Outline

- Physical model
- Equivalent Circuit
- Impedance analysis
- Digital waveguide model

### Ideal String Struck by a Mass

The ideal **struck string** is modeled as a simple initial velocity distribution along the ideal string:

- Hammer momentum transferred to string at time 0
- Hammer has bounced away after time 0 ("elastic collision")

We now derive a model in which the hammer strikes the string and remains in contact ("inelastic collision")

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**Physical Model (Ideal String Struck by a Mass)**

- String infinitely long — no gravity or air
- Wave impedance $R$
- Mass $m$ strikes string at $x = 0$ (a single point!)
- Collision speed $v_0$
- Horizontal motion neglected

At time 0, our model switches from

1. mass-in-flight and ideal string, to
2. two ideal strings joined by mass $m$ at $x = 0$

Note that the mass and two string-segment impedances are in series because they all move together

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**Equivalent Circuit (Ideal String Struck by a Mass)**

- Mass = *inductor* of $m$ Henrys (impedance $ms$)
- String endpoint = *resistor* of $R$ Ohms (impedance $R$)
- One common velocity for all three elements ⇒ series
- Series order is arbitrary
- Note that string wave impedance appears twice
- "Polarity" (reference direction) for each element defines "positive" current for that element: Current is positive when flowing from $+$ to $-$
- Positive current corresponds to a positive voltage drop
**Equivalent Circuit Analysis**

Kirchhoff’s Loop Rule:

The sum of voltages (“forces”) around any series loop is zero

So

\[ f_m(t) + f_R(t) + f_R(t) = 0 \]

Laplace transform:

\[ F_m(s) + 2F_R(s) = 0, \]

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**Laplace Transform Analysis**

From the equivalent circuit, we derived

\[ F_m(s) + 2F_R(s) = 0, \]

where

- \( F_m(s) \) = Laplace transform of upward mass force
- \( F_R(s) \) = Laplace transform of upward force on each string segment
- Recall:
  \[ F_m(s) \triangleq \mathcal{L}\{f_m\} \triangleq \int_0^\infty f_m(t)e^{-st}dt \]

Equations of motion for elements:

- String:
  \[ f_R(t) = R v(t) \leftrightarrow F_R(s) = RV(s) \]

where \( V(s) \) = \( \mathcal{L}\{v\} \) is the Laplace transform of \( v(t) \)

- Mass:
  \[ f_m(t) = m a(t) = m \dot{v}(t) \leftrightarrow F_m(s) = m[sV(s) - v_0] \]

(recall the Laplace-transform differentiation theorem)

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**Motion from Initial Conditions**

From the equivalent circuit, we derived

\[ F_m(s) + 2F_R(s) = 0 \]

Substituting gives

\[ m s V(s) - m v_0 + 2 RV(s) = 0 \]

Solving for \( V(s) \) gives

\[ V(s) = \frac{m v_0}{ms + 2R} \]

Since

\[ e^{-at}u(t) \leftrightarrow \frac{1}{s + a} \]

(where \( u(t) \) = unit step function), we find the velocity of the mass-string contact point to be

\[ v(t) = v_0 e^{-\frac{2R}{m}t}, \quad t \geq 0 \]

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**Observations**

We derived that the velocity of the mass-string contact point is

\[ v(t) = v_0 e^{-\frac{2R}{m}t}, \quad t \geq 0 \]

- At time zero the mass velocity is \( v_0 \), as it must be
- Velocity decays exponentially to zero with time-constant \( m/2R \)
- Decay rate is proportional to \( R/m \)
- Since \( R = \sqrt{K\epsilon} \), decay is faster if either string tension \( K \) or mass-density \( \epsilon \) is increased
- String displacement at \( x = 0 \) is given by the integral of transverse velocity:

\[ y(t, 0) = \int_0^t v(\tau) d\tau = v_0 \frac{m}{2R} \left[ 1 - e^{-\frac{2R}{m}t} \right] \]

where we define initial displacement to be \( y(0, 0) = 0 \)
Momentum Transfer

- Force applied to the two string endpoints by the mass is given by $-f_m(t) = 2Rv(t)$
- From Newton’s Law, $f = ma = m\ddot{v}$, momentum $mv$ is the time integral of force:

$$-\int_0^t f_m(\tau) d\tau = 2R \int_0^t v(\tau) d\tau = m v(t) \left(1 - e^{-\frac{2R}{m}t}\right)$$

- Thus, momentum delivered to the string by the mass starts at zero and grows as a relaxing exponential to $mv_0$ as $t \to \infty$
- Inelastic mass-string collision is not an instantaneous momentum transfer
- Mass momentum transfers exponentially over time
- This is why contact width can be zero

- In a real piano, the hammer, which strikes in an upward direction, falls away from the string a short time after collision, but it may remain in contact with the string for a substantial fraction of a period

Looking at the Mass from the String

Model:
- Mass divides string into two segments
- String segment model = digital waveguide
- Mass model = scattering junction

Force Equation:

$$f_m(t) + f_{1m}(t) + f_{2m}(t) = 0,$$ where

- $f_{1m}(t)$ = force applied by string-segment 1 to the mass
- $f_{2m}(t)$ = force applied by string-segment 2 to the mass
- $f_m(t)$ = mass inertial force
- Force is positive in the “up” direction

In the present problem, our force equation

$$f_m(t) + f_{1m}(t) + f_{2m}(t) = 0,$$

becomes, in terms of mass inertial force and string forces:

$$m\ddot{v}(t) + K y'_1(t, 0) - K y'_2(t, 0) = 0$$

or, using string force-wave notation:

$$f_m(t) - f_1(t) + f_2(t) = 0$$

These force relations can be checked individually:

- For string 1,

$$m\ddot{v}(t) + K y'_1(t, 0) = 0$$

$\Rightarrow$ positive slope on left accelerates mass down

- For string 2,

$$m\ddot{v}(t) - K y'_2(t, 0) = 0$$

$\Rightarrow$ positive slope on right accelerates mass up

Force Wave Variables

Force Equation:

$$f_m(t) + f_{1m}(t) + f_{2m}(t) = 0,$$

Traveling-Wave Decomposition of String Force:

$$f_1(t, x) = f_1^+(t - x/c) + f_1^-(t + x/c)$$
$$f_2(t, x) = f_2^+(t - x/c) + f_2^-(t + x/c)$$

String Force:

$$f(t, x) \triangleq -Ky'(t, x)$$

where

- $K$ = string tension
- $y'$ = string slope

Note that string force pulls up to the right.
That is, a string segment with negative slope pulls “up” to the right and “down” to the left:
Above we derived
\[ f_m(t) - f_1(t) + f_2(t) = 0 \]
We now perform the traveling-wave decompositions
\[ f_1 = f_1^+ + f_1^- \]
\[ f_2 = f_2^+ + f_2^- \]
and apply the Ohm’s law relations
\[ f_i^+ = R v_i^+ \]
\[ f_i^- = -R v_i^- , \quad i = 1, 2 \]
to obtain a digital waveguide model

- By symmetry, the mass must look identical from either string segment 1 or 2
- Therefore, let \( f_2^- \equiv 0 \) and excite from string 1

In the Laplace domain, we have
\[ 0 = F_m(s) - F_1^+(s) + F_2^-(s) \]
\[ = F_m(s) - (F_1^+ + F_1^-) + F_2^+ \quad \text{(since } F_2^- = 0) \]

By physical symmetry, the mass looks the same from string 2:

\[ \rho_2^v \triangleq \frac{V_2^+}{V_2^-} = \rho_1^v = -\frac{m_s}{m_s + 2R} \]

(reflectance of a mass \( m \) at the end of a string of wave impedance \( R \))

Limiting Behavior:
- When \( m = 0 \), reflectance is zero \( \text{ (no reflected wave) } \)
- When \( m \to \infty \), \( \rho_1^v \to -1 \) \( \text{ (rigid termination) } \)

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**Simplified Impedance Analysis**

The above results are quickly derived from the general reflection-coefficient for force waves (or voltage waves, pressure waves, etc.):

\[ \rho = \frac{R_2 - R_1}{R_2 + R_1} \]

where \( \rho \) = reflection coefficient of impedance \( R_2 \) as “seen” from impedance \( R_1 \)

When a force wave crosses from impedance \( R_1 \) to \( R_2 \) it splits into

1. a reflected wave \( f^- = \rho f^+ \) in \( R_1 \), and
2. a transmitted wave \( (1 + \rho) f^+ \) in \( R_2 \)

Therefore, a velocity wave \( v^+ \) splits into

1. reflected wave \( v^- = -\rho v^+ \) and
2. transmitted wave \( (1 - \rho)v^+ \)

These relations are of course unchanged in the Laplace domain, by linearity of the Laplace transform.
Mass Reflectance

In the mass-string-collision problem, we can immediately write down the force reflectance of the mass:

\[ \hat{\rho}(s) = \frac{R_{\text{mass+string}} - R_{\text{string}}}{R_{\text{mass+string}} + R_{\text{string}}} = \frac{(ms + R) - R}{(ms + R) + R} = \frac{ms}{ms + 2R} \]

The velocity reflectance is simply \(-\hat{\rho}(s)\), since

\[ \dot{\rho}(s) \triangleq \frac{F^-}{F^+} = -RV^- = \frac{V^-}{V^+} \]

- Simplified impedance analysis is a nice shortcut
- Be careful to correctly identify the impedance jump

Mass Transmittance

\[ \hat{\tau}_f(s) \triangleq \frac{F^+}{F^-} = \frac{F^+ + F^-}{F^+} = 1 + \frac{ms}{ms + 2R} \]

as is velocity transmittance:

\[ \hat{\tau}_v(s) \triangleq \frac{V^+}{V^-} = \frac{V^+ + V^-}{V^+} = 1 + \frac{ms}{ms + 2R} \]

For the mass-on-string problem:

\[ \hat{\tau}_f(s) = 1 + \hat{\rho}(s) = 1 + \frac{ms}{ms + 2R} = 2 \frac{ms + R}{ms + 2R} \]

\[ \hat{\tau}_v(s) = 1 - \hat{\rho}(s) = 1 - \frac{ms}{ms + 2R} = \frac{2R}{ms + 2R} \]

Limiting Behavior:

- For \( m = 0 \), both transmission filters become 1, as expected
- For \( m = \infty \), \( \hat{\tau}_v(s) \to 0 \), which makes good physical sense (infinite mass = rigid termination)
- For \( m = \infty \), \( \hat{\tau}_f(s) \to 2! \)
- Recall that signal power is force times velocity

Wave Scattering by a Mass on a String

We have derived the reflectance and transmittance of a mass \( m \) as seen from either string at impedance \( R \). We can now derive the complete scattering relations:

For force waves, the outgoing waves are:

\[ F_1^-(s) = \hat{\rho}(s)F_1^+(s) + \hat{\tau}_f(s)F_2^-(s) \]
\[ F_2^+(s) = \hat{\tau}_f(s)F_1^+(s) + \hat{\rho}(s)F_2^+(s) \]

where the incoming waves are \( F_1^+ \) and \( F_2^- \), and

\[ \hat{\rho}(s) = \frac{ms}{ms + 2R} \] (force reflectance)
\[ \hat{\tau}_f(s) = 1 + \hat{\rho}(s) = \frac{2(ms + R)}{ms + 2R} \] (force transmittance)

We may say that the mass creates a dynamic scattering junction on the string.

One-Filter Scattering Junction

The scattering relations above can be said to be in "Kelly-Lochbaum form." The general relation \( \hat{\tau}_f = 1 + \hat{\rho} \) can be used to simplify to a one-filter dynamic scattering junction

\[ F_1^- = \hat{\rho}F_1^+ + (1 + \hat{\rho})F_2^- = F_2^- + \hat{\rho} \cdot (F_1^+ + F_2^-) \]
\[ F_2^+ = (1 + \hat{\rho})F_1^+ + \hat{\rho}F_2^- = F_1^+ + \hat{\rho} \cdot (F_1^+ + F_2^-) \]

The one-filter form follows from the observation that \( \hat{\rho} \cdot (F_1^+ + F_2^-) \) appears in both computations, and therefore need only be implemented once:

\[ F^+ \triangleq \hat{\rho} \cdot (F_1^+ + F_2^-) \]
\[ F_1^- = \hat{\rho}F_1^+ + (1 + \hat{\rho})F_2^- = F_2^- + F^+ \]
\[ F_2^+ = (1 + \hat{\rho})F_1^+ + \hat{\rho}F_2^- = F_1^+ + F^+ \]

Signal Flow Diagram:
Digital Waveguide Model: Ideal String Struck by a Mass

We have derived an “analog waveguide model” for the ideal string struck by an ideal mass. We now digitize that model.

- Delays implemented using digital delay lines
- Mass reflectance digitize using the bilinear transform
- Force waves chosen

Mass Reflectance:
\[ \hat{\rho}(s) = \frac{ms}{ms + 2R} = \frac{1}{1 + \frac{2R}{ms}} \]  

Bilinear Transform:
\[ s = \frac{2}{T} \frac{1 - z^{-1}}{1 + z^{-1}} \]
where \( T \) = sampling interval, yields
\[ \hat{\rho}_{d}(z) = \frac{1}{1 + \frac{2RT}{m} \frac{1 + z^{-1}}{1 - z^{-1}}} = g \frac{1 - z^{-1}}{1 - pz^{-1}} \]

Digitized Mass Reflectance
\[ \hat{\rho}_{d}(z) = g \frac{1 - z^{-1}}{1 - pz^{-1}} \]
where
\[ g = \frac{1}{1 + \frac{RT}{m}} < 1 \]
\[ p = \frac{1 - \frac{RT}{m}}{1 + \frac{RT}{m}} < 1 \]

Thus, the mass reflectance is a one-pole, one-zero filter:
- Zero at dc
- Real pole close to dc at high sampling rates
- Unity gain at \( f_s/2 \)
- Classic dc-blocking filter

Why is the Mass Reflectance a DC-Blocker?

Physical intuition:
- The mass reflectance is zero at dc because sufficiently slow force waves can freely move a mass of any finite size \( \Rightarrow R \) dominates
- The reflectance is 1 at infinite frequency because there is no time for the mass to move before it is pushed in the opposite direction

In summary, the mass becomes a rigid termination at infinite frequency, and negligible at zero frequency — i.e., the mass reflectance is a “dc blocker”.

Final Digital Waveguide Model (Mass-Terminated String)

Final Notes
- Mass reflectance uses a warped, unaliased frequency axis (due to the bilinear transform) — it can be viewed as a “Wave Digital Filter” (WDF) mass model
- Delay lines use an unwarped, aliased frequency axis (simple sampling) — “digital waveguide model”
- Frequency axes align well at low frequencies, or given sufficient oversampling
• Mass model is 1st-order in both analog and digital domains
• A higher order digital filter can be used for the mass to improve frequency response accuracy at high frequency
  – Antisymmetric FIR preserves purely imaginary phase
  – Keep impedance phase within \((-\pi/2, \pi/2)\) for “passivity”
  – Example:
    https://ccrma.stanford.edu/~jos/FiniteDifferer