Diagonal state transition matrix = modal representation:

\[
\begin{bmatrix}
  x_1(n+1) \\
  x_2(n+1) \\
  \vdots \\
  x_{N-1}(n+1) \\
  x_N(n+1)
\end{bmatrix}
= \begin{bmatrix}
  \lambda_1 & 0 & 0 & \cdots & 0 \\
  0 & \lambda_2 & 0 & \cdots & 0 \\
  \vdots & \vdots & \ddots & \vdots & \vdots \\
  0 & 0 & 0 & \lambda_{N-1} & 0 \\
  0 & 0 & 0 & 0 & \lambda_N
\end{bmatrix}
\begin{bmatrix}
  x_1(n) \\
  x_2(n) \\
  \vdots \\
  x_{N-1}(n) \\
  x_N(n)
\end{bmatrix}
+ \begin{bmatrix}
  b_1 \\
  b_2 \\
  \vdots \\
  b_{N-1} \\
  b_N
\end{bmatrix}u(n)
\]

\[y(n) = Cx(n) + Du(n)\]

The \(N\) complex modes are decoupled:

\[x_1(n+1) = \lambda_1 x_1(n) + b_1 u(n)\]
\[x_2(n+1) = \lambda_2 x_2(n) + b_2 u(n)\]
\[\vdots\]
\[x_N(n+1) = \lambda_N x_N(n) + b_N u(n)\]

\[y(n) = c_1 x_1(n) + c_2 x_2(n) + \cdots + c_N x_N(n) + Du(n)\]

That is, diagonal state-space system consists of \(N\) parallel one-pole systems (complex, in general).
Diagonalizing a State-Space Model

- To obtain a *modal representation*, we can *diagonalize* a state-space model.
- The *similarity transformation* which diagonalizes the system is given by the *matrix of eigenvectors* of the state transition matrix $A$.
- An eigenvector $e_i$ of $A$ satisfies, by definition,

\[ A e_i = \lambda_i e_i \]

where $e_i$ and $\lambda_i$ may be complex.
- In other words, a state-space model is diagonalized by a similarity transformation matrix $E$ whose columns are given by the eigenvectors of $A$:

\[ E = [e_1 \cdots e_N] \]

- A system can be diagonalized whenever the eigenvectors of $A$ are *linearly independent*.
  - This always holds for *distinct* poles.
  - May or may not hold for *repeated* poles.
State Space Diagonalization

• Suppose we solve the equation $Ae_i = \lambda_i e_i$ and find $N$ linearly independent eigenvectors of $A$.

• Form the $N \times N$ matrix $E = [e_1 \ldots e_N]$ having these eigenvectors as columns.

• Since the eigenvectors are linearly independent, $E$ is full rank and can be inverted. This means it is one-to-one and qualifies as a linear coordinate transformation matrix.

• As derived above, the transformed state transition matrix is given by

$$\tilde{A} = E^{-1}AE$$

• Since $Ae_i = \lambda_i e_i$, we have

$$AE = E\Lambda$$

where $\Lambda$ is a diagonal matrix having the (complex) eigenvalues of $A$ along its diagonal.

• It follows that

$$\tilde{A} = E^{-1}AE = E^{-1}E\Lambda = \Lambda.$$ 

Thus, the new state transition matrix $\Lambda$ is diagonal consisting of the eigenvalues of $A$. 

4
• The transfer function of the diagonalized system is

\[
H(z) = \tilde{D} + \tilde{C} (z I - \Lambda)^{-1} \tilde{B}
\]

\[
= \tilde{D} + \frac{\tilde{c}_1 b_1 z^{-1}}{1 - \lambda_1 z^{-1}} + \frac{\tilde{c}_2 b_2 z^{-1}}{1 - \lambda_2 z^{-1}} + \cdots + \frac{\tilde{c}_N b_N z^{-1}}{1 - \lambda_N z^{-1}}
\]

\[
= \tilde{D} + \sum_{i=1}^{N} \frac{\tilde{c}_i b_i z^{-1}}{1 - \lambda_i z^{-1}}
\]

We see again that the diagonalized system (modal representation) consists of \( N \) parallel one-pole systems.

• Dynamic modes \( \lambda_i \) are decoupled

• Closely related to partial-fraction expansion of \( H(z) \):
  
  – Residue of the \( i \)th pole is \( c_i b_i \)
  – Complex-conjugate poles may be combined to form real second-order sections
Finding the Eigenvalues of $A$ in Practice

Small problems may be solved by hand by solving the system of equations

$$AE = EA$$

The Matlab built-in function `eig()` may be used to find the eigenvalues of $A$ (system poles).
Example of State-Space Diagonalization

For the previous example

\[ A \triangleq \begin{bmatrix} -\frac{1}{2} & -\frac{1}{3} \\ 1 & 0 \end{bmatrix} \quad B \triangleq \begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad C \triangleq \begin{bmatrix} 3/2 \\ 8/3 \end{bmatrix} \quad D \triangleq 1 \]

we obtain the following in Matlab:
>> eig(A) % eigenvalues of state transition matrix

ans =
   -0.2500 + 0.5204i
   -0.2500 - 0.5204i

>> roots(den) % poles of transfer function \Hmtx(z)

ans =
   -0.2500 + 0.5204i
   -0.2500 - 0.5204i

% They are the same, as they must be.
>> abs(roots(den)) % check stability

ans =
   0.5774
   0.5774

The system is stable.

Complex-conjugate poles are typically combined to produce real, second-order (2 × 2) parallel sections in the modal representation. Thus, our second-order example is already in real modal form. However, to illustrate the computations, let’s obtain the eigenvectors and compute
the *complex* modal representation:

```matlab
>> % Initial state space model from example above:
>> A = [-1/2, -1/3; 1, 0];
>> B = [1; 0];
>> C = [2-1/2, 3-1/3];
>> D = 1;

>> % Diagonalizing similarity transformation:
>> [E,L] = eig(A) % [Evects, Evals] = eig(A)

E =

   -0.4507 - 0.2165i   -0.4507 + 0.2165i
     0 + 0.8660i       0 - 0.8660i

L =

   -0.2500 + 0.5204i     0
     0          -0.2500 - 0.5204i

>> A * E - E * L % should be zero

ans =
```

9
1.0e-016 *
      0  + 0.2776i  0  - 0.2776i
      0  0

Now form the complete diagonalized state-space model (complex):

>> Ei = inv(E);  % matrix inverse
>> Ab = Ei*A*E  % diagonalized state xition mtx

Ab =
   -0.2500  + 0.5204i  0.0000  + 0.0000i
   -0.0000  -0.2500  - 0.5204i

>> Bb = Ei*B  % new input "routing vector"

Bb =
   -1.1094
   -1.1094

>> Cb = C*E  % new output linear combination

Cb =
   -0.6760  + 1.9846i  -0.6760  - 1.9846i
>> Db = D  % feed-through term unchanged

Db =
    1

Verify that we still have the same transfer function:

>> [numb,denb] = ss2tf(Ab,Bb,Cb,Db)

numb =
    1    2 + 0i    3 + 0i

denb =
    1    0.5 - 0i    0.3333

>> num = [1, 2, 3]; % original numerator
>> norm(num-numb)

ans = 1.5543e-015

>> den = [1, 1/2, 1/3]; % original denominator
>> norm(den-denb)

ans = 1.3597e-016

Close enough.
Properties of the Modal Representation

• The modal representation is not unique since $B$ and $C$ may be scaled in compensating ways to produce the same transfer function. Also, the diagonal elements of $A$ may be permuted.

• For oscillatory systems, the $\lambda_i$ are complex.

• If mode $i$ is oscillatory and undamped (lossless), the state variable $x_i(n)$ oscillates sinusoidally at some frequency $\omega_i$, where

$$\lambda_i = e^{j\omega_i T}$$

• In the damped oscillatory case, we have

$$\lambda_i = R_i e^{j\omega_i T}$$

where $R_i$ is the pole (eigenvalue) radius. For stability, we must have $|R_i| < 1$.

• In practice, we often prefer to combine complex-conjugate pole-pairs to form a real, “block-diagonal” system in which $A$ has two-by-two real matrices along its diagonal.

• Matlab function `cdf2rdf()` can be used to convert complex diagonal form to real block-diagonal form.
• The input vector \( \tilde{B} \) in the modal representation specifies \textit{how the modes are excited} by the input signal \( u(n) \):

\[
x_i(n) = \tilde{b}_i u(n)
\]

• The output vector \( \tilde{C} \) in the modal representation specifies \textit{how the modes are mixed} in the output signal \( y(n) \):

\[
y(n) = \tilde{C}\tilde{x}(n) = \tilde{c}_1\tilde{x}_1(n) + \tilde{c}_2\tilde{x}_2(n) + \cdots + \tilde{c}_N\tilde{x}_N(n)
\]

**Repeated Poles**

For repeated poles \( \lambda_i \), we have two cases:

• If the corresponding eigenvectors are \textit{linearly independent}, the modes are independent and can be decoupled (system can be diagonalized)

• Otherwise, if \( \lambda_i \) corresponds to \( k \) linearly \textit{dependent} eigenvectors, the diagonalized system will contain a \textit{Jordan block} of order \( k \) corresponding to that mode.

• Same as repeated roots in a partial-fraction expansion

• Impulse response looks like \( n\lambda^n, n^2\lambda^n \), etc.