

# The Laplace Transform

## Music 420: The Laplace Transform

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### Outline

- Definition
- Linearity and Differentiation Theorem
- Examples of Mass-Spring system analysis

The one-sided (unilateral) *Laplace transform* of a signal  $x(t)$  is defined as

$$X(s) \triangleq \mathcal{L}_s\{x\} \triangleq \int_0^\infty x(t)e^{-st} dt$$

- $t$  = time in seconds
- $s = \sigma + j\omega$  is a complex variable
- Appropriate for *causal* signals

When evaluated along the  $j\omega$  axis (*i.e.*,  $\sigma = 0$ ), the Laplace Transform reduces to the unilateral *Fourier transform*:

$$X(j\omega) = \int_0^\infty x(t)e^{-j\omega t} dt$$

Thus, the Laplace transform generalizes the Fourier transform from the real line (the frequency axis) to the entire complex plane.

The Fourier transform equals the Laplace transform evaluated along the  $j\omega$  axis in the complex  $s$  plane

The Laplace Transform can also be seen as the Fourier transform of an *exponentially windowed* causal signal  $x(t)$

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## Relation to the $z$ Transform

The Laplace transform is used to analyze *continuous-time* systems. Its discrete-time counterpart is the  $z$  transform:

$$X_d(z) \triangleq \sum_{n=0}^\infty x_d(nT)z^{-n}$$

If we define  $z = e^{sT}$ , the  $z$  transform becomes proportional to the Laplace transform of a sampled continuous-time signal:

$$X_d(e^{sT}) = \sum_{n=0}^\infty x_d(nT)e^{-snT}$$

As the sampling interval  $T$  goes to zero, we have

$$\begin{aligned} \lim_{T \rightarrow 0} X_d(e^{sT})T &= \lim_{T \rightarrow 0} \sum_{n=0}^\infty x_d(t_n) e^{-st_nT} \\ &= \int_0^\infty x_d(t) e^{-st} dt \triangleq X(s) \end{aligned}$$

where  $t_n \triangleq nT$  and  $\Delta t \triangleq t_{n+1} - t_n = T$ .

In summary,

the  $z$  transform (times the sampling interval  $T$ ) of a discrete time signal  $x_d(nT)$  approaches, as  $T \rightarrow 0$ , the Laplace Transform of the underlying continuous-time signal  $x_d(t)$ .

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Note that the  $z$  plane and  $s$  plane are related by

$$z = e^{sT}$$

In particular, the discrete-time frequency axis  $\omega_d \in (-\pi/T, \pi/T)$  and continuous-time frequency axis  $\omega_a \in (-\infty, \infty)$  are related by

$$e^{j\omega_d T} = e^{j\omega_a T}$$

For the mapping  $z = e^{sT}$  from the  $s$  plane to the  $z$  plane to be invertible, it is necessary that  $X(j\omega_a)$  be zero for all  $|\omega_a| \geq \pi/T$ . If this is true, we say  $x(t)$  is *bandlimited below half the sampling rate*. As is well known, this condition is necessary to prevent *aliasing* when sampling the continuous-time signal  $x(t)$  at the rate  $f_s = 1/T$  to produce  $x(nT)$ ,  $n = 0, 1, 2, \dots$

## Two Laplace Transform Theorems

### Linearity

The Laplace transform is a *linear operator*:

$$\alpha x(t) + \beta y(t) \longleftrightarrow \alpha X(s) + \beta Y(s)$$

*Proof:* Let

$$w(t) = \alpha x(t) + \beta y(t),$$

where  $\alpha$  and  $\beta$  are real or complex constants. Then

$$\begin{aligned} W(s) &\triangleq \mathcal{L}_s\{w\} \triangleq \mathcal{L}_s\{\alpha x(t) + \beta y(t)\} \\ &\triangleq \int_0^\infty [\alpha x(t) + \beta y(t)] e^{-st} dt \\ &= \alpha \int_0^\infty x(t) e^{-st} dt + \beta \int_0^\infty y(t) e^{-st} dt \\ &\triangleq \alpha X(s) + \beta Y(s). \end{aligned}$$

Thus, linearity of the Laplace transform follows immediately from linearity of integration

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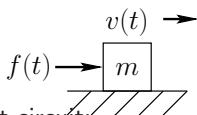
## Laplace Analysis of Linear Systems

The differentiation theorem converts *differential equations* into *algebraic* equations, which are easier to solve.

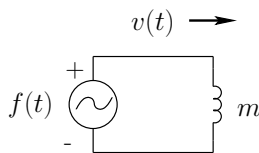
### Example: Force-Driven Mass

Consider a free mass driven by an external force along an ideal frictionless surface in one dimension:

Physical diagram:



Electrical equivalent circuit:



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## Differentiation

The *differentiation theorem* for Laplace transforms:

$$\dot{x}(t) \leftrightarrow sX(s) - x(0)$$

where  $\dot{x}(t) \triangleq \frac{d}{dt}x(t)$ , and  $x(t)$  is any differentiable function that approaches zero as  $t$  goes to infinity.

Operator notation:

$$\mathcal{L}_s\{\dot{x}\} = sX(s) - x(0).$$

*Proof:* Immediate from integration by parts:

$$\begin{aligned} \mathcal{L}_s\{\dot{x}\} &\triangleq \int_0^\infty \dot{x}(t) e^{-st} dt \\ &= x(t) e^{-st} \Big|_0^\infty - \int_0^\infty x(t) (-s) e^{-st} dt \\ &= sX(s) - x(0) \end{aligned}$$

since  $x(\infty) = 0$  by assumption

**Corollary: Integration Theorem:**

$$\mathcal{L}_s\left\{\int_0^t x(\tau) d\tau\right\} = \frac{X(s)}{s}$$

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## Force-Driven Mass Analysis

Note that in the electrical equivalent circuit

- Driving force = *voltage source* emitting  $f(t)$  volts
- Mass = *inductor* of  $L = m$  Henrys.

From Newton's second law of motion " $f = ma$ ", we have

$$f(t) = m a(t) \triangleq m \dot{v}(t) \triangleq m \ddot{x}(t).$$

Taking the unilateral Laplace transform and applying the differentiation theorem twice yields

$$\begin{aligned} F(s) &= m \mathcal{L}_s\{\ddot{x}\} \\ &= m [s \mathcal{L}_s\{\dot{x}\} - \dot{x}(0)] \\ &= m \{s [s X(s) - x(0)] - \dot{x}(0)\} \\ &= m [s^2 X(s) - s x(0) - \dot{x}(0)]. \end{aligned}$$

Thus, given

- $F(s)$  = Laplace transform of the driving force  $f(t)$ ,
- $x(0)$  = initial mass position, and
- $\dot{x}(0) \triangleq v(0)$  = initial mass velocity,

we can solve algebraically for  $X(s)$ , the Laplace transform of the mass position for all  $t \geq 0$

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## Force-Driven Mass Analysis, Continued

If the applied external force  $f(t)$  is zero, we obtain

$$X(s) = \frac{x(0)}{s} + \frac{\dot{x}(0)}{s^2} = \frac{x(0)}{s} + \frac{v(0)}{s^2}.$$

Since  $1/s$  is the Laplace transform of the Heaviside unit-step function

$$u(t) \triangleq \begin{cases} 0, & t < 0 \\ 1, & t \geq 0 \end{cases},$$

we find that the position of the mass  $x(t)$  is given for all time by

$$x(t) = x(0)u(t) + v(0)tu(t).$$

- A nonzero initial position  $x(0) = x_0$  and zero initial velocity  $v(0) = 0$  results in  $x(t) = x_0$  for all  $t \geq 0$  (mass “just sits there”)
- Similarly, any initial velocity  $v(0)$  is integrated with respect to time (mass moves forever at initial velocity)

In summary, we used the Laplace transform to solve for the motion of a simple physical system (an ideal mass) in response to initial conditions (no external driving forces).

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$x(t)$  is both the position of the mass and compression of the spring at time  $t$ .)

Taking the Laplace transform of both sides of this differential equation gives

$$\begin{aligned} 0 &= \mathcal{L}_s\{m\ddot{x} + kx\} \\ &= m\mathcal{L}_s\{\ddot{x}\} + k\mathcal{L}_s\{x\} \quad (\text{linearity}) \\ &= m[s\mathcal{L}_s\{\dot{x}\} - \dot{x}(0)] + kX(s) \quad (\text{differentiation theorem}) \\ &= m\{s[sX(s) - x(0)] - \dot{x}(0)\} + kX(s) \quad (\text{diff. thm again}) \\ &= ms^2X(s) - msx(0) - m\dot{x}(0) + kX(s) \end{aligned}$$

Let  $x(0) = x_0$  and  $\dot{x}(0) = \dot{x}_0 = v_0$  for simplicity.

Solving for  $X(s)$  gives

$$X(s) = \frac{sx_0 + v_0}{s^2 + \frac{k}{m}} \triangleq \frac{r}{s + j\omega_0} + \frac{\bar{r}}{s - j\omega_0}, \quad \omega_0 \triangleq \sqrt{k/m},$$

$$r = \frac{x_0}{2} + j\frac{v_0}{2\omega_0} \triangleq R_r e^{j\theta_r}, \quad \text{with}$$

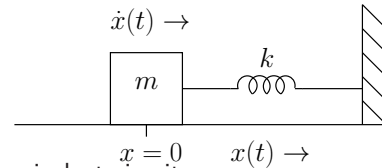
$$R_r \triangleq \frac{\sqrt{v_0^2 + \omega_0^2 x_0^2}}{2\omega_0}, \quad \theta_r \triangleq \tan^{-1}\left(\frac{v_0}{\omega_0 x_0}\right)$$

denoting the modulus and angle of the pole residue  $r$ , respectively.

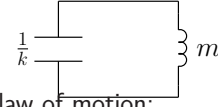
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## Mass-Spring Oscillator Time-Domain Solution

Consider now the mass-spring oscillator:



Electrical equivalent-circuit:



Newton's second law of motion:

$$f_m(t) = m\ddot{x}(t).$$

Hooke's law for ideal springs:

$$f_k(t) = kx(t)$$

Newton's third law of motion:

$$\begin{aligned} f_m(t) + f_k(t) &= 0 \\ \Rightarrow m\ddot{x}(t) + kx(t) &= 0 \end{aligned}$$

We have thus derived a second-order differential equation governing the motion of the mass and spring. (Note that

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## Mass-Spring Oscillator Analysis, Continued

We can quickly verify that

$$\boxed{e^{-at}u(t) \longleftrightarrow \frac{1}{s+a}}$$

where  $u(t)$  is the Heaviside unit step function which steps from 0 to 1 at time 0.

By linearity, the solution for the motion of the mass is

$$\begin{aligned} x(t) &= re^{-j\omega_0 t} + \bar{r}e^{j\omega_0 t} = 2\text{re}\{re^{-j\omega_0 t}\} = 2R_r \cos(\omega_0 t - \theta_r) \\ &= \frac{\sqrt{v_0^2 + \omega_0^2 x_0^2}}{\omega_0} \cos\left[\omega_0 t - \tan^{-1}\left(\frac{v_0}{\omega_0 x_0}\right)\right] \end{aligned}$$

If the initial velocity is zero ( $v_0 = 0$ ), the above formula reduces to  $x(t) = x_0 \cos(\omega_0 t)$  and the mass simply oscillates sinusoidally at frequency  $\omega_0 = \sqrt{k/m}$ , starting from its initial position  $x_0$ . If instead the initial position is  $x_0 = 0$ , we obtain

$$\begin{aligned} x(t) &= \frac{v_0}{\omega_0} \sin(\omega_0 t) \\ \Rightarrow v(t) &= v_0 \cos(\omega_0 t). \end{aligned}$$

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