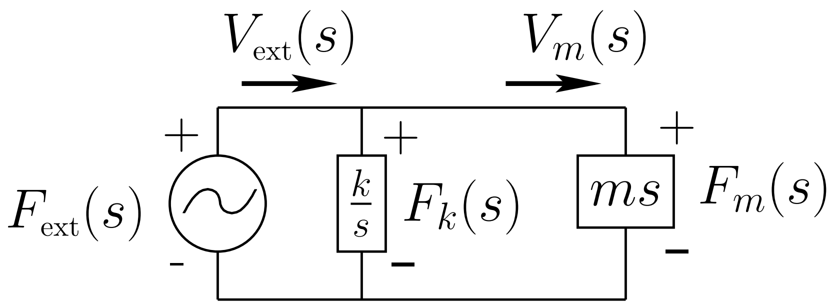
$$F_{\text{ext}}(s) \;=\; F_m(s) \;+\; F_k(s)$$

$$V_{\text{ext}}(s) \;=\; V_m(s) \;=\; V_k(s)$$

$$\Rightarrow \quad F_{\text{ext}}(s) \;=\; V_{\text{ext}}(s) \cdot \left[ms + \frac{k}{s}\right]$$

$$\Rightarrow \quad R(s) \;\stackrel{\Delta}{=}\; \frac{F_{\text{ext}}}{V_{\text{ext}}} = \; ms + \frac{k}{s}$$

Spring-Mass (Parallel) Impedance Diagram

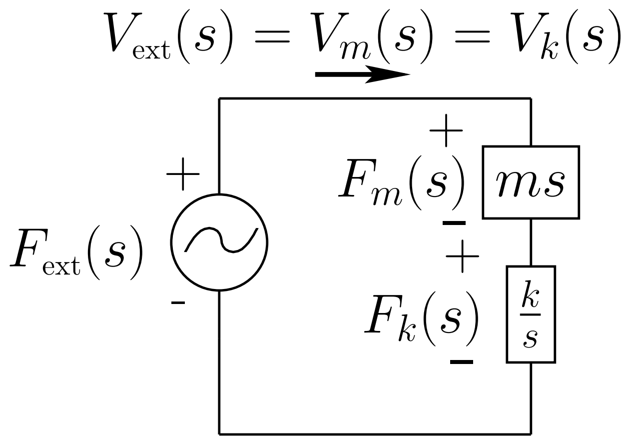


$$F_{\mathsf{ext}}(s) \;=\; F_k(s) \;=\; F_m(s)$$

$$V_{\mathsf{ext}}(s) \;=\; V_k(s) \;+\; V_m(s)$$

$$\Rightarrow \quad R(s) \;=\; \frac{\frac{k}{s} \cdot ms}{\frac{k}{s} + ms}$$

“Voltage Divider” Relations

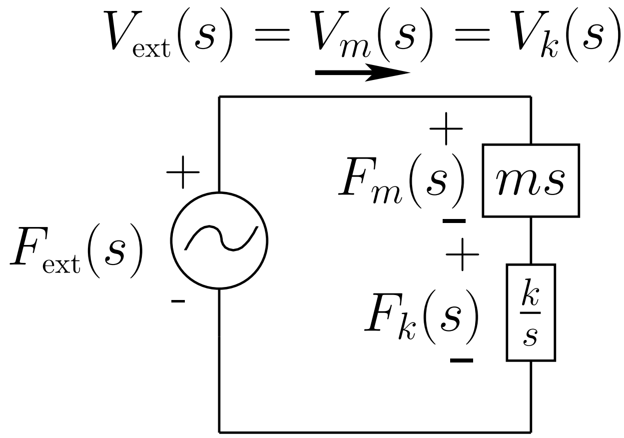


$$F_k(s) \;=\; F_{\mathsf{ext}}(s)\frac{R_k(s)}{R_m(s)+R_k(s)} \;=\; F_{\mathsf{ext}}(s)\frac{\frac{k}{s}}{ms+\frac{k}{s}}$$

$$F_m(s) \;=\; F_{\mathsf{ext}}(s)\frac{R_m(s)}{R_m(s)+R_k(s)} \;=\; F_{\mathsf{ext}}(s)\frac{ms}{ms+\frac{k}{s}}$$

Example Impulse Response Calculation

Let's take the series case:



Suppose the mass is hit with a force impulse:

$$f_{\text{ext}}(t) = \delta(t) \;\leftrightarrow\; F_{\text{ext}}(s) = 1$$

Then the Laplace transform of the force $f_m(t)$ on the mass after time 0 is given, using the "voltage divider" formula, by

$$F_m(s) = \frac{ms}{ms + \frac{k}{s}} = \frac{s^2}{s^2 + \frac{k}{m}}$$

Define $\omega_0^2 \stackrel{\Delta}{=} k/m$.

The mass *velocity* Laplace transform is then

$$\begin{aligned} V_m(s) &= \frac{F_m(s)}{ms} = \frac{1}{m}\cdot\frac{s}{s^2+\omega_0^2} \\ &= \frac{1}{m}\left[\frac{1/2}{s+j\omega_0}+\frac{1/2}{s-j\omega_0}\right] \\ &\leftrightarrow \frac{1}{m}\cos(\omega_0 t). \end{aligned}$$

Thus, the impulse response of the mass oscillates sinusoidally with

- radian frequency $\omega_0 = \sqrt{k/m}$
- amplitude $1/m$
- velocity at a maximum at time $t = 0$
- momentum $m\,v(0+) = 1$

General One-Ports

General one-port admittance

$$\Gamma(s) = \frac{b_0 s^N + b_1 s^{N-1} + \cdots + b_N}{a_0 s^N + a_1 s^{N-1} + \cdots + a_N} \stackrel{\Delta}{=} \frac{B(s)}{A(s)}$$

$\Gamma(s)$ often a *rational approximation* to a true physical admittance

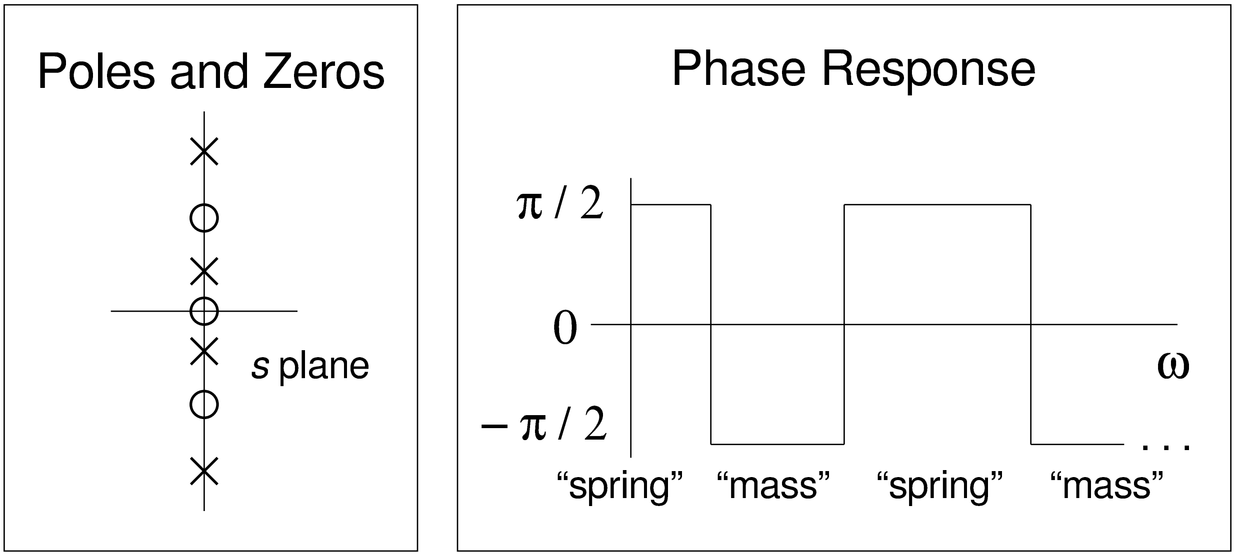
- Filter design problem
- Try to preserve fundamental physical properties

Passive One-Ports

The impedance (or admittance) of every passive one-port is *positive real*

- A complex-valued function of a complex variable $\Gamma(s)$ is said to be *positive real* (PR) if
  1) $\Gamma(s)$ is real whenever s is real
  2) Re$\{\Gamma(s)\} \geq 0$ whenever Re$\{s\} \geq 0$.
- For *strictly* PR, replace "$\geq$" with "$>$"
- The *phase* of a PR function is bounded between plus and minus 90 degrees
$$-\frac{\pi}{2} \leq \angle\Gamma(j\omega) \leq \frac{\pi}{2}$$
- Real part ≥ 0 on $j\omega$ axis
- The term "immittance" refers to either an impedance or admittance
- A *lossless* immittance (no dashpots—a *reactance* or *susceptance*) is *purely imaginary*
- For any lossless immittance, the poles and zeros must *interlace* along the $j\omega$ axis.

Poles and Zeros Interlacing along $j\omega$ Axis



Lossless Admittance $V(s)/F(s)$

Recall:

- Spring k:

$$\Gamma_k(j\omega) \triangleq \frac{V_k(j\omega)}{F_k(j\omega)} = \frac{j\omega}{k} \Rightarrow +\pi/2$$

- Mass m:

$$\Gamma_m(j\omega) \triangleq \frac{V_m(j\omega)}{F_m(j\omega)} = \frac{1}{j\omega m} \Rightarrow -\pi/2$$

Schur Functions

- A *Schur function* $S(z)$ is any stable discrete-time rational transfer function having a modulus not exceeding 1 on the unit circle,

  $$\left|S(e^{j\omega T})\right| \leq 1$$

  The corresponding continuous-time transfer function $S_c(s)$ satisfies the same condition on the $j\omega$ axis, i.e.,

  $$|S_c(j\omega)| \leq 1$$

- The *reflection transfer function* ("reflectance") corresponding to any positive-real immittance is a *Schur function*
  - Reflectance for "force" waves (pressure, voltage) is

    $$S_f(z) = \frac{R(z) - R_0}{R(z) + R_0}$$

    where R_0 is the impedance of the adjacent wave-propagation medium
  - Reflectance for "velocity" waves is

    $$S_v(z) = \frac{\Gamma(z) - \Gamma_0}{\Gamma(z) + \Gamma_0} = -S_f(z)$$

  - Duality: $R \leftrightarrow \Gamma$, '$+$' $\leftrightarrow$ '$-$'
  - Every passive reflectance is Schur
  - Every lossless reflectance is an allpass filter (modulus 1 on the $j\omega$ axis)
  - Every lossless immittance has an allpass reflectance

Guitar Bridge Admittance

Consider the limiting case of a *lossless* bridge (zero damping, mass-spring model only):

- Bridge = pure reactance (no sound gets out)
- String-body coupling is *energy conserving*
- All poles on $j\omega$ axis (infinite-Q resonances)
- All zeros on $j\omega$ axis (infinitely deep notches)
- Resonances and anti-resonances *alternate*
- Admittance looks like a spring at dc ($\omega = 0$) ("stiffness controlled")
- Admittance looks like a mass at $\omega = \infty$ ("mass controlled")
- As frequency increases, driving-point impedance alternates between looking like a mass and looking like a spring, switching at each pole/zero
- Looks like a spring at frequencies above an anti-resonance and below a resonance
- Looks like a mass at frequencies above a resonance and below an anti-resonance

Matlab for Synthetic Guitar Bridge Admittance

Method 1:

$$
\begin{aligned}
R(z) &= R_0 \frac{1+S(z)}{1-S(z)} \\
S(z) &= g\frac{\tilde{A}(z)}{A(z)}, \quad g < 1 \\
\tilde{A}(z) &= z^{-N}A(z^{-1}) = \textsc{Flip}(A)(z) \\
A(z) &= \text{polynomial with roots at desired poles}
\end{aligned}
$$

- $R(z)$ guaranteed positive real
- No independent control of residues and bandwidths

Method 2:

$$
\begin{aligned}
R(z) &= R_0(1-z^{-1})\sum_{i=1}^{N/2} \frac{1}{1+a_1(i)z^{-1}+a_2(i)z^{-2}} \\
a_1(i) &= -2R_i\cos(\theta_i), \quad a_2(i) = R_i^2 \\
R_i &\approx e^{-\pi B_i T}, \quad \theta_i = 2\pi F_i T
\end{aligned}
$$

- Not guaranteed positive real (or is it?)
  Nulls between resonances make us hopeful
- Have independent control of residues and bandwidths

Other Methods:

- General filter design problem, approximating $R(e^{j\omega T})$ as the "desired frequency response"
- How to enforce the positive real constraint?

Matlab for Method 1:

```
fs = 8192;  % Sampling rate in Hz
fc = 300;   % Upper frequency to look at
nfft = 8192;% FFT size (spectral grid density)

nspec = nfft/2+1;
nc = round(nfft*fc/fs);
f = ([0:nc-1]/nfft)*fs;

% Measured guitar body resonances
F = [4.64 96.52 189.33 219.95]; % frequencies
B = [ 10   20     40     50  ]; % bandwidths

nsec = length(F);

R = exp(-pi*B/fs);     % Pole radii
theta = 2*pi*F/fs;     % Pole angles
poles = R .* exp(j*theta);
```

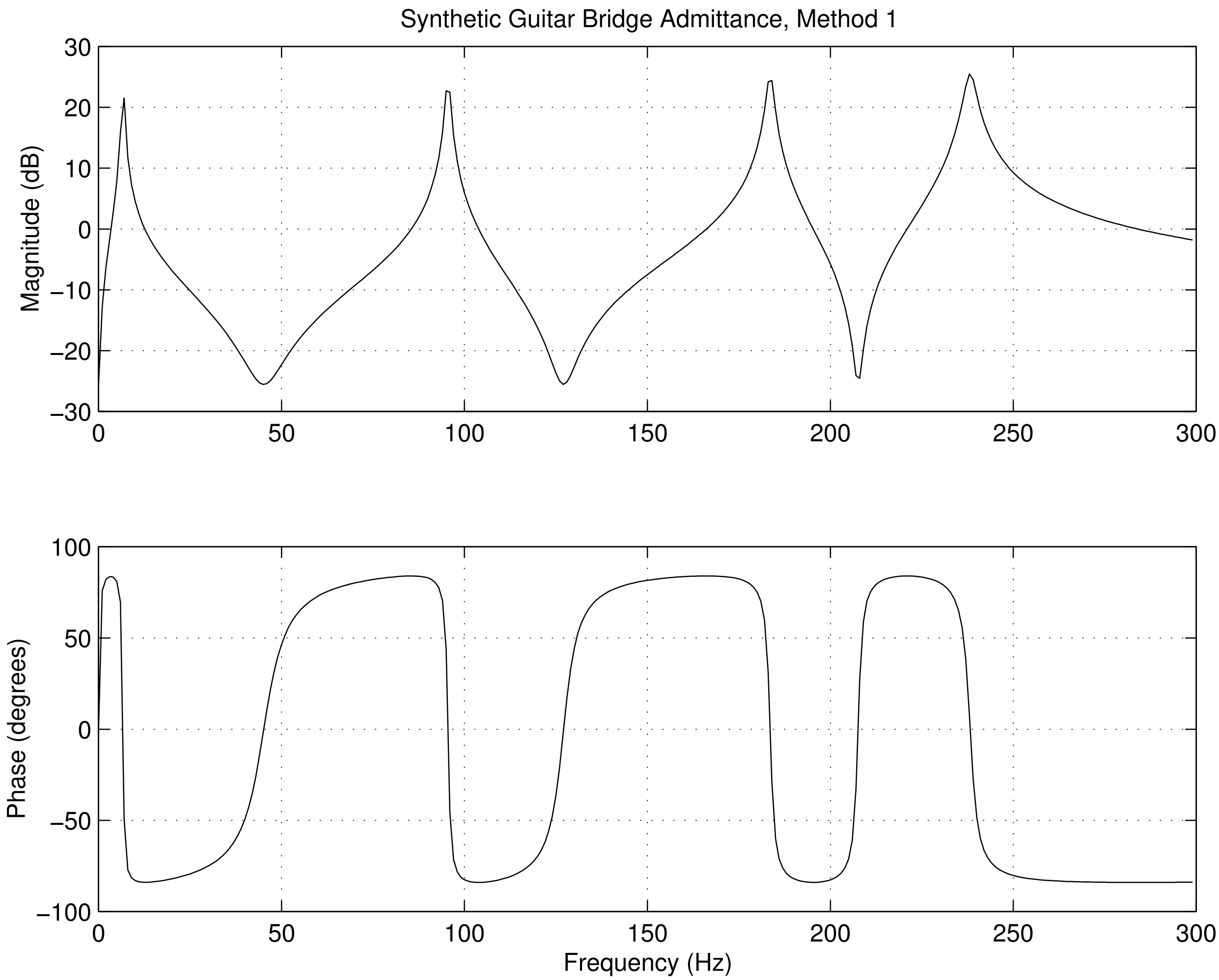
```
A1 = -2*R.*cos(theta);  % 2nd-order section coeff
A2 = R.*R;              % 2nd-order section coeff
denoms = [ones(size(A1)); A1; A2]'
A = [1,zeros(1,2*nsec)];

for i=1:nsec,
    % polynomial multiplication = FIR filtering:
    A = filter(denoms(i,:),1,A);
end;
```

Now A contains the (stable) denominator of the desired bridge admittance. We want now to construct a numerator which gives a positive-real result. We'll do this by first creating a passive reflectance and then computing the corresponding PR admittance.

```
g = 0.9;            % Uniform loss factor
B = g*fliplr(A);    % Flip(A)/A = desired allpass

Badm = A - B;
Aadm = A + B;
Badm = Badm/Aadm(1); % Renormalize
Aadm = Aadm/Aadm(1);

% Plots
```

```
fr = freqz(Badm,Aadm,nfft,'whole');

nc = round(nfft*fc/fs);
spec = fr(1:nc);
f = ([0:nc-1]/nfft)*fs;
dbmag = db(spec);
phase = angle(spec)*180/pi;

subplot(2,1,1);
plot(f,dbmag); grid;
title('Synthetic Guitar Bridge Admittance');
ylabel('Magnitude (dB)');
subplot(2,1,2);

plot(f,phase); grid;
ylabel('Phase (degrees)');
xlabel('Frequency (Hz)');
```

Synthetic Guitar-Bridge Admittance, Method 1



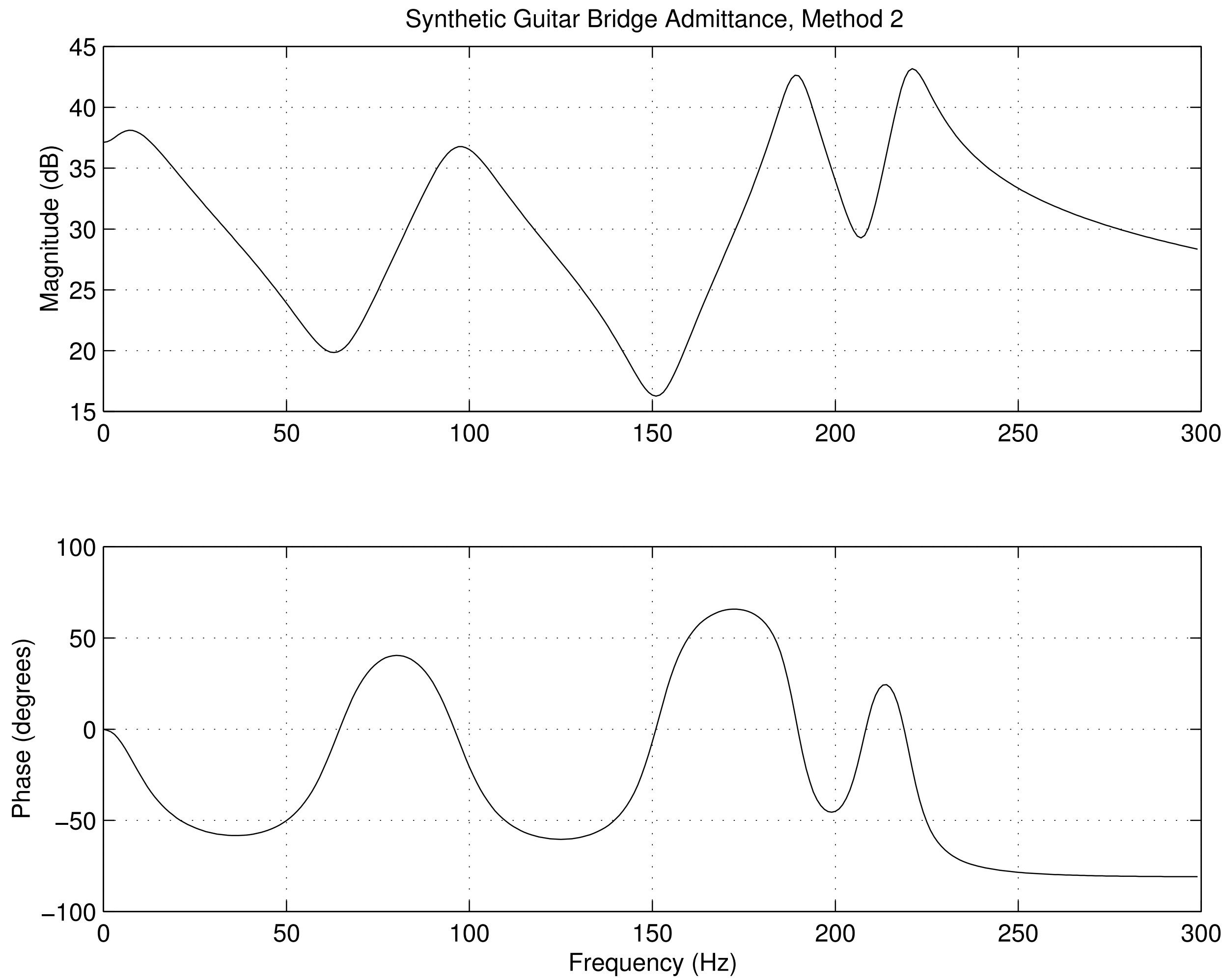
- All resonances reach same height, by construction
- Phase min/max also uniform
- Resonance tuning is arbitrary
- Anti-resonances determined by resonances
- Guaranteed positive real

Matlab for Method 2:

```
... as in Method 1 until ...

% Measured guitar body resonances
F = [4.64 96.52 189.33 219.95]; % Hz
B = [ 20   20     10     10  ]; % Hz
Rd= [  0    5     40     40  ]; % Heights in dB
R = 10.^(Rd/20); nsec = length(F);

R = exp(-pi*B/fs);      % Pole radii
theta = 2*pi*F/fs;      % Pole angles
poles = R .* exp(j*theta); % Complex poles
A1 = -2*R.*cos(theta);
A2 = R.*R;
denoms = [ones(size(A1)); A1; A2]'
A = [1,zeros(1,2*nsec)];
for i=1:nsec,
    % polynomial multiplication = FIR filtering
    A = filter(denoms(i,:),1,A);
end;

% Construct a resonator as a sum of
% arbitrary modes with unit residues,
% adding a near-zero at dc.
```

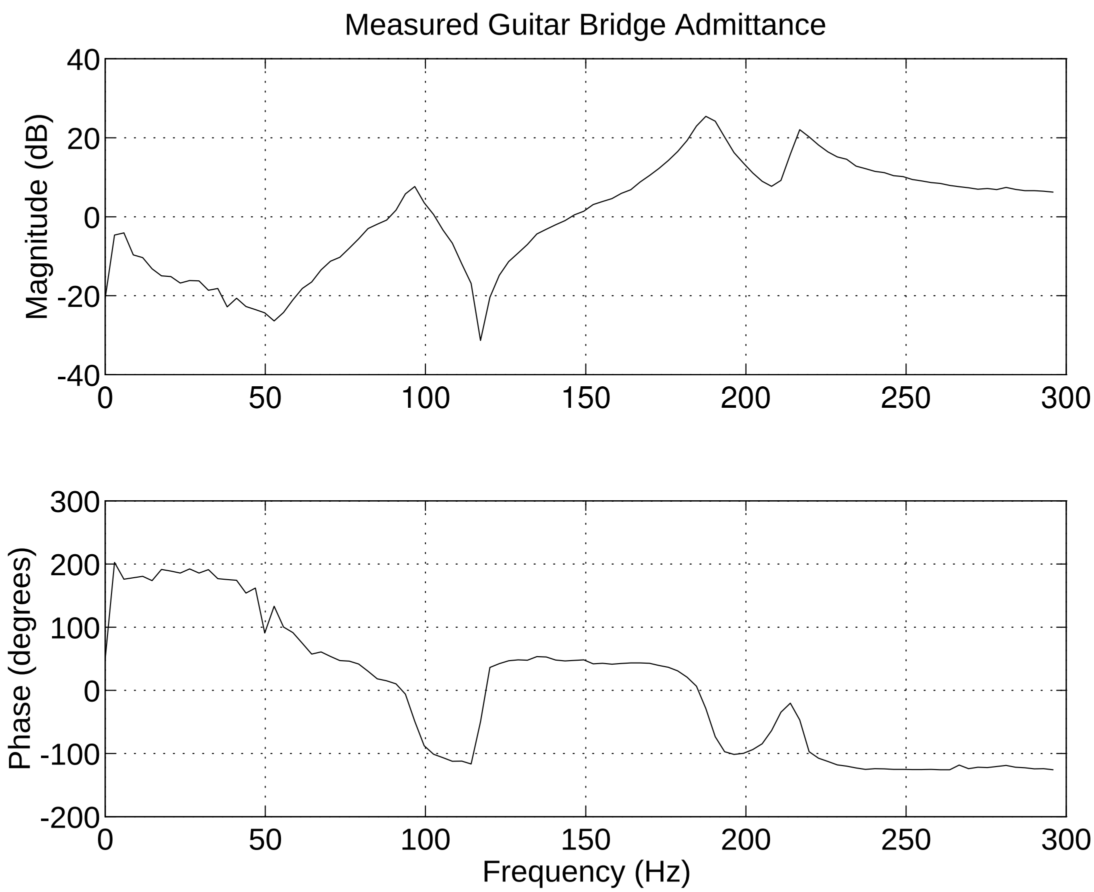
```
B = zeros(1,2*nsec+1);
impulse = [1,zeros(1,2*nsec)];
for i=1:nsec,
    % polynomial multiplication
    B = B + filter(A,denoms(i,:),impulse);
end;

B = filter([1 -0.995],1,B); % near-zero at dc

    % filter() does polynomial division here:
    B = B + filter(A,denoms(i,:),impulse);
end;
% add a near-zero at dc
B = filter([1 -0.995],1,B);

... as in Method 1 for display ...
```

Synthetic Guitar-Bridge Admittance, Method 2



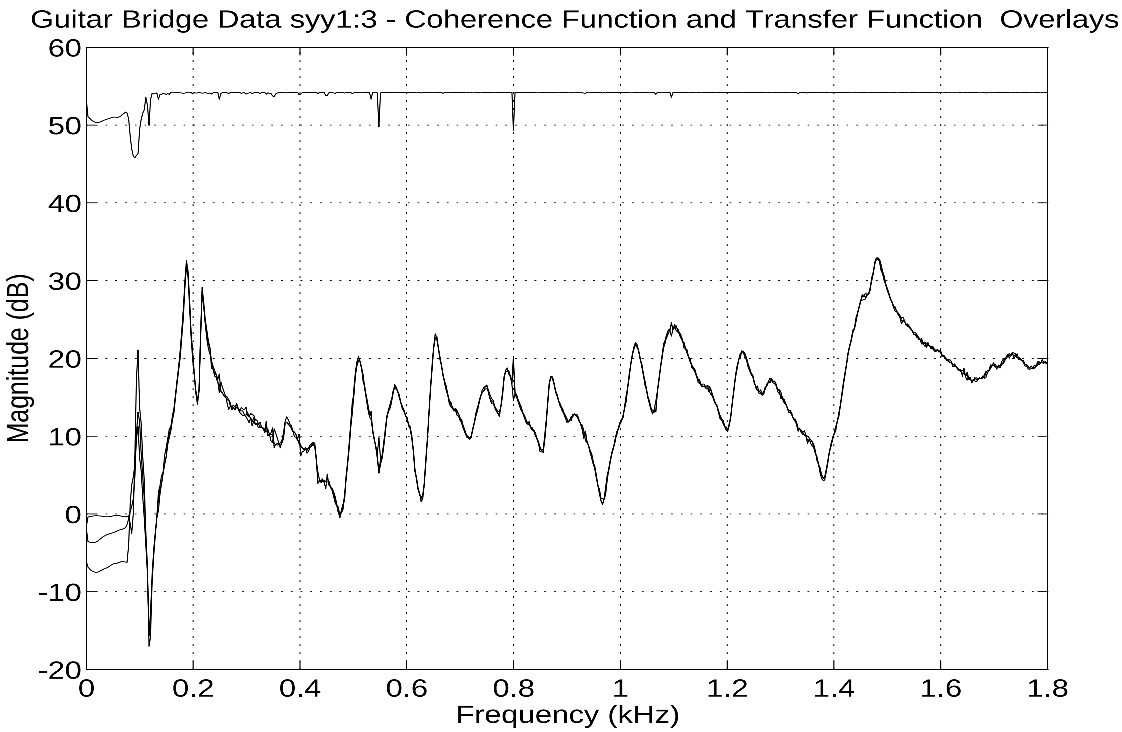
- Resonances have arbitrary height
- Phase min/max non-uniform
- Resonances arbitrarily tunable
- Anti-resonances arbitrarily tunable
- Must check positive real condition

Measured Guitar-Bridge Admittance



- Looks similar to method 2 case
- Anti-resonance near 115 Hz can be achieved in synthetic case (method 2) by adjusting residues
- Phase looks "impossible" below 50 Hz

Coherence and Overlaid Amplitude Responses of Measured Guitar-Bridge Admittance

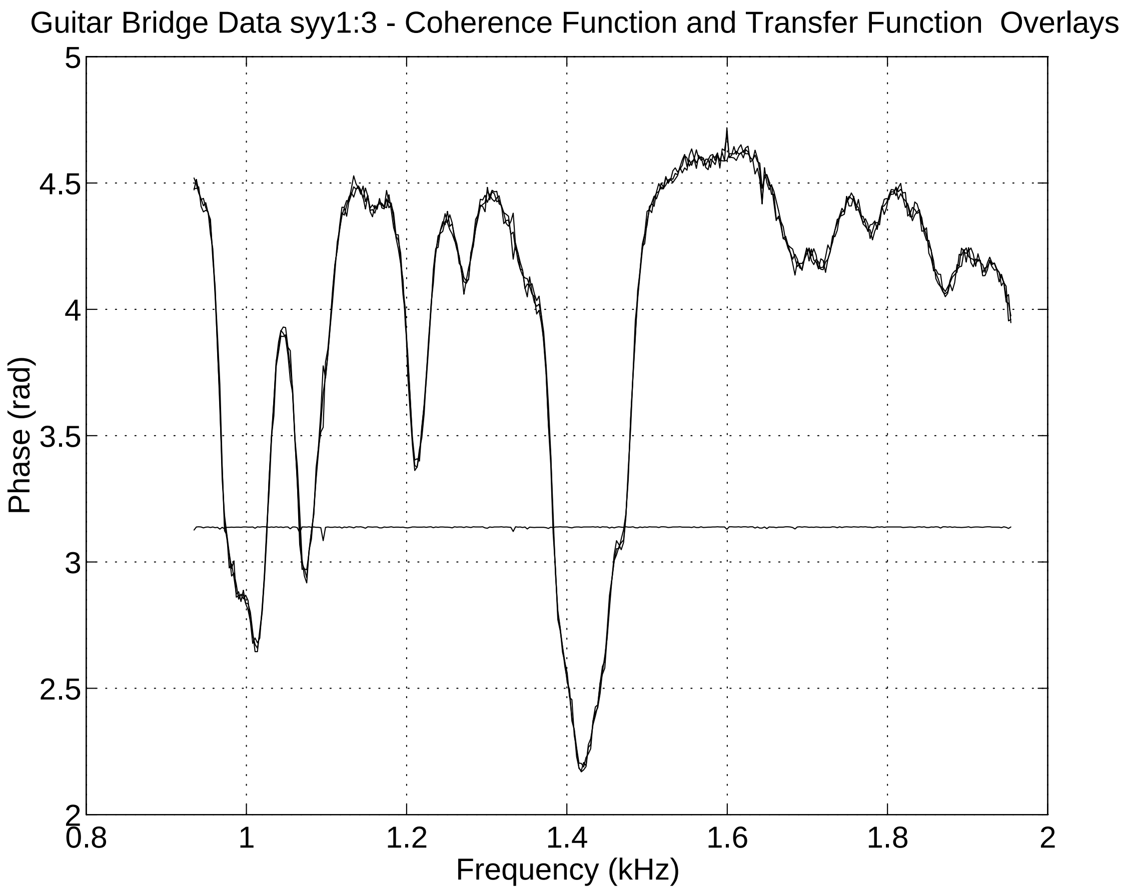


- Note overlay of three separate admittance *magnitude* measurements (easy to see below 50 Hz)
- Coherence function is the top line (max = 1):

$$C_{xy}(\omega) \stackrel{\Delta}{=} \frac{|S_{xy}(\omega)|^2}{S_{xx}(\omega)S_{yy}(\omega)}$$

- Measurements less reliable where coherence is small
- Data not good below 120 Hz or so
- Air modes numerous above 500 Hz or so

Coherence and Overlaid Phase Responses of Measured Guitar-Bridge Admittance



- Three admittance *phase* measurements overlaid
- Only high-coherence frequency interval shown
- Coherence function is the horizontal line, max $= 1$
- Peak-to-peak excursion is less than π radians, as required for passivity