**State Space Modal Representation**

*Diagonal* state transition matrix = *modal representation:*

\[
\begin{bmatrix}
    x_1(n+1) \\
    x_2(n+1) \\
    \vdots \\
    x_{N-1}(n+1) \\
    x_N(n+1)
\end{bmatrix}
= \begin{bmatrix}
    \lambda_1 & 0 & 0 & \cdots & 0 \\
    0 & \lambda_2 & 0 & \cdots & 0 \\
    \vdots & \vdots & \ddots & \vdots & \vdots \\
    0 & 0 & 0 & \lambda_{N-1} & 0 \\
    0 & 0 & 0 & 0 & \lambda_N
\end{bmatrix}
\begin{bmatrix}
    x_1(n) \\
    x_2(n) \\
    \vdots \\
    x_{N-1}(n) \\
    x_N(n)
\end{bmatrix}
+ \begin{bmatrix}
    b_1 \\
    b_2 \\
    \vdots \\
    b_{N-1} \\
    b_N
\end{bmatrix}u(n)
\]

\[
y(n) = Cx(n) + Du(n)
\]

The *N* complex modes are *decoupled:*

\[
x_1(n+1) = \lambda_1 x_1(n) + b_1 u(n)
\]
\[
x_2(n+1) = \lambda_2 x_2(n) + b_2 u(n)
\]
\[
\vdots
\]
\[
x_N(n+1) = \lambda_N x_N(n) + b_N u(n)
\]

\[
y(n) = c_1 x_1(n) + c_2 x_2(n) + \cdots + c_N x_N(n) + Du(n)
\]

That is, diagonal state-space system consists of *N* parallel one-pole systems (complex, in general).
Diagonalizing a State-Space Model

• To obtain a modal representation, we can diagonalize a state-space model

• The similarity transformation which diagonalizes the system is given by the matrix of eigenvectors of the state transition matrix $A$

• An eigenvector $e_i$ of $A$ satisfies, by definition,

$$Ae_i = \lambda_i e_i$$

where $e_i$ and $\lambda_i$ may be complex

• In other words, a state-space model is diagonalized by a similarity transformation matrix $E$ whose columns are given by the eigenvectors of $A$:

$$E = [e_1 \cdots e_N]$$

• A system can be diagonalized whenever the eigenvectors of $A$ are linearly independent.
  – This always holds for distinct poles
  – May or may not hold for repeated poles
State Space Diagonalization

• Suppose we solve the equation $Ae_i = \lambda_i e_i$ and find $N$ linearly independent eigenvectors of $A$.

• Form the $N \times N$ matrix $E = [e_1 \ldots e_N]$ having these eigenvectors as columns.

• Since the eigenvectors are linearly independent, $E$ is full rank and can be inverted. This means it is one-to-one and qualifies as a linear coordinate transformation matrix.

• As derived above, the transformed state transition matrix is given by

$$\tilde{A} = E^{-1}AE$$

• Since $Ae_i = \lambda_i e_i$, we have

$$AE = E\Lambda$$

where $\Lambda$ is a diagonal matrix having the (complex) eigenvalues of $A$ along its diagonal.

• It follows that

$$\tilde{A} = E^{-1}AE = E^{-1}E\Lambda = \Lambda.$$  

Thus, the new state transition matrix $\Lambda$ is diagonal consisting of the eigenvalues of $A$. 
• The transfer function of the diagonalized system is

\[
\mathbf{H}(z) = \tilde{D} + \tilde{C}(zI - \Lambda)^{-1} \tilde{B}
\]

\[
= \tilde{D} + \frac{\tilde{c}_1 b_1 z^{-1}}{1 - \lambda_1 z^{-1}} + \frac{\tilde{c}_2 b_2 z^{-1}}{1 - \lambda_2 z^{-1}} + \cdots + \frac{\tilde{c}_N b_N z^{-1}}{1 - \lambda_N z^{-1}}
\]

\[
= \tilde{D} + \sum_{i=1}^{N} \frac{\tilde{c}_i \tilde{b}_i z^{-1}}{1 - \lambda_i z^{-1}}
\]

We see again that the diagonalized system (modal representation) consists of \(N\) parallel one-pole systems.

• Dynamic modes \(\lambda_i\) are decoupled

• Closely related to partial-fraction expansion of \(\mathbf{H}(z)\):
  
  – Residue of the \(i\)th pole is \(c_i b_i\)
  
  – Complex-conjugate poles may be combined to form real second-order sections
Finding the Eigenvalues of $A$ in Practice

Small problems may be solved by hand by solving the system of equations

$$AE = E\Lambda$$

The Matlab built-in function `eig()` may be used to find the eigenvalues of $A$ (system poles).
Example of State-Space Diagonalization

For the previous example

\[ A \triangleq \begin{bmatrix} -\frac{1}{2} & -\frac{1}{3} \\ 1 & 0 \end{bmatrix} \quad B \triangleq \begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad C \triangleq \begin{bmatrix} 3/2 \\ 8/3 \end{bmatrix} \quad D \triangleq 1 \]

we obtain the following in Matlab:
>> eig(A) % eigenvalues of state transition matrix

ans =
    -0.2500 + 0.5204i
    -0.2500 - 0.5204i

>> roots(den) % poles of transfer function \Hmtx(z)

ans =
    -0.2500 + 0.5204i
    -0.2500 - 0.5204i

% They are the same, as they must be.

>> abs(roots(den)) % check stability

ans =
    0.5774
    0.5774

The system is stable.

Complex-conjugate poles are typically combined to produce real, second-order \((2 \times 2)\) parallel sections in the modal representation. Thus, our second-order example is already in real modal form. However, to illustrate the computations, let’s obtain the eigenvectors and compute
the *complex* modal representation:

```matlab
>> % Initial state space model from example above:
>> A = [-1/2, -1/3; 1, 0];
>> B = [1; 0];
>> C = [2-1/2, 3-1/3];
>> D = 1;

>> % Diagonalizing similarity transformation:
>> [E,L] = eig(A) % [Evecs, Evals] = eig(A)

E =

-0.4507 - 0.2165i  -0.4507 + 0.2165i
  0 + 0.8660i       0 - 0.8660i

L =

-0.2500 + 0.5204i     0
  0             -0.2500 - 0.5204i

>> A * E - E * L      % should be zero

ans =
```

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Now form the complete diagonalized state-space model (complex):

```matlab
>> Ei = inv(E); % matrix inverse
>> Ab = Ei*A*E % diagonalized state xition mtx

Ab =
-0.2500 + 0.5204i  0.0000 + 0.0000i
-0.0000 -0.2500 - 0.5204i

>> Bb = Ei*B % new input "routing vector"

Bb =
-1.1094
-1.1094

>> Cb = C*E % new output linear combination

Cb =
-0.6760 + 1.9846i  -0.6760 - 1.9846i
Db = D % feed-through term unchanged

Db =
    1

Verify that we still have the same transfer function:

>> [numb,denb] = ss2tf(Ab,Bb,Cb,Db)

numb =
    1      2 + 0i      3 + 0i

denb =
    1      0.5 - 0i      0.3333

>> num = [1, 2, 3]; % original numerator
>> norm(num-numb)

ans = 1.5543e-015

>> den = [1, 1/2, 1/3]; % original denominator
>> norm(den-denb)

ans = 1.3597e-016

Close enough.
Properties of the Modal Representation

- The modal representation is not *unique* since $B$ and $C$ may be scaled in compensating ways to produce the same transfer function. Also, the diagonal elements of $A$ may be permuted.

- For oscillatory systems, the $\lambda_i$ are *complex*.

- If mode $i$ is oscillatory and *undamped* (lossless), the state variable $x_i(n)$ oscillates *sinusoidally* at some frequency $\omega_i$, where

$$
\lambda_i = e^{j\omega_i T}
$$

- In the damped oscillatory case, we have

$$
\lambda_i = R_i e^{j\omega_i T}
$$

where $R_i$ is the pole (eigenvalue) radius. For stability, we must have $|R_i| < 1$.

- In practice, we often prefer to combine complex-conjugate pole-pairs to form a real, “block-diagonal” system in which $A$ has two-by-two real matrices along its diagonal.

- Matlab function `cdf2rdf()` can be used to convert complex diagonal form to real block-diagonal form.
• The input vector $\tilde{B}$ in the modal representation specifies *how the modes are excited* by the input signal $u(n)$:

$$x_i(n) = \tilde{b}_i u(n)$$

• The output vector $\tilde{C}$ in the modal representation specifies *how the modes are mixed* in the output signal $y(n)$:

$$y(n) = \tilde{C}\tilde{x}(n) = \tilde{c}_1\tilde{x}_1(n) + \tilde{c}_2\tilde{x}_2(n) + \cdots + \tilde{c}_N\tilde{x}_N(n)$$

### Repeated Poles

For repeated poles $\lambda_i$, we have two cases:

• If the corresponding eigenvectors are *linearly independent*, the modes are independent and can be decoupled (system can be diagonalized)

• Otherwise, if $\lambda_i$ corresponds to $k$ linearly *dependent* eigenvectors, the diagonalized system will contain a *Jordan block* of order $k$ corresponding to that mode.

• Same as repeated roots in a partial-fraction expansion

• Impulse response looks like $n\lambda^n$, $n^2\lambda^n$, etc.