MUS420 Supplement
Wave Digital Filters and Waveguide Networks for Numerical Integration of Time-Dependent PDEs

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February 5, 2019

Outline:

- Wave Digital Filtering Overview
- Multidimensional Wave Digital Filters
- Waveguide Meshes and Numerical Integration
- Beams
- Plates
- Elastic Solids

Reference:

http://ccrma.stanford.edu/~bilbao/

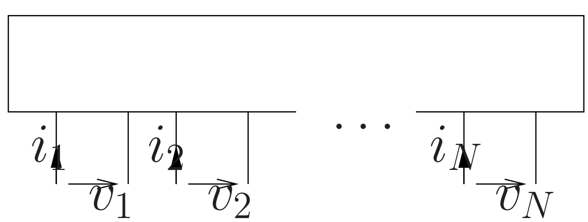
Wave Digital Filtering Overview

- A digital filter design procedure (1971)
- Transforms lumped analog (classical) filters to discrete time
- Excellent numerical properties:
  - low sensitivity of response to mutliplier coefficients
  - stable under coefficient quantization
  - elimination of limit cycles

  because we have available a discrete-time energy.

Classical Kirchoff Network

- Basic component: N-port



- Characterized by port voltages v_k and currents i_k, $k = 1 \dots N$.

- N-ports to be connected portwise
- Instantaneous power absorbed by the N-port through the ports is

$$p_{inst} = \sum_{j=1}^{N} v_j i_j$$

Passivity

A given N-port satisfies an energy balance

$$\int_{t_1}^{t_2} p_{inst} dt + \int_{t_1}^{t_2} p_s dt = W(t_2) - W(t_1) + \int_{t_1}^{t_2} p_l dt$$

where we have:

$$\begin{aligned} W(t) &= \text{stored energy} \\ p_s(t) &= \text{internal source power} \\ p_l(t) &= \text{internal power loss} \end{aligned}$$

- *Passive* if we have

$$\int_{t_1}^{t_2} p_{inst} dt \geq W(t_2) - W(t_1) \qquad \forall t_1, t_2$$

- *Lossless* if we have

$$\int_{t_1}^{t_2} p_{inst} dt = W(t_2) - W(t_1) \qquad \forall t_1, t_2$$

Passivity for LTI N-ports

For an LTI N-port, we (sometimes) have an impedance relationship

$$\hat{\mathbf{v}}(\mathbf{s}) = \mathbf{Z}(\mathbf{s})\hat{\mathbf{i}}(\mathbf{s})$$

at steady state

- Passive (for real-valued network) if $\mathbf{Z}$ is positive real, i.e.

$$Re\left(\mathbf{Z} + \mathbf{Z}^*\right) \geq 0 \qquad \text{for} \qquad Re(s) \geq 0$$

- Lossless if, in addition,

$$Re\left(\mathbf{Z} + \mathbf{Z}^*\right) = 0 \qquad \text{for} \qquad Re(s) = 0$$

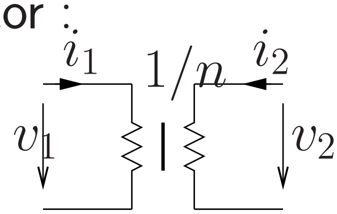
Circuit Elements

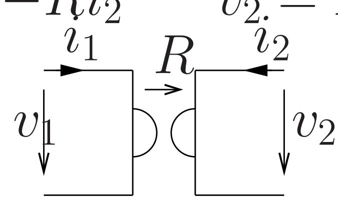
The standard *passive* LTI circuit elements are

Inductor :	$v = L\dfrac{di}{dt}$	$Z = Ls$
Resistor :	$v = R_0 i$	$Z = R_0$
Capacitor :	$i = C\dfrac{dv}{dt}$	$Z = \dfrac{1}{Cs}$

Ideal Transformer : $v_2 = nv_1$ $\quad i_1 = -ni_2$

Gyrator : $v_1 = -Ri_2$ $\quad v_2 = Ri_1$





Also, have *active* elements like current, voltage sources etc.

Kirchoff's Laws

Any number k of individual ports can be connected in

- Series (KCL),

$$\begin{aligned} i_1 &= i_2 = \ldots = i_k \\ v_1 &+ v_2 + \ldots + v_k = 0 \end{aligned}$$

- Parallel (KVL)

$$\begin{aligned} v_1 &= v_2 = \ldots = v_k \\ i_1 &+ i_2 + \ldots + i_k = 0 \end{aligned}$$

A series or parallel connection is a lossless k-port.

- *Tellegen's Theorem* implies that a network made up of passive N-ports will itself be passive.

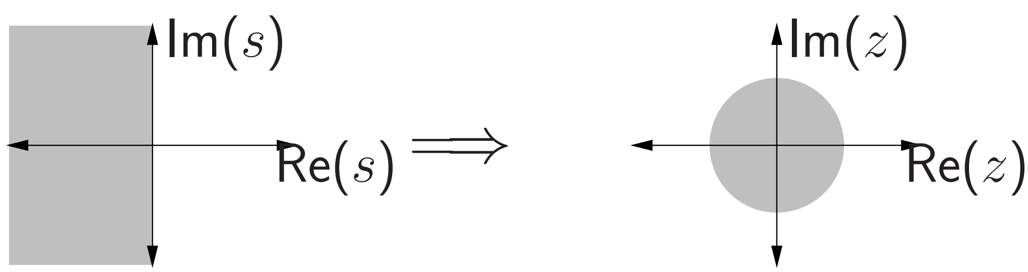
Discretization

- Closed network defines a set of ODEs in the state variables (solution may not exist or be unique)
- Need a discretization method (for digital filtering, or simulation)

Wave Digital Filters are based on the application of a *spectral mapping* or bilinear transform:

$$s \to \frac{2}{T}\frac{1-z^{-1}}{1+z^{-1}}$$

which takes the s RHP to the z outer disk:



T is the time step and $\frac{1}{T}$ is the sampling frequency.

- Stable causal analog filters mapped to stable causal digital filters. Also: DC$\rightarrow$ DC, $\infty \rightarrow$ Nyquist.
- Discrete "impedances" inherit passivity property (now called pseudo-passivity), and are called positive real in the outer disk.

Trapezoid Rule

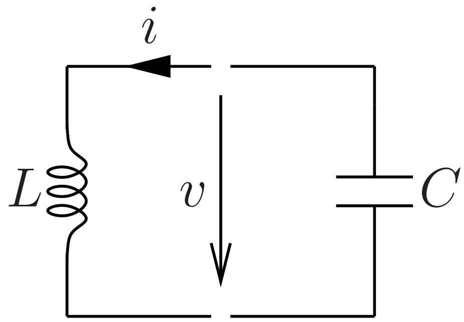
In the time domain, this bilinear transform is equivalent to applying the *trapezoid rule* in order to integrate (or differentiate).

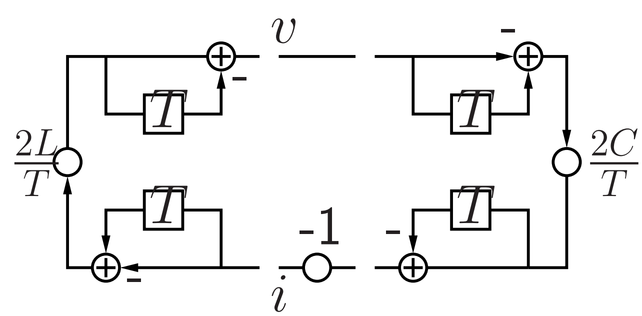
- Example: for an inductor
$$\hat{v} = L\frac{d\hat{i}}{dt}$$
Under bilinear transformation:
$$Z(s) = Ls \rightarrow \frac{2L}{T}\frac{1-z^{-1}}{1+z^{-1}}$$
In time domain:
$$\frac{v(n)+v(n-1)}{2} = L\frac{i(n)-i(n-1)}{T}$$
- Accurate to order T^2

- The above *difference equation* defines a digital $one - port$, with discrete-time voltage and current v and i.
- Only *reactive* elements are affected by this transformation (i.e. not resistors, ideal transformers, or Kirchoff's Laws).

Connecting Digital N-ports

- Can connect digital one-ports using Kirchoff's Laws, and still have power conservation. If elements are pseudopassive, then so is a network constructed from such elements.
- Example: parallel LC connection





- Problem: Delay-free loops.
- Result: Signal-flow diagrams are non-realizable (unless one is willing to perform matrix inversions).

Wave Variables

It is possible to get around these realizability problems by introducing *wave variables* (1971) borrowed from microwave engineering

Introduce, for any port variables v and i, the quantities

$$\begin{aligned} a &= v + iR && \text{Input Wave} \\ b &= v - iR && \text{Output Wave} \end{aligned}$$

R is an arbitrary positive constant, called the *port resistance*. We can also define *power-normalized* wave variables as

$$\begin{aligned} a' &= \frac{v + iR}{2\sqrt{R}} \\ b' &= \frac{v - iR}{2\sqrt{R}} \end{aligned}$$

The two types of waves are simply related to eachother by

$$\begin{aligned} a &= 2\sqrt{R}a' \\ b &= 2\sqrt{R}b' \end{aligned}$$

Power waves are useful in dealing with *time-varying* and *nonlinear* circuit elements.

Example: Wave Digital Inductor

From the trapezoid rule (bilinear transform) we have

$$\frac{v(n)+v(n-1)}{2} = L\frac{i(n)-i(n-1)}{T}$$

Inserting wave variables, we get:

$$a(n+1)+b(n+1)+a(n)+b(n) = \frac{2L}{RT}(a(n+1)-b(n+1)-a(n)$$

And under the choice $R = \frac{2L}{T}$, we get

$$b(n) = -a(n-1)$$

A strictly causal input/output relationship. (Same in power-normalized case).

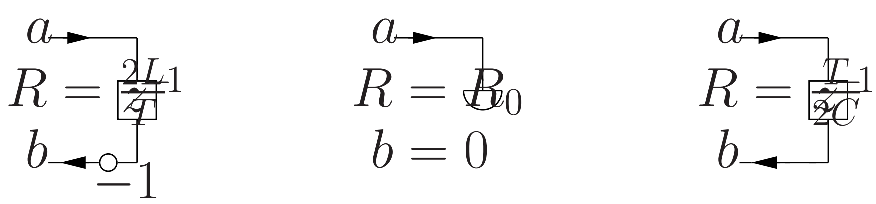
- Energetic interpretation:

$$\begin{aligned} p_{inst}(n) &= v(n)i(n) \\ &= a'(n)^2 - b'(n)^2 \\ &= a'(n)^2 - a'(n-1)^2 \end{aligned}$$

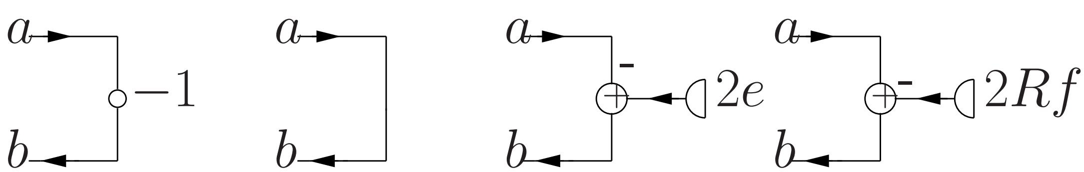
- Square of value in delay register has interpretation as *stored energy*

Wave-Digital Elements

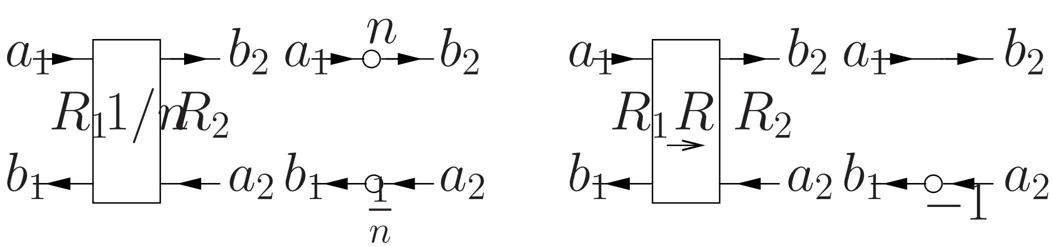
We can derive wave digital equivalents of the standard circuit elements.



Inductor Resistor Capacitor



Short Circuit Open Circuit Voltage Source Current Source



Ideal Transformer Gyrator

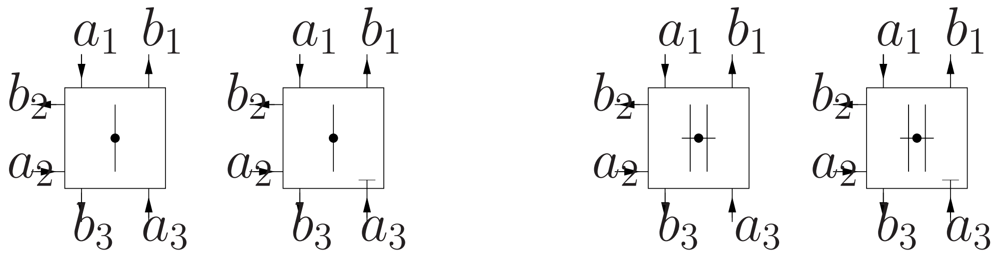
Adaptors

Connections between the elements are governed, as before, by Kirchoff's Laws. In terms of wave variables, we have, for a connection of k ports:

$$b_m = a_m - \frac{2R_m}{\sum_{j=1}^{k} R_j} \sum_{j=1}^{k} a_j, \qquad m = 1 \dots k \ \text{Series Connection}$$

$$b_m = -a_m + \frac{2}{\sum_{j=1}^{k} G_j} \sum_{j=1}^{k} G_j a_j, \ \ m = 1 \dots k \ \text{Parallel Connectio}$$

where $G_j = \frac{1}{R_j}$ is the *conductance* at port j. The signal processing block which carries out this operation is called an *adaptor*



Series Adaptor Parallel Adaptor

Scattering Matrices

The series and parallel adaptor equations can be written as

$$\mathbf{b} = \mathbf{S}\mathbf{a}$$

where

$$\begin{aligned}\mathbf{S} &= \quad \mathbf{I} - \alpha^T \mathbf{1} && \text{(Series)}\\ \mathbf{S} &= -\ \mathbf{I} + \mathbf{1}^T \beta && \text{(Parallel)}\end{aligned}$$

where $\alpha = \frac{2}{\sum_{i=1}^{k} R_i}(R_1 \ldots R_k)$, $\beta = \frac{2}{\sum_{i=1}^{k} G_i}(G_1 \ldots G_k)$
we have also: $\mathbf{S^2} = \mathbf{I}$ in either case. For power-normalized waves:

$$\mathbf{S}' = \mathbf{I} - \alpha'^T \alpha' \qquad \text{Series}$$

$$\mathbf{S}' = -\mathbf{I} + \beta'^T \beta' \qquad \text{Parallel}$$

where $\alpha' = \sqrt{\alpha}$ and $\beta' = \sqrt{\beta}$ (over all components). We have $\mathbf{S}'^T\mathbf{S}' = \mathbf{S}'\mathbf{S}'^T = \mathbf{I}$ (orthonormal, unitary).

- Form of the adaptor equation is simple (O(N) adds, multiplies)
- Easy to apply rounding rules so that junction behaves passively, even in finite arithmetic:
  - signals:Extended precision within junction. Magnitude truncation on outputs
  - reflection parameters (α, β) may also be truncated without affecting passivity (though accuracy will of course suffer).

Multidimensional Wave Digital Filters and Numerical Integration of Partial Differential Equations

Distributed Problems and Coordinate Changes

Symmetric Hyperbolic Systems

Multidimensional time and space-dependent physical systems are often described by systems of PDEs which are symmetric hyperbolic. For example, in 1D, a typical situation is:

$$\mathbf{P}\frac{\partial \mathbf{u}}{\partial t} = \mathbf{A}\frac{\partial \mathbf{u}}{\partial x} + \mathbf{B}\mathbf{u}$$

where $\mathbf{P}(\mathbf{x})$ is symmetric, > 0, and $\mathbf{A}(\mathbf{x})$ is symmetric (both square, real). Can write:

$$\begin{aligned}
\mathbf{u}^{\mathbf{T}}\mathbf{P}\frac{\partial \mathbf{u}}{\partial t} &= \mathbf{u}^{\mathbf{T}}\mathbf{A}\frac{\partial \mathbf{u}}{\partial x} + \mathbf{u}^{\mathbf{T}}\mathbf{B}\mathbf{u} \\
\frac{1}{2}\frac{\partial \mathbf{u}^{\mathbf{T}}\mathbf{P}\mathbf{u}}{\partial t} &= \frac{1}{2}\frac{\partial \mathbf{u}^{\mathbf{T}}\mathbf{A}\mathbf{u}}{\partial x} - \frac{1}{2}u^{T}\frac{\partial \mathbf{A}}{\partial x}u + \frac{1}{2}\mathbf{u}^{\mathbf{T}}(\mathbf{B}+\mathbf{B}^{\mathbf{T}})\mathbf{u}
\end{aligned}$$

and integrating over the real line (assuming no boundaries,

$$\frac{\partial}{\partial t}\int_x \mathbf{u}^{\mathbf{T}}\mathbf{P}\mathbf{u}dx = \int_x \mathbf{u}^{\mathbf{T}}\left(-\mathbf{A}' + \mathbf{B} + \mathbf{B}^{\mathbf{T}}\right)\mathbf{u}$$

- $\int_x \mathbf{u}^{\mathrm{T}}\mathbf{P}\mathbf{u}dx$ is usually called the total energy of the system.
- like an ODE describing evolution of energy.
- if right-hand side is zero, then the system is lossless (in weighted $\mathbf{P}$ norm).
- if right-hand side is never positive, then the energy can only decrease.
- symmetric hyperbolic $\rightarrow$ reciprocal networks (almost).

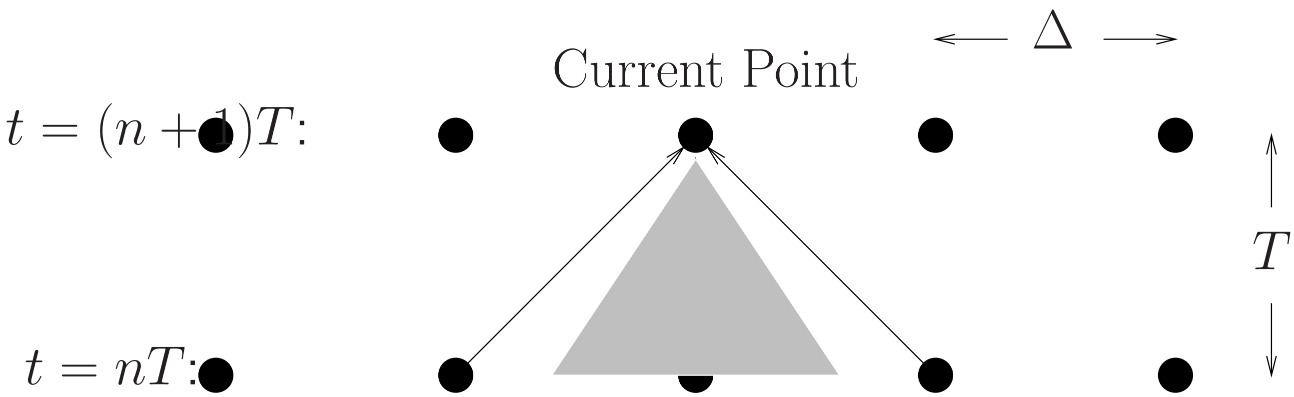
CFL Criterion

Hyperbolic systems $\rightarrow$ finite propagation speeds.

For explicit numerical methods on a grid, have a necessary stability condition on the time-step space-step ratio. In 1D, on a regular grid, only neighboring points:

$$v_0 \equiv \frac{\mathrm{space-step}}{\mathrm{time-step}} \geq \qquad \mathrm{maximum} \qquad \mathrm{speed.}$$

(Courant-Friedrichs-Lewy Condition).



In higher-D, same general result, but extra factors appear (solution does not necessarily move along a grid direction).

In the network approaches, CFL appears in an elementary way as a constraint on the positivity of the circuit elements (for passivity).

Multidimensional Problems and Coordinate Transformation

Multidimensional (distributed) systems$\rightarrow$ system of PDEs.

Time-dependent systems derived from conservation laws:

- may be of hyperbolic type (finite speeds)
- quantities conserved with respect to time alone.

In order to obtain a multidimensional circuit representation of a system of PDEs, useful to consider coordinate changes

$$\mathbf{u} = (\mathbf{x}, t)^T \rightarrow \mathbf{t} = (t_1 \dots t_k)^T$$

New coordinates should be causal, in the sense that:

- Any positive change Δt in the variable t (time) must be reflected by a similar positive change Δt_i in all the new coordinates t_i, $i = 1 \dots k$.
- Conversely, any positive change Δt_i in any of the new coordinates must produce a positive change in the time variable t.

A simple set of linear coordinate transformations:

$$\mathbf{t} = \mathbf{H}^{-1}\mathbf{V}\mathbf{u}$$

- $\mathbf{V}$ is a diagonal square matrix of the form diag$(1, \dots, v_0)$ used to nondimensionalize the original coordinates. v_0 has dimension of a velocity.
- $\mathbf{H}$ is a square (preferably orthonormal) matrix. Elements in bottom row are all positive (to satisfy above criteria).

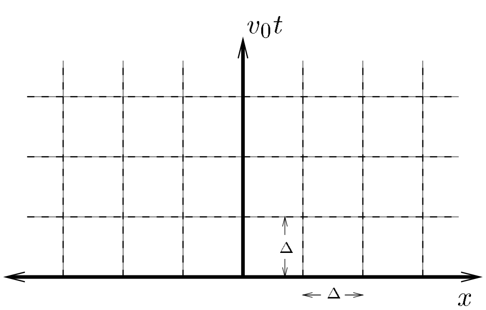
Remarks: A theoretical "hack" for extending passivity ideas to multi-D

A Simple Coordinate Change and Sampling

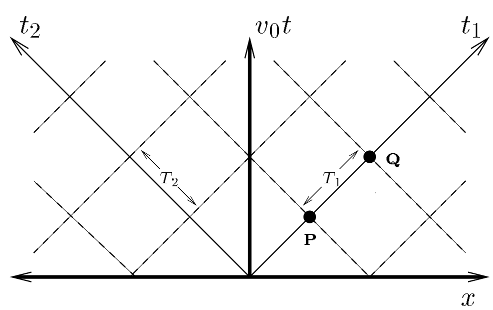
If we have only one spatial dimension, then coordinate change options are limited. The only suitable one (in this context) is

$$\begin{bmatrix} t_1 \\ t_2 \end{bmatrix} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 1 & \\ & v_0 \end{bmatrix} \begin{bmatrix} x \\ t \end{bmatrix}$$

A simple rotation of the coordinates by 45 degrees. If we now define uniform grids in the two coordinate systems, we get:



(a)



(b)

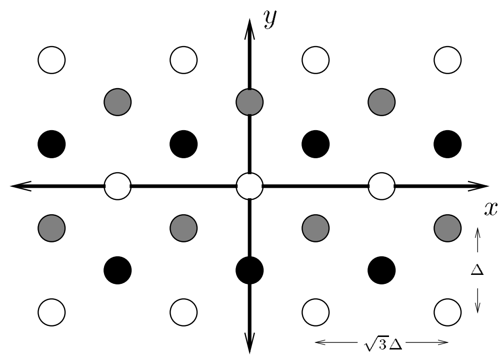
- grid spacings $(\Delta, \frac{1}{v_0}\Delta)$ in original coordinates, $(T_1, T_2 = T_1)$ in new.
- grids partially align, if we have $T_1 = \sqrt{2}\Delta$.

Coordinate Changes in Higher Dimensions: Hexagonal Coordinates

The transformation defined by

$$\mathbf{H} = \begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} & -\sqrt{\frac{2}{3}} \\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \end{bmatrix}$$

when uniformly "sampled" in the new coordinates gives rise to the following grid pattern (viewed in the original coordinate system):



Three different grids (white, grey, black points) which exist every third time step.

Rectangular coordinates

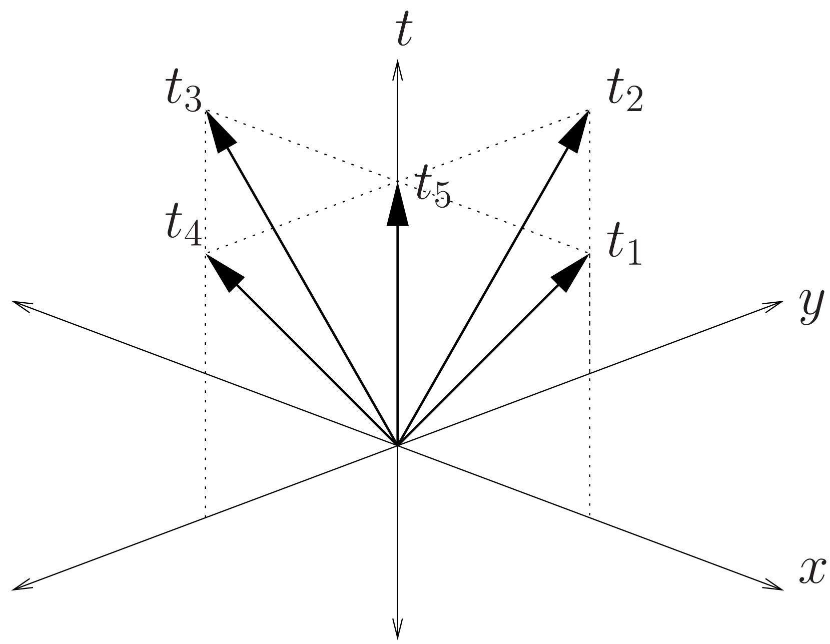
In order to generate a standard rectangular grid by uniformly sampling in the causal coordinates, need to use an embedding:

$$\mathbf{H} = \begin{bmatrix} 1 & 0 & -1 & 0 & 0 \\ 0 & 1 & 0 & -1 & 0 \\ 1 & 1 & 1 & 1 & 1 \end{bmatrix}$$

which maps (x, y, t) to a 5-dimensional space $\mathbf{t} =$ (north+time , south+time , east+time, west+time, time alone), and we get a normal rectangular grid in (x, y) if we sample uniformly in the five directions

- Ugly theoretical manipulation to do something simple; not actually going to solve a problem in 5D. Need them to define directions of energy flow.
- Now need to define a particular right pseudo-inverse $\mathbf{H}^{-\mathbf{R}}$...in practice, choice is relatively immaterial (but still must have elements of right column positive).
- Crux is: need five dimensions for a rectilinear grid, because in a simulation, energy can approach a grid point from any of the four compass points (and also from a past grid point at the same location).
- in 3D need to embed in a 7-D system.

Picture is:



Multi-Dimensional Elements and Wave Digital Filters

MD N-ports

Need to extend the notion of passivity to distributed networks.

Basic element, the N-port has the same form in multi-D:

- "Port" is no longer localized in space, and thus we have, for any port j,

$$
\begin{aligned}
v_j &= v_j(\mathbf{x}, t) \\
i_j &= i_j(\mathbf{x}, t)
\end{aligned}
$$

- instantaneous power (or power density) absorbed at any point $(\mathbf{x}, t)$ is

$$p_{inst}(\mathbf{x}, t) = \sum_{j=1}^{N} v_j i_j$$

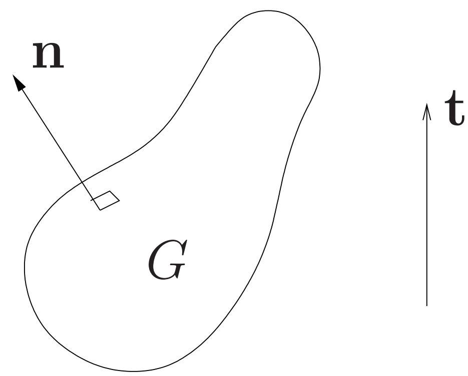
- Also have distributed *source* power p_s and *dissipated power* p_d
- N-ports must be connected portwise
- stored energy requires a generalized treatment...

MD N-ports Continued

- If the N-port is reactive, we have an associated *stored energy flux*. In terms of causal coordinates $\mathbf{t} = (t_1 \ldots t_k)$, this is a vector function

$$W_{st}(\mathbf{x}, t) = (W_1, W_2, \ldots, W_k)$$

For an causal $MD - passive$ N-port defined over a region G, we need $W_s \geq 0$ everywhere in G. Stored energy flows forward in physical time (and thus forward in all the causal coordinates)



- Integral energy balance (with respect to a region G) is

$$\int_G (p_{inst} + p_s)\, d\mathbf{t} = \int_G p_d d\mathbf{t} + \int_{\partial G} W_{st} \cdot \mathbf{n} d\sigma$$

MD-Passivity

An MD N-port is called integrally MD-passive (in G) if we have

$$\int_G p_{inst} d\mathbf{t} - \int_{\partial G} W_{st} \cdot \mathbf{n} d\sigma \geq 0$$

or differentially MD passive, if we have

$$p_{inst} - \nabla \cdot W_{st} \geq 0$$

and MD-lossless if equality holds.

- Kirchoff's Laws (treated as an N-port) are the same in MD, so that a Kirchoff connection of MD N-ports will itself behave passively (Tellegen).

- The resistor, defined by $v = iR$, R constant is also the same in MD, so MD-passive as well.
- Transformers, gyrators lossless as well, provided we define $W_{st} \equiv \mathbf{0}$
- Reactive elements need a more involved treatment...

The MD Inductor

Consider the following relation:

$$v(\mathbf{t}) = L\frac{\partial i(\mathbf{t})}{\partial t_j}$$

where L is a positive constant, and $t_j, j = 1 \ldots k$ is an causal coordinate.

- Defined over entire $\mathbf{t}$ space, but is really just a set of $1D$ differential relations.
- v obtained from i by integrating in t_j direction (forward in time).

We can define a discrete approximation by applying the trapezoid rule in the t_j direction:

$$\frac{v_{\ldots,n_j,\ldots} + v_{\ldots,n_j-1,\ldots}}{2} = \frac{L}{T_1}\left(i_{\ldots,n_j,\ldots} - i_{\ldots,n_j-1,\ldots}\right)$$

And introducing wave variables,

$$
\begin{aligned}
a_{n_1,\ldots,n_k} &= v_{n_1,\ldots,n_k} + R i_{n_1,\ldots,n_k} \\
b_{n_1,\ldots,n_k} &= v_{n_1,\ldots,n_k} - R i_{n_1,\ldots,n_k}
\end{aligned}
$$

can get the MD-equivalent of the wave-digital inductance one-port:

$$
b_{\ldots,n_j,\ldots} = -a_{\ldots,n_j-1,\ldots}, \qquad , R = \frac{2L}{T_j} \tag{1}
$$

MD-Inductor (Continued)

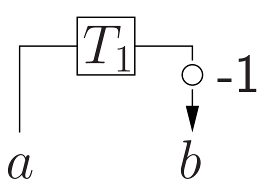
- If the coordinate is causal, then the MD inductance one-port takes an input (at every grid point), shifts and delays it, then outputs.
- Discretization (in LSI case) can be viewed as a multidimensional passivity-preserving spectral mapping. (need to define multi-D positive realness).
- graphical representations are:

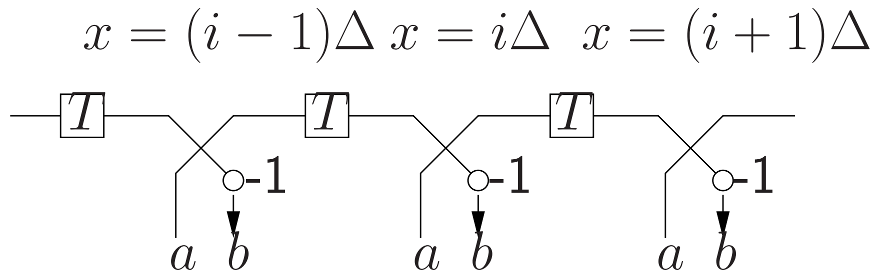
(a) (b)

- stored energy flux (of continuous element) is $\frac{1}{2}L\mathbf{e_j}i^2$ where $\mathbf{e_j}$ is a unit vector in the t_j direction.
- MD-passive if L is positive (lossless)
- capacitors...dual case.

MD Inductor: Expanding Out Spatial Dependence

In order to see how an MD Inductor can be programmed, consider an inductor of inductance L and direction t_1 and time step T_1:

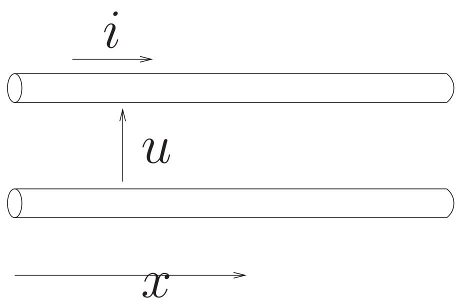




The 1D Transmission Line

- Described by the *telegrapher's equations*:

$$
\begin{aligned}
l\frac{\partial i}{\partial t} + \frac{\partial u}{\partial x} + ri + e &= 0 \\
c\frac{\partial u}{\partial t} + \frac{\partial i}{\partial x} + gu + h &= 0
\end{aligned}
$$



- Material parameters:
  - $l(x) =$ inductance/unit length
  - $c(x) =$ shunt capacitance/unit length
  - $r(x) =$ resistance/unit length
  - $g(x) =$ shunt conductance/unit length
- $e(x,t)$, $h(x,t)$ are sources (possibly distributed, time-varying).

The 1D Transmission Line (cont'd)

- For l, c constant, $r = g = e = f = 0$, reduces to the *wave equation*

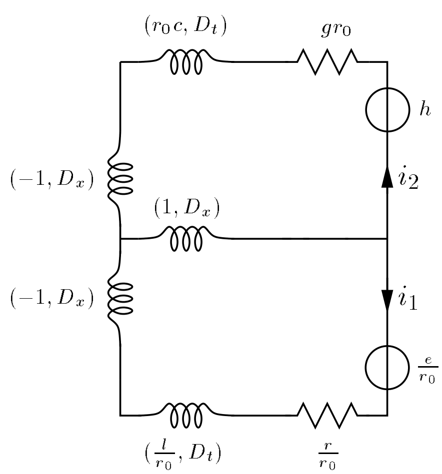
$$\frac{\partial^2 u}{\partial t^2} = \gamma^2 \frac{\partial^2 u}{\partial x^2} \qquad \gamma = \sqrt{\frac{1}{lc}}$$

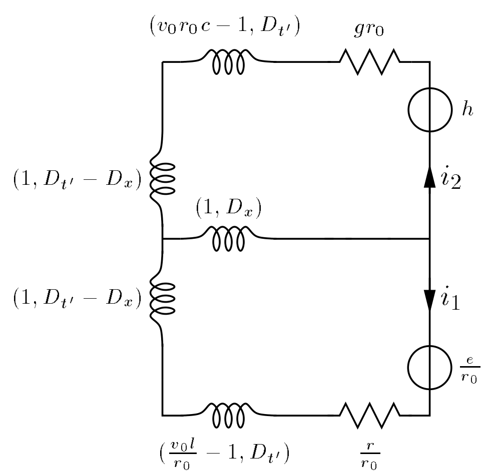
  Otherwise, describes dispersive, lossy 1D wave propagation.

- A simple model problem.

First Attempt at a Kirchoff Circuit (cont'd)

We can write the telegrapher's equations down directly as a two-loop circuit, with currents i and $\frac{u}{r_0}$ $r_0 =$ scaling parameter, dimensions of resistance:





Problem: Not MD-passive.

Fix: Classical network theory manipulations.

Passive MD Circuit

To any T-junction corresponds a lattice or Jaumann equivalent:

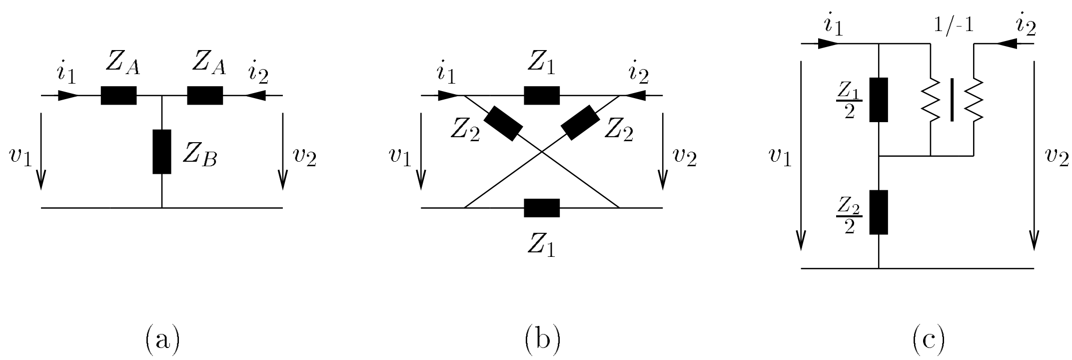
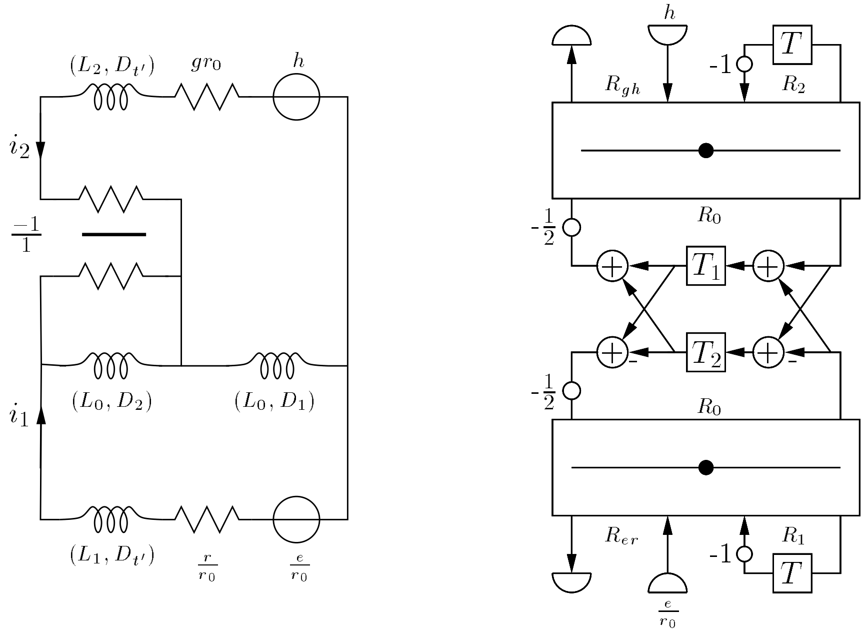


Figure 1: *Equivalent Two-ports*: (a) T-junction, with impedances Z_A and Z_B and (b) and (c), lattice and Jaumann equivalent two-ports, both with $Z_1 = Z_A$ and $Z_2 = Z_A + 2Z_B$.

Employing this equivalence gives a concretely MD-passive structure



(a) (b)

Comments

- Energetically, the system of equations has been broken into several smaller interacting parts, each of which is passive on its own.
- The passive part of the circuit (RL) dissipates MD-power.
- Source-free case: with power-normalized waves, all operations (scattering, shifting) are norm-reducing (in finite arithmetic as well).
- With the judicious choice of $r_0 = \sqrt{\frac{l_{min}}{c_{min}}}$, the stability bound is

$$v_0 \geq \sqrt{\frac{1}{l_{min}c_{min}}}$$

which is different from (worse than) the optimal bound

$$v_0 \geq \sqrt{\frac{1}{(lc)_{min}}}$$

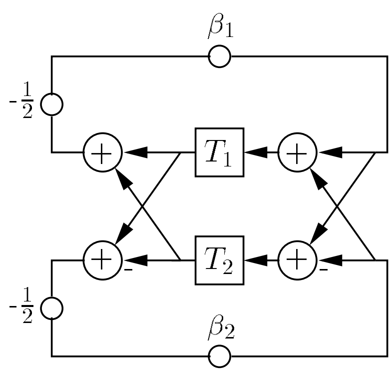
Fix: preconditioning of equations

- Memory requirements are double that of centered differences (a multistep method)

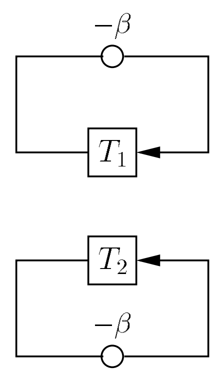
Simplifications

If l and c are constant, and we have no sources then we can derive a simplified form (a).

If in addition, we have $lg = cr$, then the line is distorionless, and we have simple travelling waves, which are attenuated (b).



(a)



(b)

2D "Parallel Plate"

- Equations are:

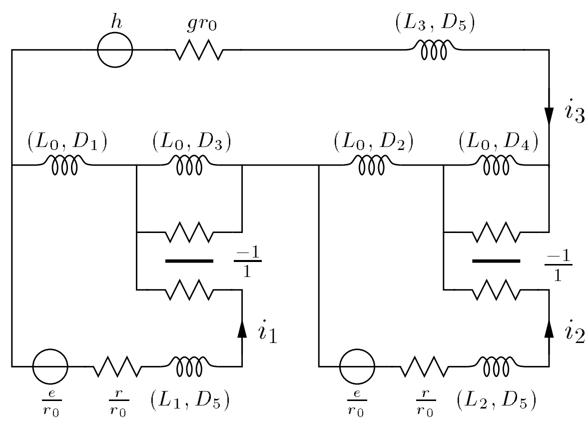
$$\begin{aligned} l\frac{\partial i_x}{\partial t} + \frac{\partial u}{\partial x} + ri_x + e &= 0 \\ l\frac{\partial i_y}{\partial t} + \frac{\partial u}{\partial y} + ri_y + f &= 0 \\ c\frac{\partial u}{\partial t} + \frac{\partial i_x}{\partial x} + \frac{\partial i_y}{\partial y} + gu + h &= 0 \end{aligned}$$

- – $(i_x, i_y) =$ current density in plate
  – $u =$ voltage across plates
  – other quantities as before; in general, l, c, r and g all may be spatially varying.

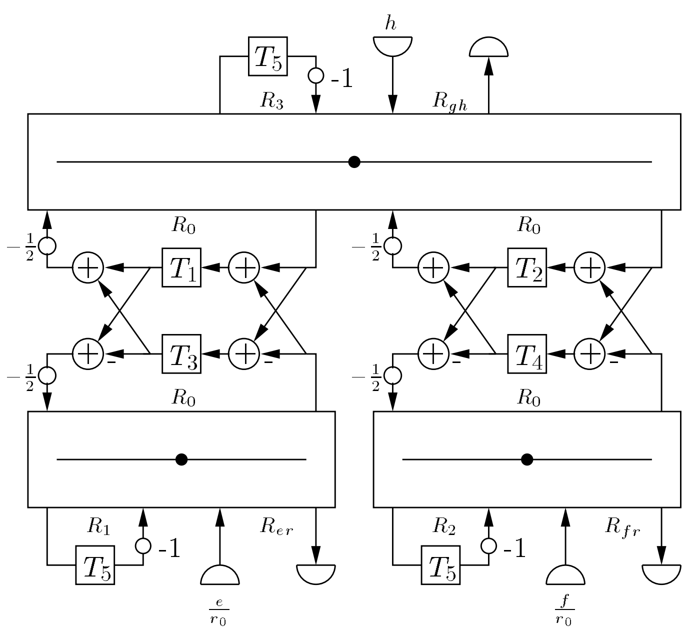
- A useful, general model problem: identical to systems of
  – an ideal membrane (in forces and transverse velocity)
  – TE or TM mode
  – 2D linear acoustic medium (velocities and pressure)

A Passive Circuit and Wave Digital Network

Applying a coordinate transformation (for rectangular coordinates), we get the following circuit and network:



$$L_0 = \frac{1}{2}$$
$$L_1 = L_2 = \frac{v_0 l}{r_0} - 1$$
$$L_3 = v_0 r_0 c - 2$$



$$R_0 = \frac{2}{\Delta}$$
$$R_3 = \frac{2(v_0 rc - 2)}{\Delta}$$
$$R_1 = R_2 = \frac{2(\frac{v_0 l}{r_0} - 1)}{\Delta}$$
$$R_{er} = R_{fr} = \frac{r}{r_0}$$
$$R_{gh} = gr_0$$

Other Systems

The circuit approach is applicable to a wide range of problems including:

- 3D Maxwell's Equations (inhomogeneous, lossy materials)
- 3D Linear Elasticity Equations (inhomogeneous)

  and with some tampering, to parabolic/borderline parabolic problems like
- Heat Equation (nD)
- Schrodinger's Equation (nD)
- Euler-Bernoulli Equations for a beam (1D) and plate (2D)

  Most interestingly, the method can also be applied to systems of nonlinear conservation laws, in particular
- Euler Equations (lossless nonlinear fluid dynamics), with an extension to the lossy case (Navier-Stokes).

In Sum

- Physical system $\rightarrow$ circuit representation (concretely passive)

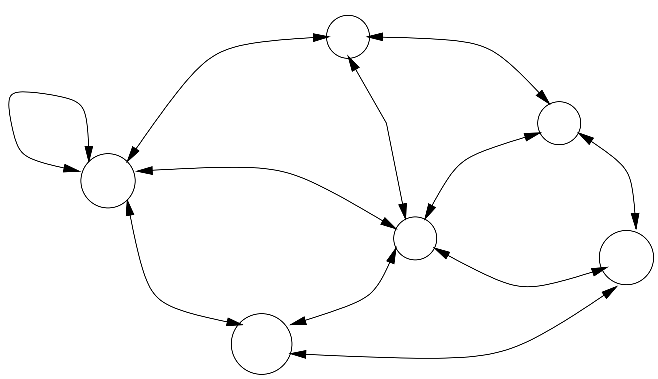
- Circuit representation $\rightarrow$ Signal Flow Graph

Question: Is there another way of deriving similar structures?

The Waveguide Mesh and Numerical Integration

Digital Waveguide Networks

- Networks made up of connections of transmission-line like "unit element" filters, connected at scattering junctions.

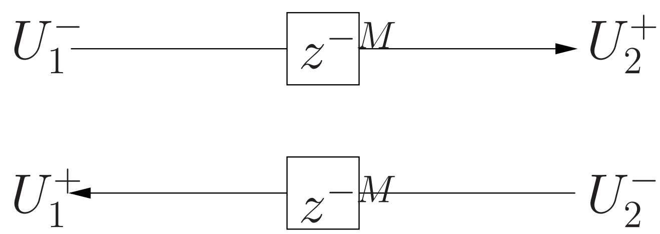


- Originally conceived as a stable means of doing artificial reverberation, and modelling room acoustics.
- Impedances as well as line lengths may be time-varying, (audio effects, such as chorusing, flanging, etc.)
- Same passivity properties as circuit-based method (under finite arithmetic as well).

- a different point of view from the multi-D circuits: Begin in a discrete setting, no spectral mapping is used (explicitly).
- when applied to E+M problems, is roughly equivalent to TLM (Transmission Line Matrix method).

The Bidirectional Delay Line

The central element in a digital waveguide network is a bidirectional delay line.



A bidirectional delay line has associated with it

- an impedance $Z > 0$ ($Y = \frac{1}{Z}$ is the admittance
- a delay (usually an integer $M \geq 1$) number of samples), same *in both directions*.
- two incoming and outgoing (voltage) wave variables
- (optional) a physical length.

can be considered a discrete time lossless (LBR) two-port.(Indeed, it is included among the original wave digital filtering elements).

If the delay line pair has a physical length Δ associated with it, then the two wave variables may be interpreted as travelling wave components which solve the 1D wave equation, where the speed is $c = \Delta/T$, and T is the sample period.

Traveling Wave Variables

We can define a related set of wave variables (current) in the bidirectional delay line by

$$\begin{aligned} I_j^+ &= \frac{1}{Z} U_j^+ \\ I_j^- &= -\frac{1}{Z} U_j^- \end{aligned}$$

for $j = 1, 2$.

- sign inversion of incoming current wave with respect to outgoing (not left/right, but could be defined this way).
- current waves are auxiliary; need them only to define scattering junctions (not stored variables).

"Physical" voltages and currents at the endpoints of the delay line pair:

$$\begin{aligned} U_j &= U_j^+ + U_j^- \\ I_j &= I_j^+ + I_j^- \end{aligned}$$

for $j = 1, 2$.

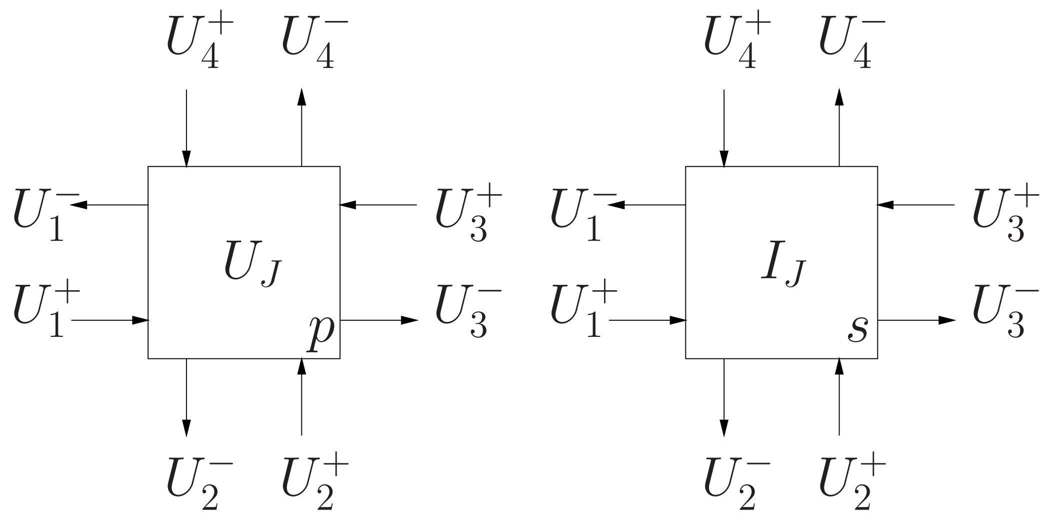
- A different formulation from MD wave digital filters: wave variables are instantaneously related to physical quantities (not to derivatives of them)

Scattering Junctions

Scattering Junctions (series and parallel) are defined identically to the WDF case.

$$U_j^- = U_j^+ - \frac{2Z_j}{\sum_{j=1}^k Z_j} \sum_{j=1}^k U_j^+, \qquad m = 1 \ldots k \ \text{Series Connection}$$

$$U_j^- = -U_j^+ + \frac{2}{\sum_{j=1}^k Y_j} \sum_{j=1}^k Y_j U_j^+, \ m = 1 \ldots k \ \text{Parallel Connection}$$



Comments

- Passivity follows from power conservation at the junctions (Kirchoff's Laws) and LBR property of delay lines. No need to invoke MD-passivity, since we are dealing with, in a sense, lumped elements (unit element filters).
- Passivity under coefficient and signal truncation follows just as in WDF case.
- Generalizations include:
  - Power-normalization of waves (as per WDFs)
  - allowing Z to be time-varying. Still passive, if power-normalized variables are used. (audio effects algorithms)
  - non-integer (and also posibly time-varying) delay line lengths. (audio effects algorithms)

– delays can be replaced by arbitrary bounded-real functions of z^{-1} (dispersion).

– vector-valued signals. For an n-component "waveguide", we now require:
$n \times n$ matrix impedance $\mathbf{Z}$, with $Z > 0$. $2n \times 2n$ matrix reflectance $\mathbf{S}$, $\mathbf{S}$ paraunitary. (comes in handy for solids, also as framework for treating TLM "super-symmetric condensed node")

1D Transmission Line Revisited

The lossless source-free 1D transmission line equations are

$$
\begin{aligned}
l\frac{\partial i}{\partial t} + \frac{\partial u}{\partial x} &= 0 \\
c\frac{\partial u}{\partial t} + \frac{\partial i}{\partial x} &= 0
\end{aligned}
$$

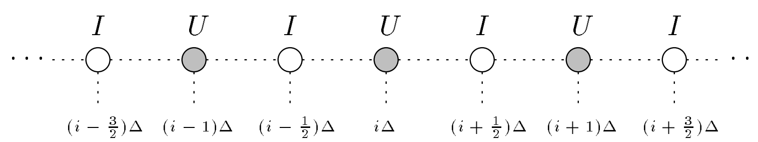
Can apply centered differences over a uniform grid:

$$
I_{i+\frac{1}{2}}(n+\tfrac{1}{2}) - I_{i+\frac{1}{2}}(n-\tfrac{1}{2}) \;+\; \frac{1}{v_0\bar{l}_{i+\frac{1}{2}}}\left(U_{i+1}(n) - U_i(n)\right) = 0
$$

$$
U_i(n) - U_i(n-1) \;+\; \frac{1}{v_0\bar{c}_i}\Big(I_{i+\frac{1}{2}}(n-\tfrac{1}{2}) - I_{i-\frac{1}{2}}(n-\tfrac{1}{2})\Big) =
$$

$$U_i(n) \qquad \text{approximates} \qquad u(i\Delta, nT)$$
$$I_{i+\frac{1}{2}}(n+\frac{1}{2}) \qquad \text{approximates} \qquad i((i+\frac{1}{2})\Delta, (n+\frac{1}{2})T)$$

- Grid variables are staggered in time and space (Yee, FDTD).



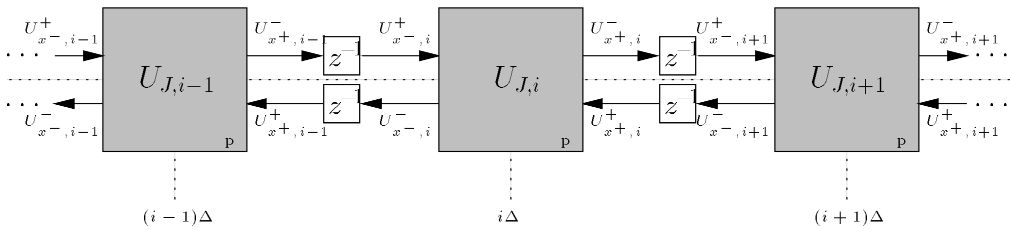
Constant Parameters

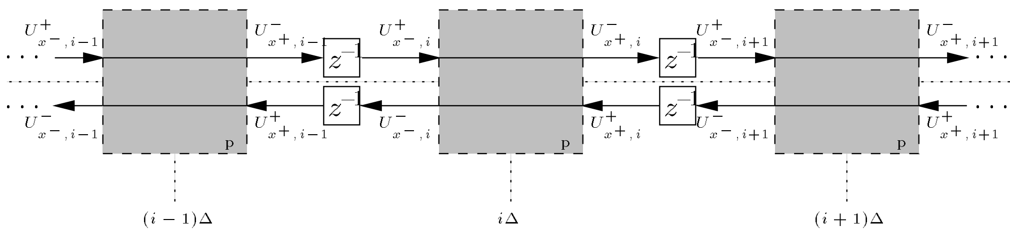
If l and c are constant, and we choose $v_0 = \sqrt{\frac{1}{lc}}$, then the difference equations simplify to a single equation in the voltages alone:

$$U_i(n+1) + U_i(n-1) = U_{i+1}(n) + U_{i-1}(n)$$

Just solving the wave equation, at the CFL limit.

Consider a chain of bidirectional delay lines, of length Δ, and operating at time step T connected by parallel junctions:

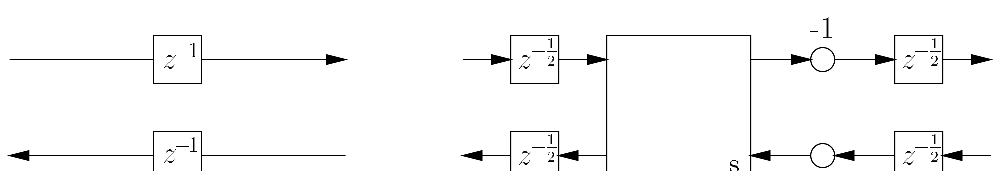




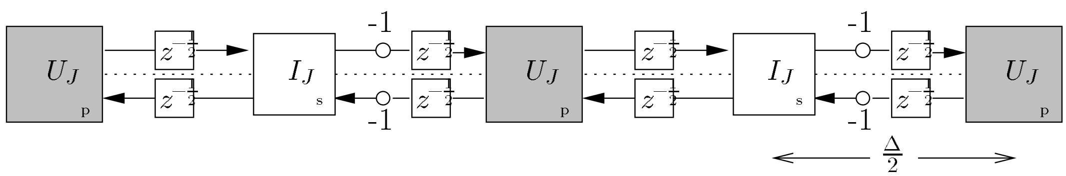
- Junction voltages solve the wave equation at CFL, if all impedances are identical (no scattering)

An Interleaved Mesh

Notice that it is possible to split a bidirectional delay line in the following way:



- Now have two half-sample, half-length delay lines of equal impedance (series junction functions, for the moment,as merely a sign inverter).
- Can employ this identity in the previous structure:



- Solves constant-parameter Transmission Line Equations, at CFL, on interleaved grid, if we choose $Z = \sqrt{\frac{l}{c}}$ for all subsections.
- Junction currents at series junctions give physical current.

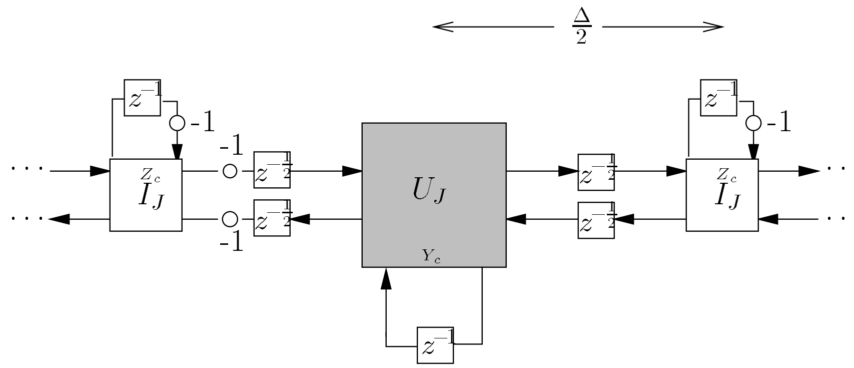
Varying Material Parameters

Now we have $l(x)$ and $c(x)$.

- Expect dispersion (scattering due to varying line impedance) Natural fix: Delay line impedances should be different.

- Local propagation speed, $\sqrt{\frac{1}{lc}}$ also varies Fix: Insert passive "storage" registers at every junction, in order to slow down local propagation speed in mesh.

Examine a mesh of the following form:



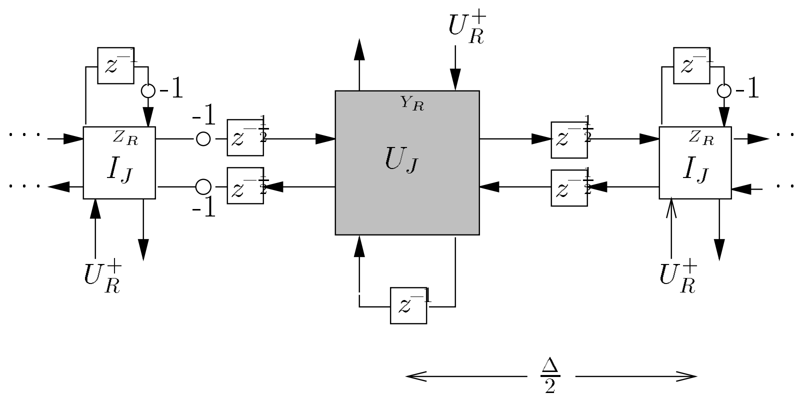
- will solve T-line equations if line impedances chosen properly
- Get a family of difference methods, each with different stability properties.
- stability constraints derive from positivity condition on impedances (as per WDFs).

Losses and Sources

The full Transmission Line Equations are, including losses (resistive, and shunt conductive):

$$
\begin{aligned}
l\frac{\partial i}{\partial t} + \frac{\partial u}{\partial x} + ri + e &= 0 \\
c\frac{\partial u}{\partial t} + \frac{\partial i}{\partial x} + gu + h &= 0
\end{aligned}
$$

Can treat this by adding a "resistive source" (analogous to WDF counterpart) at every junction:



Simplifies considerably in certain cases...

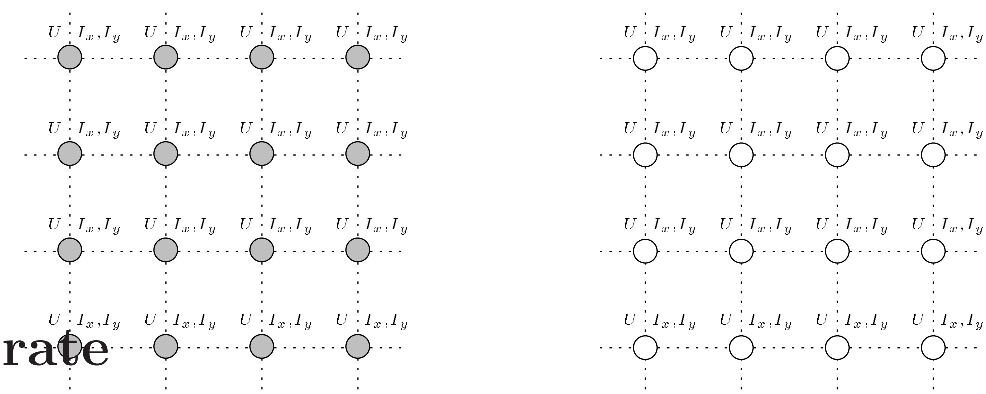
2D Parallel Plate Revisited

Full equations are, again,

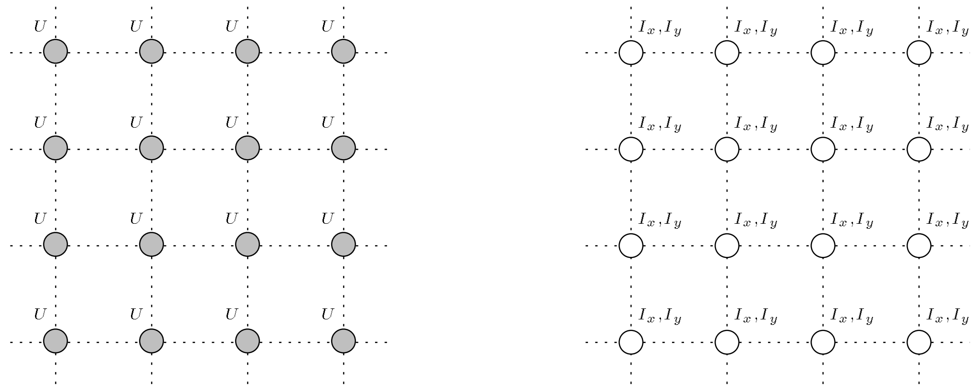
$$
\begin{aligned}
l\frac{\partial i_x}{\partial t} + \frac{\partial u}{\partial x} + ri_x + e &= 0 \\
l\frac{\partial i_y}{\partial t} + \frac{\partial u}{\partial y} + ri_y + f &= 0 \\
c\frac{\partial u}{\partial t} + \frac{\partial i_x}{\partial x} + \frac{\partial i_y}{\partial y} + gu + h &= 0
\end{aligned}
$$

Now, various levels of downsampling the computational grid (interleaving), for centered differences:

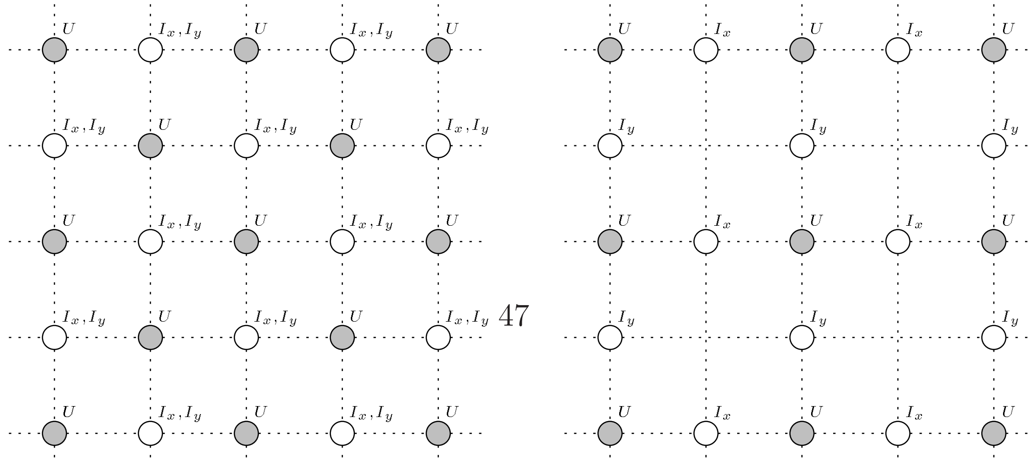
Full rate



Half-rate

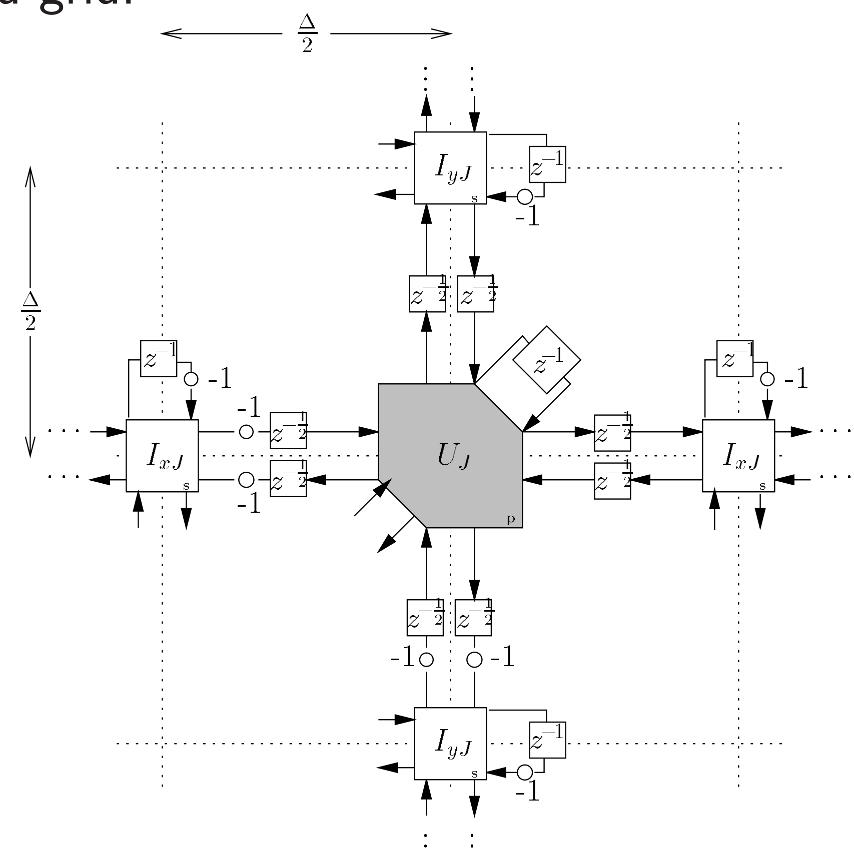


Quarter and One-eighth Rate:



Mesh for 2D Transmission Line

Can construct a mesh operating at the maximally interleaved grid:



- Can juggle impedances so that this solves the 2D transmission line
- Some configurations operate at CFL: $v_0 = \max_{grid} \sqrt{\frac{2}{lc}}$

FDTD and TLM: A Messy Mystery

We have, so far:

- These meshes calculate FDTD solutions (with special treatment to discretization of material parameters).
- An energetic interpretation of the method.
- Stability takes form of positivity constraint on impedances (instead of an eigenvalue condition). Gives rise to CFL criterion.

But, this mesh seems to be equivalent to the so-called "expanded" or "extended" node formulation of TLM.

TLM=FDTD? (in infinite-precision machine arithmetic)

A few vague statements in the literature...

More on TLM

Expanded node TLM solves for interleaved, or staggered (in space and time) field components.

Other "condensed" formulations, such as the Hybrid Symmetric, Super Symmetric etc., solve for the field quantities at the same grid location.

Still just centered differences (but not interleaved). Some comments:

- TLM: a hunt for unitary scattering junctions Observation: In waveguide (or WDF) formulation, do not require scattering matrices to be unitary for power conservation. But if needs be, then we have a Quick fix: Power-normalized waves (unknown in TLM).
- Condensed formulations. Observation: Scattering junctions derived using field-theoretic manipulations (vaguely like MD-circuit approach). Observation: Can also approach the problem by using vector delay lines. Build network, then set impedances (now matrices) accordingly.
- Passive arithmetic properties Observation: unknown in TLM.

Lesson: parochialism is a sad thing.

WDFs vs. Transmission line Meshes

- Stability, passivity, numerical properties: Identical.
- Finite-difference interpretation:

- meshes: “two-step” method.
- WDFs: “multistep” method.

both rewritten as one-step methods in the wave variables.

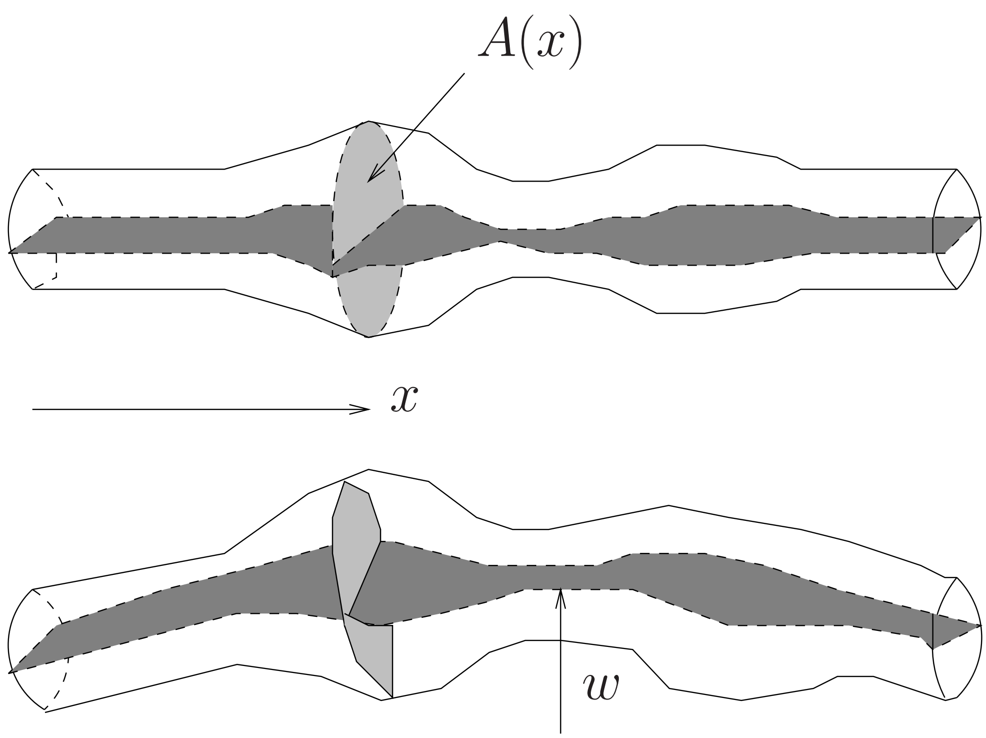
- Accuracy:
  - meshes:can be made more accurate (and faster converging) by identification with higher-order difference methods.
  - WDFs:passivity-preserving spectral mapping (A-stable method) precludes this.
- Upwind methods:
  - meshes: not easy (because delay lines are bidirectional)
  - WDFs: directions are split. Can be applied to advective problems (supersonic flow etc.)
- Irregular grids:
  - meshes: Possible.
  - WDFs: Difficult, because we require a smooth coordinate system at the outset (and integrate along specific directions).
- Distributed Nonlinear problems (fluid flow)

  - meshes: unknown.
  - WDFs: works.

- Interleaving:
  - meshes: by construction.
  - WDFs: difficult, because we are modelling pointwise behavior of PDEs (possible to get around this by incorporating multidimensional transmission lines into formulation).

Beams

- support transverse elastic wave propagation in a medium which is essentially 1D (longitudinal and torsional modes also exist)
- dispersive (even in homogeneous medium)
- medium has its own restoring stiffness
- transverse motion assumed to lie in one direction



Euler-Bernoulli Beams

- a first approximation: neglects effects of rotational lateral inertia. Plane subsections remain plane and parallel to the normal.
- Relevant quantities:
  - $\rho =$ density
  - $A =$ cross-subsectional area
  - $E =$ Young's modulus
  - $I =$ moment of inertia
- Get a set of equations:

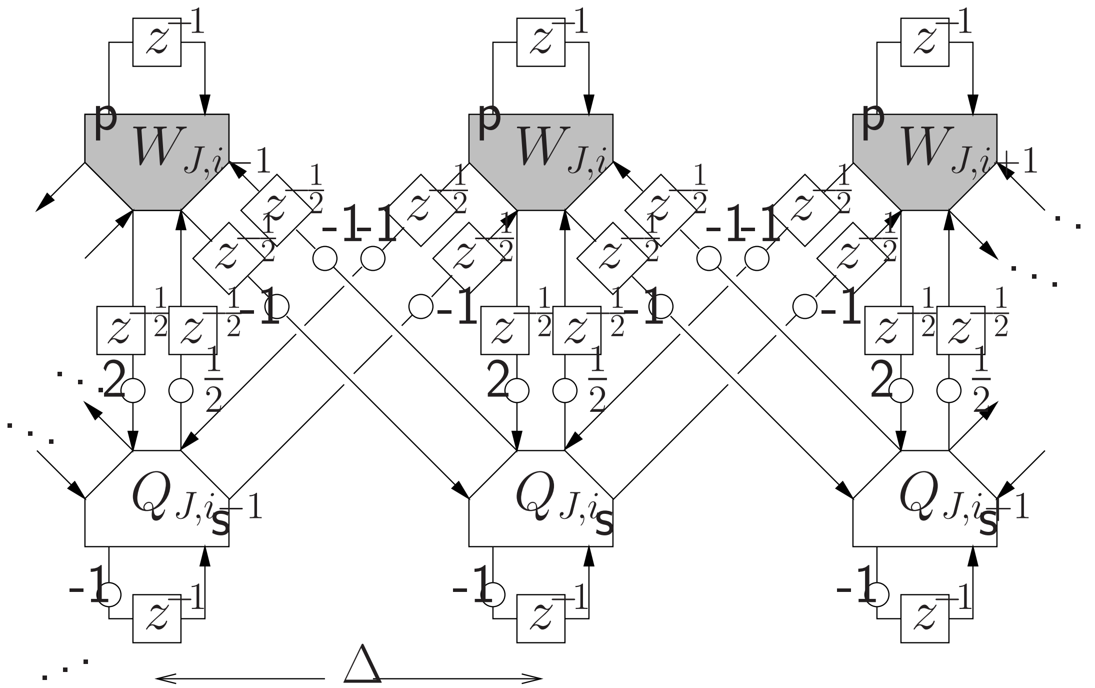
$$\begin{aligned} \frac{\partial w}{\partial t} &= \frac{1}{\rho A}\frac{\partial^2 q}{\partial x^2} \\ \frac{\partial q}{\partial t} &= -EI\frac{\partial^2 w}{\partial x^2} \end{aligned}$$

  In constant parameter case:

$$\frac{\partial^2 w}{\partial t^2} = -b^2\frac{\partial^4 w}{\partial x^4}, \qquad b = \sqrt{\frac{EI}{\rho A}} \tag{2}$$

- Phase velocity unbounded

Mesh for Euler-Bernoulli Beam



- Problem: unbounded phase velocity gives a very restrictive constraint on the time step:

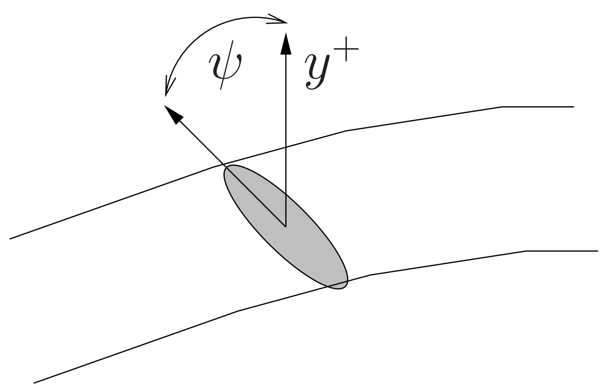
$$T \leq \frac{1}{2} \min_{x=i\Delta} \sqrt{\frac{\rho_x A_x}{E_x I_x}} \Delta^2 \tag{3}$$

- A typical problem for explicit methods for parabolic problems (though E-B are borderline parabolic). Similar problems crop up in the WDF approach (also possible).

- Need a better model...

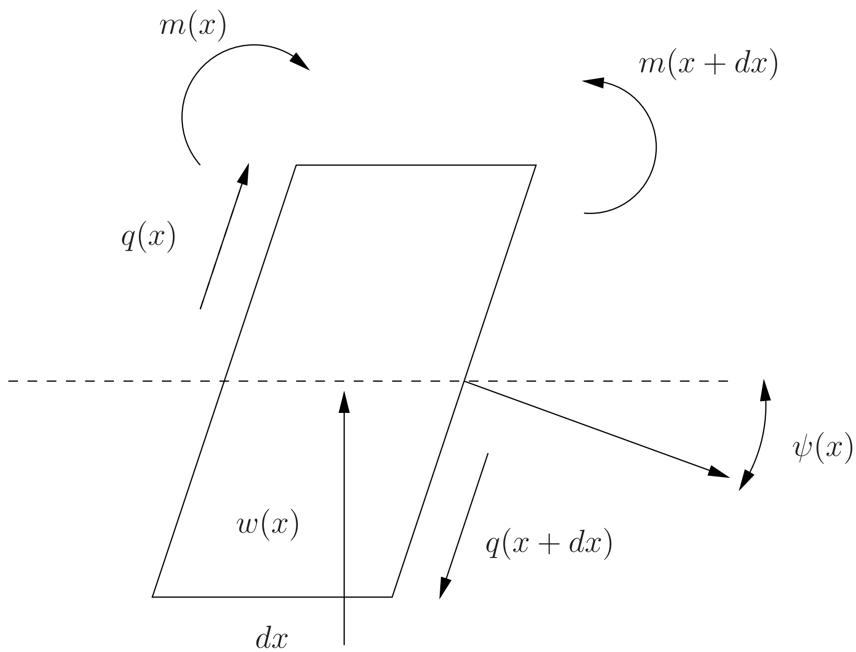
Timoshenko's Model (1921)

- Take into account shear and lateral inertia effects
- Transverse motion still constrained to lie in one perpendicular direction
- Plane subsections remain plane, but no longer parallel to the normal direction:



- New degree of freedom, $\psi(x)$ (in addition to w.

Timoshenko's Equations



- Dependent Variables:
  - $w =$ vertical displacement
  - $q =$ shear force
  - $\psi =$ angular deviation
  - $m =$ bending moment
- New material parameters:
  - $G =$ shear modulus
  - $\kappa =$ "Timoshenko coefficient" (geometrical)

Timoshenko's Equations Cont'd.

System:

$$\begin{bmatrix} \rho A & 0 & 0 & 0 \\ 0 & \frac{1}{A\kappa G} & 0 & 0 \\ 0 & 0 & \rho I & 0 \\ 0 & 0 & 0 & \frac{1}{EI} \end{bmatrix} \frac{\partial}{\partial t} \begin{bmatrix} v \\ q \\ \omega \\ m \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix} \frac{\partial}{\partial x} \begin{bmatrix} v \\ q \\ \omega \\ m \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} v \\ q \\ \omega \\ m \end{bmatrix}$$

with

$$v = \frac{\partial w}{\partial t} \quad \omega = \frac{\partial \psi}{\partial t} \quad m = EI\frac{\partial \psi}{\partial x} \quad q = A\kappa G\left(\frac{\partial w}{\partial x} - \psi\right)$$

- A symmetric hyperbolic system (bounded velocities)
- A pair of antisymmetrically coupled lossless transmission lines

Networks for Timoshenko's Equations

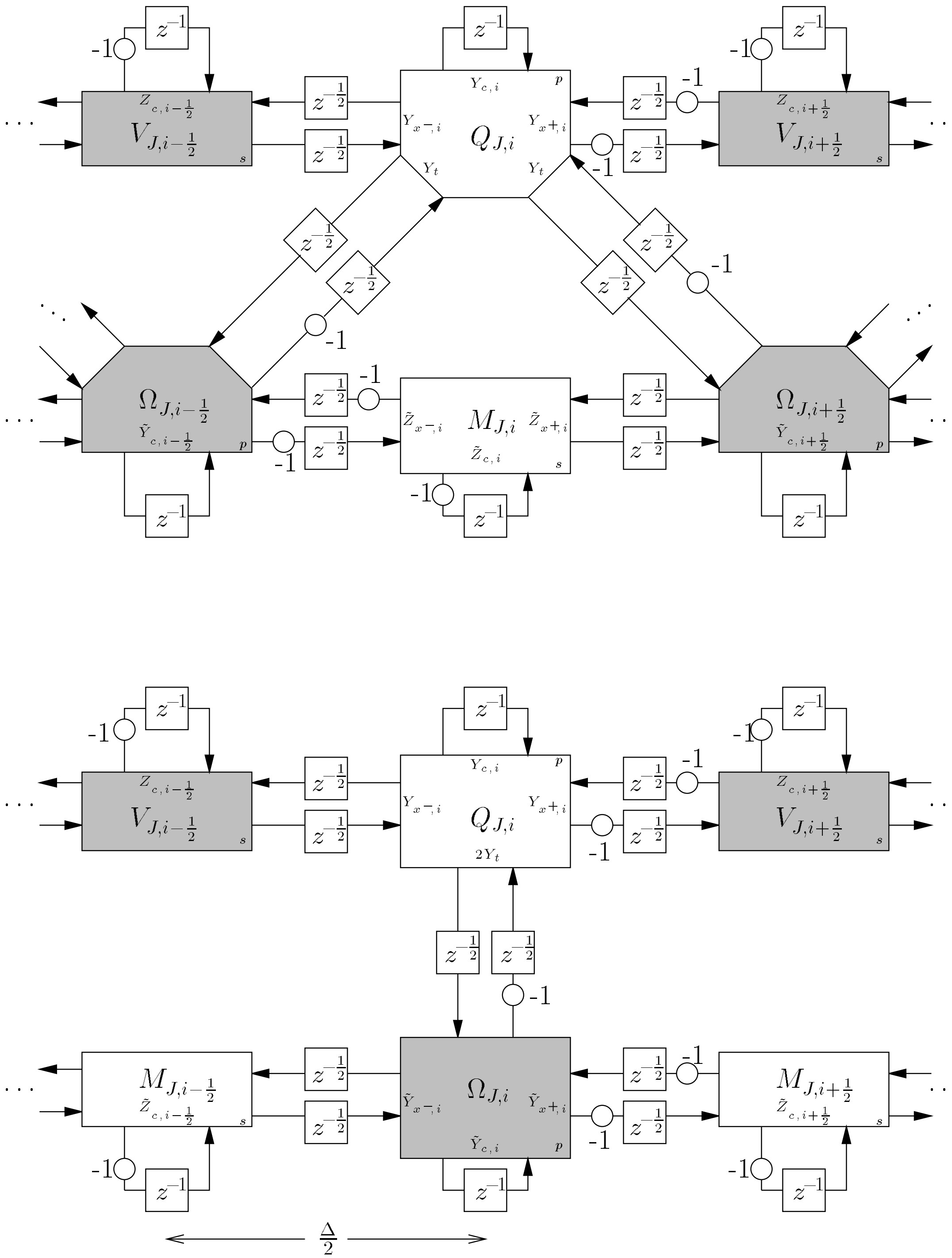


Figure 1.8: *Two Waveguide Networks for Timoshenko's Equations*

Multidimensional Circuit and Wave Digital Network for Timoshenko's Equations

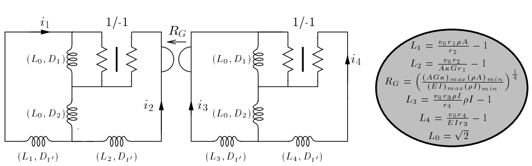


Figure 1.9: *MDKC for Timoshenko's Equations*

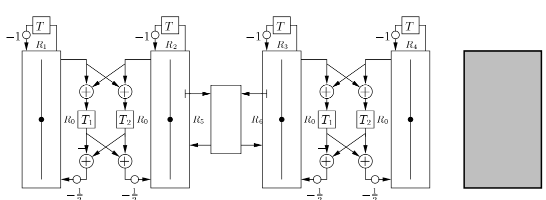


Figure 1.10: *MDWDF for Timoshenko's Equations*

Stability Condition for Waveguide Network for Timoshenko System

The maximum speeds of the Timoshenko Beam are

$$c_{1,max} = \max_x \sqrt{\frac{G\kappa}{\rho}} \quad c_{2,max} = \max_x \sqrt{\frac{E}{\rho}}$$

Stability conditions for the staggered waveguide mesh (TLM) are:

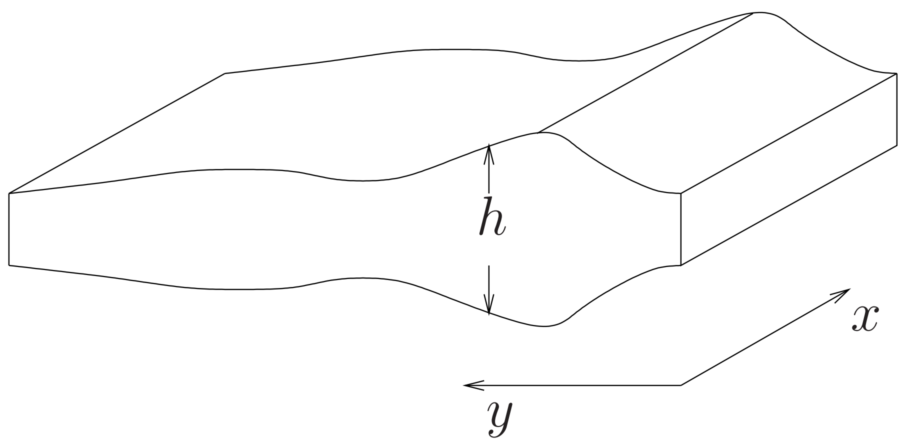
$$\Delta \; \geq \; T\max\left(\max_{x=\Delta i}\sqrt{\frac{1}{(\frac{\rho}{G\kappa}) - \frac{T}{2(\rho A)}}}, \max_{x=\Delta i}\sqrt{\frac{EI}{\rho I - \frac{T}{2}}}\right)$$

Also: A maximum permissible time-step (independent of Δ) Aproaches CFL bound as $T \to 0$.

- Stability bound is complicated by the staggering (in contrast to the transmission line case)
- Fix: Can make stability bound optimal by introducing vector travelling waves

Plates

- a 2D generalization of the beam.
- motion assumed transverse
- dispersive



- There is a direct 2D analogue of the classical Euler-Bernoulli Equations; same unbounded velocities, same strict stability bounds.

Mindlin's Plate Equations (1950s)

Mindlin generalized Timoshenko's Equations to 2D.

- The new dependent variables are w (height) and (ψ_x, ψ_y) (angles).
- plate thickness is $h(x, y)$
- Can be written as a system of eight PDEs:

$$\frac{\rho h^3}{12}\frac{\partial \omega_x}{\partial t} = \frac{\partial m_x}{\partial x} + \frac{\partial m_{xy}}{\partial y} - q_x$$

$$\rho h \frac{\partial v}{\partial t} = \frac{\partial q_x}{\partial x} + \frac{\partial q_y}{\partial y} \qquad \frac{\rho h^3}{12}\frac{\partial \omega_y}{\partial t} = \frac{\partial m_{xy}}{\partial x} + \frac{\partial m_y}{\partial y} - q_y$$

$$\frac{1}{\kappa^2 G h}\frac{\partial q_x}{\partial t} = \frac{\partial v}{\partial x} + \omega_x \qquad \frac{1}{D}\frac{\partial m_x}{\partial t} = \frac{\partial \omega_x}{\partial x} + \nu \frac{\partial \omega_y}{\partial y}$$

$$\frac{1}{\kappa^2 G h}\frac{\partial q_y}{\partial t} = \frac{\partial v}{\partial y} + \omega_y \qquad \frac{1}{D}\frac{\partial m_y}{\partial t} = \frac{\partial \omega_y}{\partial y} + \nu \frac{\partial \omega_x}{\partial x}$$

$$\frac{2}{D(1-\nu)}\frac{\partial m_{xy}}{\partial t} = \frac{\partial \omega_y}{\partial x} + \frac{\partial \omega_x}{\partial y}$$

with:

$$\omega_x = \frac{\partial \psi_x}{\partial t} \qquad \omega_y = \frac{\partial \psi_y}{\partial t} \qquad v = \frac{\partial w}{\partial t}$$

$$D = \frac{E h^3}{12(1-\nu^2)} \qquad \nu = \mathrm{Poisson's ratio} \qquad (4)$$

• A 2D "parallel plate" connected to a 5-variable system

Multidimensional Passive Circuit for Mindlin's Equations

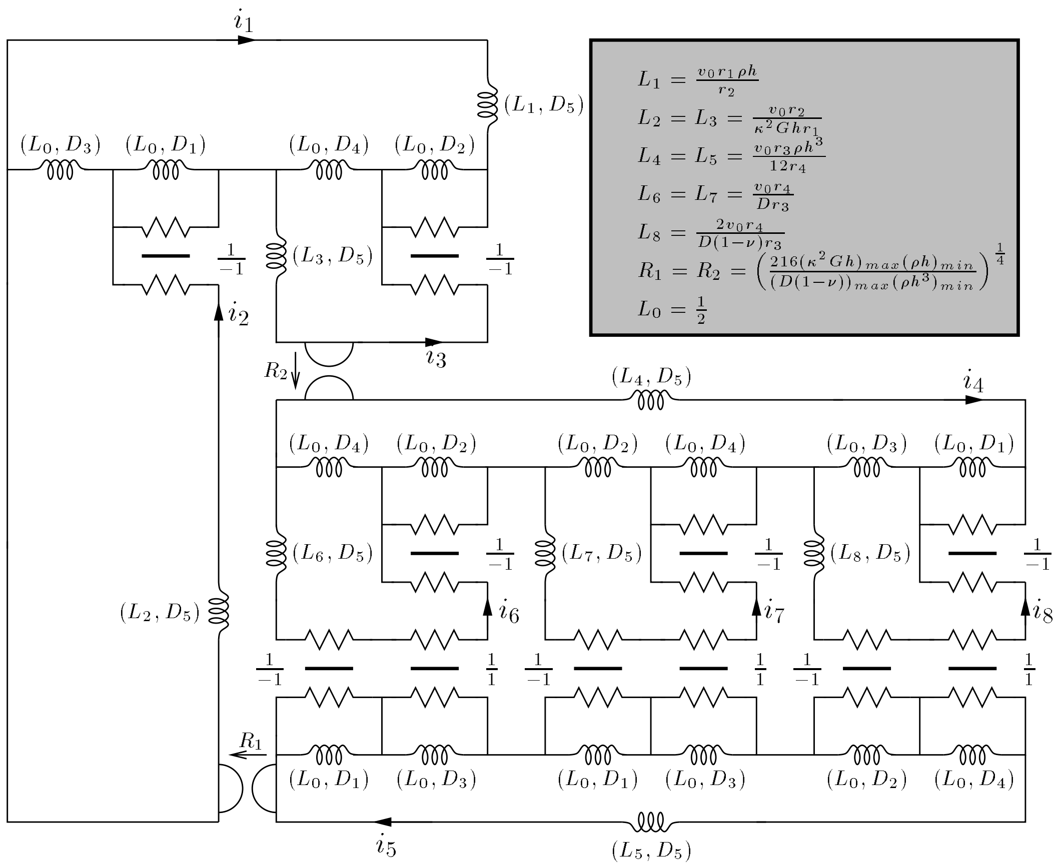


Figure 1.11: *MDKC for Mindlin's Plate Equations*

Multidimensional Wave Digital Network for Mindlin's Equations

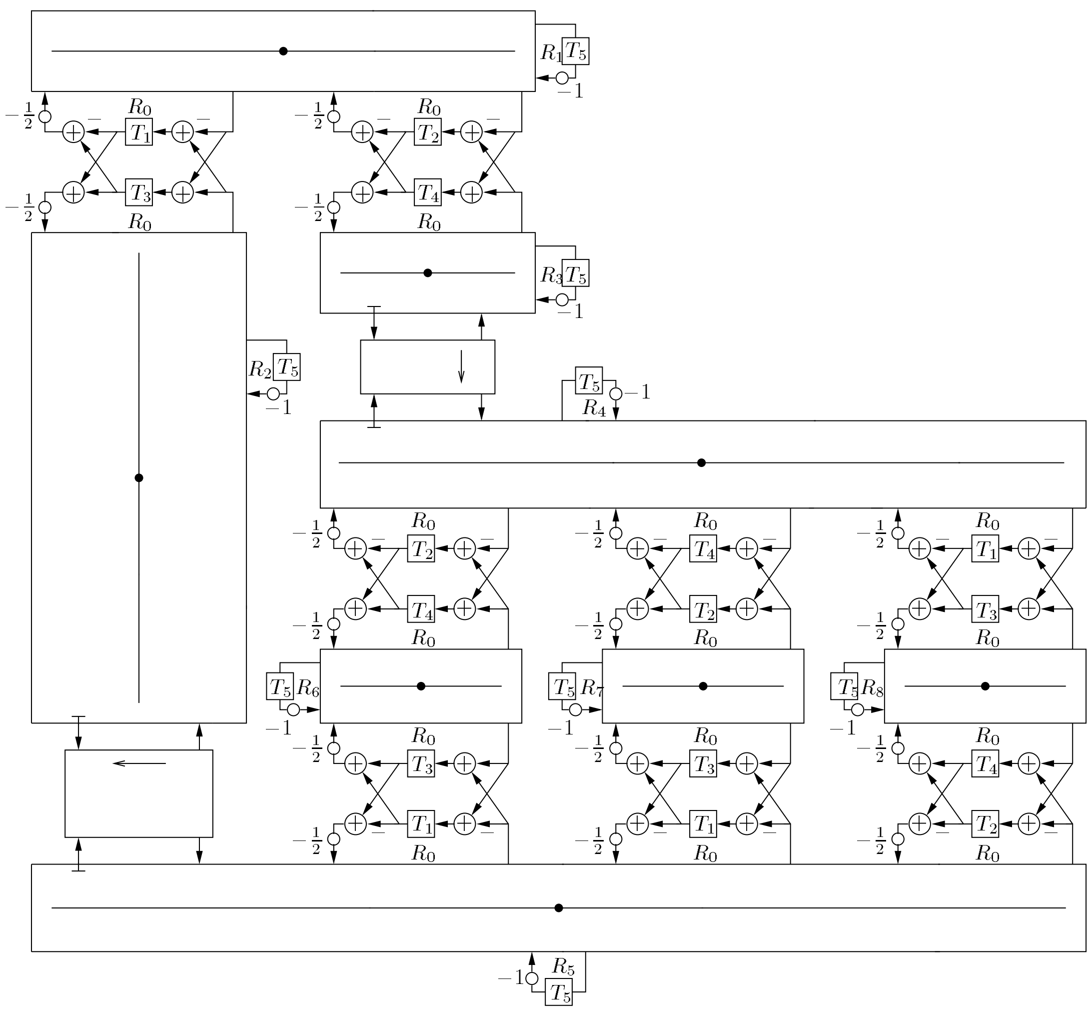


Figure 1.12: *MDWDF for Mindlin's Plate Equations*

Elastic Solids

In a linear, isotropic (but not necessarily homogeneous) elastic medium we have

- Stress-strain relation:

$$\begin{bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{zz} \\ \sigma_{xy} \\ \sigma_{xz} \\ \sigma_{yz} \end{bmatrix} = \begin{bmatrix} \lambda + 2\mu & \lambda & \lambda & 0 & 0 & 0 \\ \lambda & \lambda + 2\mu & \lambda & 0 & 0 & 0 \\ \lambda & \lambda & \lambda + 2\mu & 0 & 0 & 0 \\ 0 & 0 & 0 & 2\mu & 0 & 0 \\ 0 & 0 & 0 & 0 & 2\mu & 0 \\ 0 & 0 & 0 & 0 & 0 & 2\mu \end{bmatrix} \begin{bmatrix} e_{xx} \\ e_{yy} \\ e_{zz} \\ e_{xy} \\ e_{xz} \\ e_{yz} \end{bmatrix}$$

with

$$e_{ij} = \frac{1}{2}\left(u_{i,j} + u_{j,i}\right), \qquad i, j = x, y, z.$$

λ,μ are the Lame constants. (u_x, u_y, u_z) is the displacement vector at any point in the medium. Also, $\sigma_{ij} = \sigma_{ji}$ (Cons. of angular momentum)

- Force balance

$$
\begin{aligned}
\rho\frac{\partial^2 u_x}{\partial t^2} &= \frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \sigma_{xy}}{\partial y} + \frac{\partial \sigma_{xz}}{\partial z} \\
\rho\frac{\partial^2 u_y}{\partial t^2} &= \frac{\partial \sigma_{yx}}{\partial x} + \frac{\partial \sigma_{yy}}{\partial y} + \frac{\partial \sigma_{yz}}{\partial z} \\
\rho\frac{\partial^2 u_z}{\partial t^2} &= \frac{\partial \sigma_{zx}}{\partial x} + \frac{\partial \sigma_{zy}}{\partial y} + \frac{\partial \sigma_{zz}}{\partial z}
\end{aligned}
$$

(neglecting body forces)

First-order system

Rewriting as a first-order system in the stresses and velocities $(v_x, v_y, v_z) = \frac{\partial}{\partial t}(u_x, u_y, u_z)$ gives

$$\begin{bmatrix} \rho D_t & 0 & 0 & -D_x & 0 & 0 & -D_y & 0 & -D_z \\ 0 & \rho D_t & 0 & 0 & -D_y & 0 & -D_x & -D_z & 0 \\ 0 & 0 & \rho D_t & 0 & 0 & 0 & -D_z & -D_y & -D_x \\ -D_x & 0 & 0 & \beta D_t & \gamma D_t & \gamma D_t & 0 & 0 & 0 \\ 0 & -D_y & 0 & \gamma D_t & \beta D_t & \gamma D_t & 0 & 0 & 0 \\ 0 & 0 & -D_z & \gamma D_t & \gamma D_t & \beta D_t & 0 & 0 & 0 \\ -D_y & -D_x & 0 & 0 & 0 & 0 & \alpha D_t & 0 & 0 \\ 0 & -D_z & -D_y & 0 & 0 & 0 & 0 & \alpha D_t & 0 \\ -D_z & 0 & -D_x & 0 & 0 & 0 & 0 & 0 & \alpha D_t \end{bmatrix} \begin{bmatrix} v_x \\ v_y \\ v_z \\ \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{zz} \\ \sigma_{xy} \\ \sigma_{xz} \\ \sigma_{yz} \end{bmatrix} =$$

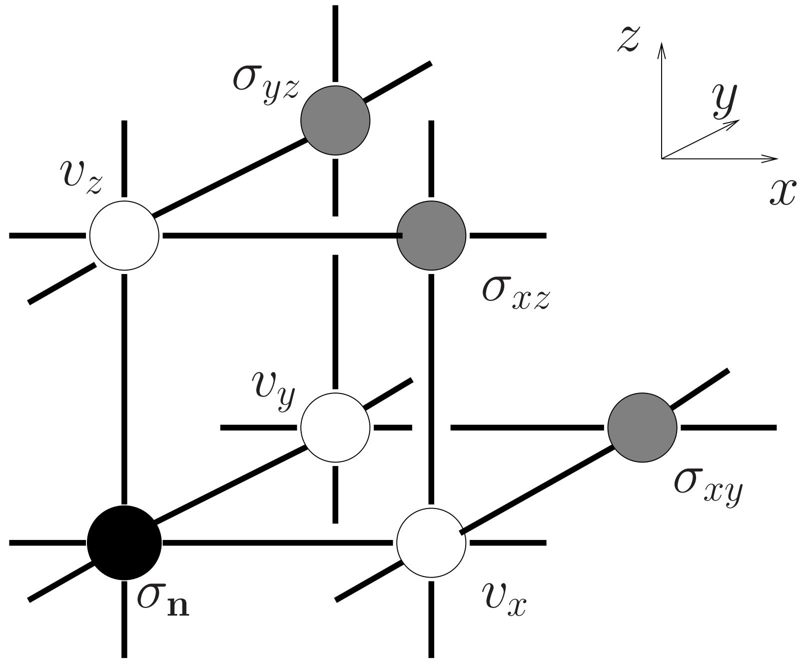
with

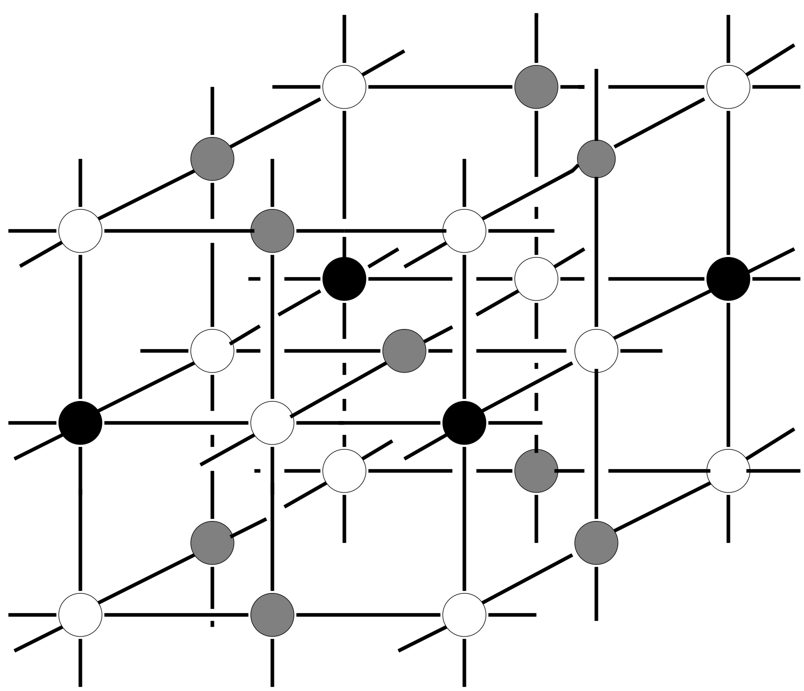
$$\alpha = \frac{1}{\mu} \qquad \beta = \frac{\mu + \lambda}{\mu\,(2\mu + 3\lambda)} \qquad \gamma = \frac{\lambda}{2\mu\,(2\mu + 3\lambda)}$$

- Symmetric hyperbolic even if α, β, λ and ρ are spatially varying.
- Coupling of time derivatives (this does not occur in the other systems seen so far). Thus a scalar staggered grid is out of the question (need σ_{xx}, σ_{yy} and σ_{zz} at same grid point, at same time step).

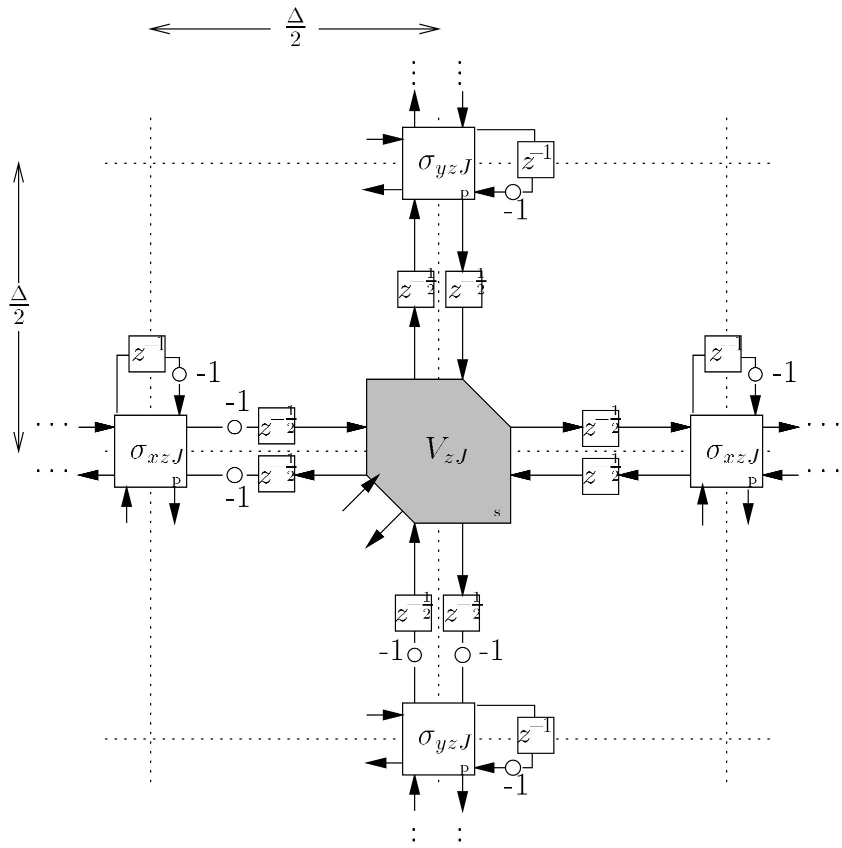
Fix: vectorized differences.

Computational Grid for Elastic Solids

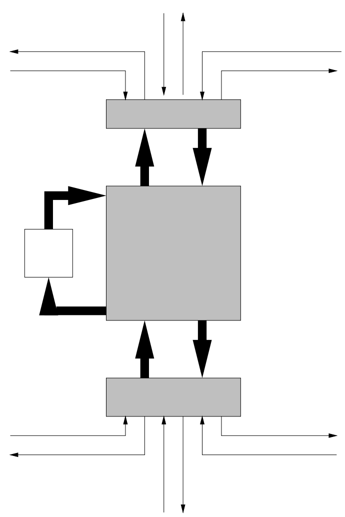




Solid Mesh



$\frac{\Delta}{2}$
$\frac{\Delta}{2}$
σ_{yzJ}
z^{-1}
-1
$z^{-\frac{1}{2}}$
σ_{xzJ}
V_{zJ}
p
s



Comments

- maximum allowable time-step is optimal:

$$v_0 \geq \max_{grid} \left(\sqrt{\frac{3(\lambda + 2\mu)}{\rho}} \right) = \sqrt{3} c_P$$

  under an appropriate choice of impedances in the network.

- Required vector wave variables, so need to pay attention to implementation of vector junction (in finite arithmetic).

- Multidimensional circuit approach works as well, but time-step is sub-optimal.

- Free boundary is simple to implement, in a perfectly lossless manner.

- Need to perform a time-integration (somehow) in order to get displacements.