MUS420/EE367A Supplementary:
The Laplace Transform

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Outline

- Definition
- Linearity and Differentiation Theorem
- Examples of Mass-Spring system analysis

The Laplace Transform

The one-sided (unilateral) *Laplace transform* of a signal $x(t)$ is defined as

$$X(s) \triangleq \mathcal{L}_s\{x\} \triangleq \int_0^\infty x(t)e^{-st}dt$$

- $t =$ time in seconds
- $s = \sigma + j\omega$ is a complex variable
- Appropriate for *causal* signals

When evaluated along the $j\omega$ axis (*i.e.*, $\sigma = 0$), the Laplace Transform reduces to the unilateral *Fourier transform*:

$$X(j\omega) = \int_0^\infty x(t)e^{-j\omega t}dt$$

Thus, the Laplace transform generalizes the Fourier transform from the real line (the frequency axis) to the entire complex plane.

The Fourier transform equals the Laplace transform evaluated along the $j\omega$ axis in the complex s plane

The Laplace Transform can also be seen as the Fourier transform of an *exponentially windowed* causal signal $x(t)$

Relation to the z Transform

The Laplace transform is used to analyze *continuous-time* systems. Its discrete-time counterpart is the z transform:

$$X_d(z) \triangleq \sum_{n=0}^{\infty} x_d(nT)z^{-n}$$

If we define $z = e^{sT}$, the z transform becomes proportional to the Laplace transform of a sampled continuous-time signal:

$$X_d(e^{sT}) = \sum_{n=0}^{\infty} x_d(nT)e^{-snT}$$

As the sampling interval T goes to zero, we have

$$\begin{aligned} \lim_{T\to 0} X_d(e^{sT})T &= \lim_{\Delta t\to 0} \sum_{n=0}^{\infty} \left[\frac{x_d(t_n)}{\Delta t}\right] e^{-st_n}\Delta t \\ &= \int_0^\infty x_d(t)e^{-st}dt \triangleq X(s) \end{aligned}$$

where $t_n \triangleq nT$ and $\Delta t \triangleq t_{n+1} - t_n = T$.

In summary,

the z transform (times the sampling interval T) of a discrete time signal $x_d(nT)$ approaches, as $T \to 0$, the Laplace Transform of the underlying continuous-time signal $x_d(t)$.

Note that the z plane and s plane are related by

$$\boxed{z = e^{sT}}$$

In particular, the discrete-time frequency axis $\omega_d \in (-\pi/T, \pi/T)$ and continuous-time frequency axis $\omega_a \in (-\infty, \infty)$ are related by

$$\boxed{e^{j\omega_d T} = e^{j\omega_a T}}$$

For the mapping $z = e^{sT}$ from the s plane to the z plane to be invertible, it is necessary that $X(j\omega_a)$ be zero for all $|\omega_a| \geq \pi/T$. If this is true, we say $x(t)$ is *bandlimited below half the sampling rate*. As is well known, this condition is necessary to prevent *aliasing* when sampling the continuous-time signal $x(t)$ at the rate $f_s = 1/T$ to produce $x(nT)$, $n = 0, 1, 2, \ldots$

Two Laplace Transform Theorems

Linearity

The Laplace transform is a *linear operator*:

$$\boxed{\alpha x(t) + \beta y(t) \longleftrightarrow \alpha X(s) + \beta Y(s)}$$

Proof: Let

$$w(t) = \alpha x(t) + \beta y(t),$$

where α and β are real or complex constants. Then

$$\begin{aligned} W(s) &\triangleq \mathcal{L}_s\{w\} \triangleq \mathcal{L}_s\{\alpha x(t) + \beta y(t)\} \\ &\triangleq \int_0^\infty [\alpha x(t) + \beta y(t)]\, e^{-st} dt \\ &= \alpha \int_0^\infty x(t) e^{-st} dt + \beta \int_0^\infty y(t) e^{-st} dt \\ &\triangleq \alpha X(s) + \beta Y(s). \end{aligned}$$

Thus, linearity of the Laplace transform follows immediately from linearity of integration

Differentiation

The *differentiation theorem* for Laplace transforms:

$$\boxed{\dot{x}(t) \leftrightarrow sX(s) - x(0)}$$

where $\dot{x}(t) \triangleq \frac{d}{dt} x(t)$, and $x(t)$ is any differentiable function that approaches zero as t goes to infinity.

Operator notation:

$$\boxed{\mathcal{L}_s\{\dot{x}\} = sX(s) - x(0).}$$

Proof: Immediate from integration by parts:

$$\begin{aligned} \mathcal{L}_s\{\dot{x}\} &\triangleq \int_0^\infty \dot{x}(t) e^{-st} dt \\ &= x(t) e^{-st}\Big|_0^\infty - \int_0^\infty x(t)(-s) e^{-st} dt \\ &= sX(s) - x(0) \end{aligned}$$

since $x(\infty) = 0$ by assumption

Corollary: *Integration Theorem:*

$$\boxed{\mathcal{L}_s\left\{\int_0^t x(\tau) d\tau\right\} = \frac{X(s)}{s}}$$

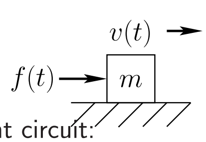
Laplace Analysis of Linear Systems

The differentiation theorem converts *differential equations* into *algebraic* equations, which are easier to solve.

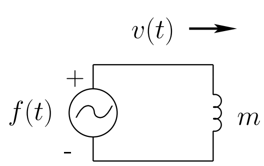
Example: Force-Driven Mass

Consider a free mass driven by an external force along an ideal frictionless surface in one dimension:

Physical diagram:



Electrical equivalent circuit:



Force-Driven Mass Analysis

Note that in the electrical equivalent circuit

- Driving force $=$ *voltage source* emitting $f(t)$ *volts*
- Mass $=$ *inductor* of $L = m$ *Henrys*.

From Newton's second law of motion "$f = ma$", we have

$$f(t) = m\, a(t) \triangleq m\, \dot{v}(t) \triangleq m\, \ddot{x}(t).$$

Taking the unilateral Laplace transform and applying the differentiation theorem twice yields

$$\begin{aligned} F(s) &= m\, \mathcal{L}_s\{\ddot{x}\} \\ &= m\, [\, s\mathcal{L}_s\{\dot{x}\} - \dot{x}(0)] \\ &= m\, \{\, s\, [s\, X(s) - x(0)] - \dot{x}(0)\} \\ &= m\, \left[s^2\, X(s) - s\, x(0) - \dot{x}(0)\right]. \end{aligned}$$

Thus, given

- $F(s) =$ Laplace transform of the driving force $f(t)$,
- $x(0) =$ initial mass position, and
- $\dot{x}(0) \triangleq v(0) =$ initial mass velocity,

we can solve algebraically for $X(s)$, the Laplace transform of the mass position for all $t \geq 0$

Force-Driven Mass Analysis, Continued

If the applied external force $f(t)$ is zero, we obtain

$$X(s) = \frac{x(0)}{s} + \frac{\dot{x}(0)}{s^2} = \frac{x(0)}{s} + \frac{v(0)}{s^2}.$$

Since $1/s$ is the Laplace transform of the Heaviside unit-step function

$$u(t) \triangleq \begin{cases} 0, & t < 0 \\ 1, & t \geq 0 \end{cases},$$

we find that the position of the mass $x(t)$ is given for all time by

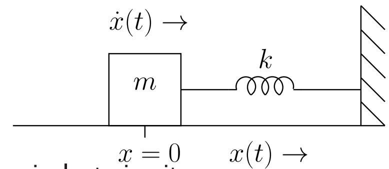
$$x(t) = x(0)\,u(t) + v(0)\,t\,u(t).$$

- A nonzero initial position $x(0) = x_0$ and zero initial velocity $v(0) = 0$ results in $x(t) = x_0$ for all $t \geq 0$ (mass "just sits there")
- Similarly, any initial velocity $v(0)$ is integrated with respect to time (mass moves forever at initial velocity)

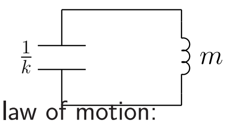
In summary, we used the Laplace transform to solve for the motion of a simple physical system (an ideal mass) in response to initial conditions (no external driving forces).

Mass-Spring Oscillator Time-Domain Solution

Consider now the mass-spring oscillator:



Electrical equivalent-circuit:



Newton's second law of motion:

$$f_m(t) = m\ddot{x}(t).$$

Hooke's law for ideal springs:

$$f_k(t) = kx(t)$$

Newton's third law of motion:

$$\begin{aligned} f_m(t) + f_k(t) &= 0 \\ \Rightarrow\ m\ddot{x}(t) + kx(t) &= 0 \end{aligned}$$

We have thus derived a second-order differential equation governing the motion of the mass and spring. (Note that

$x(t)$ is both the position of the mass and compression of the spring at time t.)

Taking the Laplace transform of both sides of this differential equation gives

$$\begin{aligned} 0 &= \mathcal{L}_s\{m\ddot{x} + kx\} \\ &= m\mathcal{L}_s\{\ddot{x}\} + k\mathcal{L}_s\{x\} \quad \text{(linearity)} \\ &= m\left[s\mathcal{L}_s\{\dot{x}\} - \dot{x}(0)\right] + kX(s) \quad \text{(differentiation theorem)} \\ &= m\left\{s\left[sX(s) - x(0)\right] - \dot{x}(0)\right\} + kX(s) \quad \text{(diff. thm again)} \\ &= ms^2X(s) - msx(0) - m\dot{x}(0) + kX(s) \end{aligned}$$

Let $x(0) = x_0$ and $\dot{x}(0) = \dot{x}_0 = v_0$ for simplicity.
Solving for $X(s)$ gives

$$X(s) = \frac{sx_0 + v_0}{s^2 + \frac{k}{m}} \triangleq \frac{r}{s + j\omega_0} + \frac{\overline{r}}{s - j\omega_0}, \quad \omega_0 \triangleq \sqrt{k/m},$$

$$r = \frac{x_0}{2} + j\frac{v_0}{2\omega_0} \triangleq R_r e^{j\theta_r}, \quad \text{with}$$

$$R_r \triangleq \frac{\sqrt{v_0^2 + \omega_0{}^2 x_0^2}}{2\omega_0}, \qquad \theta_r \triangleq \tan^{-1}\left(\frac{v_0}{\omega_0 x_0}\right)$$

denoting the modulus and angle of the pole residue r, respectively.

Mass-Spring Oscillator Analysis, Continued

We can quickly verify that

$$\boxed{e^{-at}u(t) \longleftrightarrow \frac{1}{s+a}}$$

where $u(t)$ is the Heaviside unit step function which steps from 0 to 1 at time 0.

By linearity, the solution for the motion of the mass is

$$\begin{aligned} x(t) &= re^{-j\omega_0 t} + \overline{r}e^{j\omega_0 t} = 2\text{re}\left\{re^{-j\omega_0 t}\right\} = 2R_r\cos(\omega_0 t - \theta_r) \\ &= \frac{\sqrt{v_0^2 + \omega_0{}^2 x_0^2}}{\omega_0}\cos\left[\omega_0 t - \tan^{-1}\left(\frac{v_0}{\omega_0 x_0}\right)\right] \end{aligned}$$

If the initial velocity is zero ($v_0 = 0$), the above formula reduces to $x(t) = x_0\cos(\omega_0 t)$ and the mass simply oscillates sinusoidally at frequency $\omega_0 = \sqrt{k/m}$, starting from its initial position x_0. If instead the initial position is $x_0 = 0$, we obtain

$$\begin{aligned} x(t) &= \frac{v_0}{\omega_0}\sin(\omega_0 t) \\ \Rightarrow\ v(t) &= v_0\cos(\omega_0 t). \end{aligned}$$