Nonlinearities

Many musical instrument models require nonlinear elements:

- Amplifier distortion (electric guitar)
- Reed model (woodwinds)
- Bowed string contact friction

Since a nonlinear element generally expands signal bandwidth, it can cause aliasing in a discrete-time implementation.

In the above examples, the nonlinearity also appears inside a feedback loop. This means the bandwidth expansion compounds over time, causing more and more aliasing.
Memoryless Nonlinearities

Memoryless or instantaneous nonlinearities are the simplest and most commonly implemented form of nonlinear element:

\[ y(n) = f(x(n)) \]

excluding the special case \( f(x) = \alpha x \) which defines a simple linear gain of \( \alpha \).

Example: Arctan Nonlinearity

An example of an invertible memoryless nonlinearity is the arctangent mapping:

\[ f(x) = \frac{2}{\pi} \arctan(\alpha x), \quad x \in [-1, 1] \]

where normally \( \alpha \gg 1 \).

Cubic Soft Clipper

We used the cubic soft-clipper in simulating amplifier distortion:

\[
 f(x) = \begin{cases} 
 -\frac{2}{3}, & x \leq -1 \\
 x - \frac{x^3}{3}, & -1 \leq x \leq 1 \\
 \frac{2}{3}, & x \geq 1 
\end{cases}
\]

It is non-invertible when driven into “hard clipping”.
Series Expansions

Any “smooth” function $f(x)$ can be expanded as a Taylor series expansion:

$$f(x) = f(0) + \frac{f'(0)}{1} x + \frac{f''(0)}{1 \cdot 2} x^2 + \frac{f'''(0)}{1 \cdot 2 \cdot 3} x^3 + \cdots,$$

Arctangent example:

$$\arctan(x) = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots$$

Note that all even-order terms are zero, because $\arctan(x)$ is an odd function of $x$.

Bandwidth Expansion

The series expansion of a memoryless nonlinearity gives a useful handle on aliasing.

Square Law

The “gentlest” nonlinearity is quadratic:

$$y(n) = x(n) + \alpha x^2(n)$$

The Fourier transform of the output signal is easily found using the dual of the convolution theorem:

$$Y(\omega) = X(\omega) + \alpha (X * X)(\omega)$$

where “$*$” denotes convolution.

In general, the bandwidth of $X * X$ is double that of $X$.

More generally,

$$x^k(n) \longleftrightarrow \underbrace{(X * X * \cdots * X)}_{k \text{ times}}(\omega)$$

so that the spectral bandwidth of $x^k(n)$ is $k$ times that of $x(n)$, in general.
**Arctangent Nonlinearity**

Since the series expansion of the arctangent nonlinearity is

\[
\arctan(x) = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots
\]

bandwidth expansion is *infinite* (in continuous time).

**Cubic Soft-Clipper**

The cubic soft-clipper, like any polynomial nonlinearity, is defined directly by its series expansion:

\[
f(x) = \begin{cases} 
-\frac{2}{3}, & x \leq -1 \\
 x - \frac{x^3}{3}, & -1 \leq x \leq 1 \\
 \frac{2}{3}, & x \geq 1 
\end{cases}
\]

In the absence of hard-clipping (\(|x| \leq 1\)), bandwidth expansion is limited to a factor of *three*.

This is the slowest aliasing rate obtainable for an *odd* nonlinearity.

**Practical Advice**

- Verify that aliasing sounds bad before getting rid of it
- Aliasing (bandwidth expansion) is reduced by smoothing the “corner” in the clipping nonlinearity
- Consider a healthy *oversampling factor* for nonlinear subsystems
- Make sure there is adequate lowpass filtering in a feedback loop containing a nonlinearity

Example: Cubic Nonlinearity in a Feedback Loop:

- 3X oversampling
- Lowpass filter to \([-\frac{\pi}{3}; \frac{\pi}{3}]\) after the nonlinearity
- Optionally downsample by 3 after LPF and upsample by 3 before nonlinearity