Nonlinearities

Many musical instrument models require *nonlinear elements*:

- Amplifier distortion (electric guitar)
- Reed model (woodwinds)
- Bowed string contact friction

Since a nonlinear element generally *expands signal bandwidth*, it can cause *aliasing* in a discrete-time implementation.

In the above examples, the nonlinearity also appears inside a *feedback loop*. This means the bandwidth expansion *compounds* over time, causing more and more aliasing.
Memoryless Nonlinearities

Memoryless or instantaneous nonlinearities are the simplest and most commonly implemented form of nonlinear element:

\[ y(n) = f(x(n)) \]

excluding the special case \( f(x) = \alpha x \) which defines a simple linear gain of \( \alpha \).

Example: Arctan Nonlinearity

An example of an invertible memoryless nonlinearity is the arctangent mapping:

\[ f(x) = \frac{2}{\pi} \arctan(\alpha x), \quad x \in [-1, 1] \]

where normally \( \alpha \gg 1 \).

Cubic Soft Clipper

We used the cubic soft-clipper in simulating amplifier distortion:

\[
 f(x) = \begin{cases} 
 -\frac{2}{3}, & x \leq -1 \\ 
 x - \frac{x^3}{3}, & -1 \leq x \leq 1 \\ 
 \frac{2}{3}, & x \geq 1 
\end{cases}
\]

It is non-invertible when driven into “hard clipping”.

\[
x = -1:0.01:1; \quad \text{plot}([-2/3*ones(1,100), x - x^3/3, (2/3)*ones(1,100)])
\]
Series Expansions

Any “smooth” function \( f(x) \) can be expanded as a Taylor series expansion:

\[
f(x) = f(0) + \frac{f'(0)}{1} x + \frac{f''(0)}{1 \cdot 2} x^2 + \frac{f'''(0)}{1 \cdot 2 \cdot 3} x^3 + \cdots,
\]

Arctangent example:

\[
\arctan(x) = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots
\]

Note that all even-order terms are zero, because \( \arctan(x) \) is an odd function of \( x \).

Bandwidth Expansion

The series expansion of a memoryless nonlinearity gives a useful handle on aliasing.

Square Law

The “gentlest” nonlinearity is quadratic:

\[
y(n) = x(n) + \alpha x^2(n)
\]

The Fourier transform of the output signal is easily found using the dual of the convolution theorem:

\[
Y(\omega) = X(\omega) + \alpha (X \ast X)(\omega)
\]

where “\( \ast \)” denotes convolution.

In general, the bandwidth of \( X \ast X \) is double that of \( X \).

More generally,

\[
x^k(n) \longleftrightarrow (X \ast X \ast \cdots \ast X)(\omega) \quad \text{k times}
\]

so that the spectral bandwidth of \( x^k(n) \) is \( k \) times that of \( x(n) \), in general.
Arctangent Nonlinearity

Since the series expansion of the arctangent nonlinearity is
\[
\arctan(x) = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots
\]
bandwidth expansion is *infinite* (in continuous time).

Cubic Soft-Clipper

The cubic soft-clipper, like any polynomial nonlinearity, is defined directly by its series expansion:
\[
f(x) = \begin{cases} 
-\frac{2}{3}, & x \leq -1 \\
-x^3, & -1 \leq x \leq 1 \\
\frac{2}{3}, & x \geq 1 
\end{cases}
\]
In the absence of hard-clipping (\(|x| \leq 1\)), bandwidth expansion is limited to a factor of *three*. This is the slowest aliasing rate obtainable for an *odd* nonlinearity.

Practical Advice

- Verify that aliasing sounds bad before getting rid of it
- Aliasing (bandwidth expansion) is reduced by smoothing the “corner” in the clipping nonlinearity
- Consider a healthy *oversampling factor* for nonlinear subsystems
- Make sure there is adequate lowpass filtering in a feedback loop containing a nonlinearity

Example: Cubic Nonlinearity in a Feedback Loop:
- 3X oversampling
  (2X suffices for full-band audio, since aliasing into the guard-band above 20 kHz is inaudible)
- Lowpass filter to \([-\frac{\pi}{3}, \frac{\pi}{3}\] after the nonlinearity
- Optionally downsample by 3 after LPF and upsample by 3 before nonlinearity