Music 420: The Laplace Transform

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Outline

• Definition
• Linearity and Differentiation Theorem
• Examples of Mass-Spring system analysis
The Laplace Transform

The one-sided (unilateral) Laplace transform of a signal $x(t)$ is defined as

$$X(s) \doteq \mathcal{L}_s\{x\} \doteq \int_0^\infty x(t)e^{-st}dt$$

- $t =$ time in seconds
- $s = \sigma + j\omega$ is a complex variable
- Appropriate for causal signals

When evaluated along the $j\omega$ axis (i.e., $\sigma = 0$), the Laplace Transform reduces to the unilateral Fourier transform:

$$X(j\omega) = \int_0^\infty x(t)e^{-j\omega t}dt$$

Thus, the Laplace transform generalizes the Fourier transform from the real line (the frequency axis) to the entire complex plane.

The Fourier transform equals the Laplace transform evaluated along the $j\omega$ axis in the complex $s$ plane

The Laplace Transform can also be seen as the Fourier transform of an exponentially windowed causal signal $x(t)$
Relation to the z Transform

The Laplace transform is used to analyze continuous-time systems. Its discrete-time counterpart is the \( z \) transform:

\[
X_d(z) \triangleq \sum_{n=0}^{\infty} x_d(nT)z^{-n}
\]

If we define \( z = e^{sT} \), the \( z \) transform becomes proportional to the Laplace transform of a sampled continuous-time signal:

\[
X_d(e^{sT}) = \sum_{n=0}^{\infty} x_d(nT)e^{-snT}
\]

As the sampling interval \( T \) goes to zero, we have

\[
\lim_{T \to 0} X_d(e^{sT})T = \lim_{\Delta t \to 0} \sum_{n=0}^{\infty} \left[ \frac{x_d(t_n)}{\Delta t} \right] e^{-st_n} \Delta t
\]

\[
= \int_{0}^{\infty} x_d(t)e^{-st}dt \triangleq X(s)
\]

where \( t_n \triangleq nT \) and \( \Delta t \triangleq t_{n+1} - t_n = T \).

In summary,

the \( z \) transform (times the sampling interval \( T \)) of a discrete time signal \( x_d(nT) \) approaches, as \( T \to 0 \), the Laplace Transform of the underlying continuous-time signal \( x_d(t) \).
Note that the $z$ plane and $s$ plane are related by

$$z = e^{sT}$$

In particular, the discrete-time frequency axis $\omega_d \in (-\pi/T, \pi/T)$ and continuous-time frequency axis $\omega_a \in (-\infty, \infty)$ are related by

$$e^{j\omega_d T} = e^{j\omega_a T}$$

For the mapping $z = e^{sT}$ from the $s$ plane to the $z$ plane to be invertible, it is necessary that $X(j\omega_a)$ be zero for all $|\omega_a| \geq \pi/T$. If this is true, we say $x(t)$ is \textit{bandlimited below half the sampling rate}. As is well known, this condition is necessary to prevent \textit{aliasing} when sampling the continuous-time signal $x(t)$ at the rate $f_s = 1/T$ to produce $x(nT)$, $n = 0, 1, 2, \ldots$
Two Laplace Transform Theorems

Linearity

The Laplace transform is a linear operator:

\[ \alpha x(t) + \beta y(t) \leftrightarrow \alpha X(s) + \beta Y(s) \]

Proof: Let

\[ w(t) = \alpha x(t) + \beta y(t), \]

where \( \alpha \) and \( \beta \) are real or complex constants. Then

\[ W(s) \overset{\Delta}{=} \mathcal{L}_s\{w\} \overset{\Delta}{=} \mathcal{L}_s\{\alpha x(t) + \beta y(t)\} \]

\[ \overset{\Delta}{=} \int_0^\infty [\alpha x(t) + \beta y(t)] e^{-st} dt \]

\[ = \alpha \int_0^\infty x(t)e^{-st} dt + \beta \int_0^\infty y(t)e^{-st} dt \]

\[ \overset{\Delta}{=} \alpha X(s) + \beta Y(s). \]

Thus, linearity of the Laplace transform follows immediately from linearity of integration.
Differentiation

The differentiation theorem for Laplace transforms:

\[
\dot{x}(t) \leftrightarrow sX(s) - x(0)
\]

where \(\dot{x}(t) \triangleq \frac{d}{dt}x(t)\), and \(x(t)\) is any differentiable function that approaches zero as \(t\) goes to infinity.

Operator notation:

\[
\mathcal{L}_s\{\dot{x}\} = sX(s) - x(0).
\]

**Proof:** Immediate from integration by parts:

\[
\mathcal{L}_s\{\dot{x}\} \triangleq \int_0^\infty \dot{x}(t)e^{-st}dt
\]

\[
= x(t)e^{-st}\bigg|_0^\infty - \int_0^\infty x(t)(-s)e^{-st}dt
\]

\[
= sX(s) - x(0)
\]

since \(x(\infty) = 0\) by assumption

**Corollary:** Integration Theorem:

\[
\mathcal{L}_s\left\{ \int_0^t x(\tau)d\tau \right\} = \frac{X(s)}{s}
\]
Laplace Analysis of Linear Systems

The differentiation theorem converts *differential equations* into *algebraic* equations, which are easier to solve.

**Example: Force-Driven Mass**

Consider a free mass driven by an external force along an ideal frictionless surface in one dimension:

Physical diagram:

\[
\begin{align*}
&f(t) \rightarrow m \\
&v(t) \rightarrow
\end{align*}
\]

Electrical equivalent circuit:

\[
\begin{align*}
&v(t) \rightarrow \\
&f(t) \rightarrow m
\end{align*}
\]
Force-Driven Mass Analysis

Note that in the electrical equivalent circuit

- Driving force = \textit{voltage source} emitting $f(t)$ \textit{volts}
- Mass = \textit{inductor} of $L = m$ \textit{Henrys}.

From Newton’s second law of motion “$f = ma$”, we have

$$f(t) = ma(t) \overset{\Delta}{=} m \dot{v}(t) \overset{\Delta}{=} m \ddot{x}(t).$$

Taking the unilateral Laplace transform and applying the differentiation theorem twice yields

$$F(s) = m \mathcal{L}_s\{\ddot{x}\}
= m \left[ s \mathcal{L}_s\{\dot{x}\} - \dot{x}(0) \right]
= m \left\{ s \left[ s X(s) - x(0) \right] - \dot{x}(0) \right\}
= m \left[ s^2 X(s) - s x(0) - \dot{x}(0) \right].$$

Thus, given

- $F(s)$ = Laplace transform of the driving force $f(t)$,
- $x(0)$ = initial mass position, and
- $\dot{x}(0) \overset{\Delta}{=} v(0)$ = initial mass velocity,

we can solve algebraically for $X(s)$, the Laplace transform of the mass position for all $t \geq 0$.
Force-Driven Mass Analysis, Continued

If the applied external force $f(t)$ is zero, we obtain

$$X(s) = \frac{x(0)}{s} + \frac{\dot{x}(0)}{s^2} = \frac{x(0)}{s} + \frac{v(0)}{s^2}.$$ 

Since $1/s$ is the Laplace transform of the Heaviside unit-step function

$$u(t) \triangleq \begin{cases} 
0, & t < 0 \\
1, & t \geq 0 
\end{cases},$$

we find that the position of the mass $x(t)$ is given for all time by

$$x(t) = x(0) u(t) + v(0) t u(t).$$

- A nonzero initial position $x(0) = x_0$ and zero initial velocity $v(0) = 0$ results in $x(t) = x_0$ for all $t \geq 0$ (mass “just sits there”)

- Similarly, any initial velocity $v(0)$ is integrated with respect to time (mass moves forever at initial velocity)

In summary, we used the Laplace transform to solve for the motion of a simple physical system (an ideal mass) in response to initial conditions (no external driving forces).
Mass-Spring Oscillator Time-Domain Solution

Consider now the mass-spring oscillator:

\[ m \ddot{x}(t) + kx(t) = 0 \]

Electrical equivalent-circuit:

\[ \frac{1}{k} \quad \text{m} \]

Newton’s second law of motion:

\[ f_m(t) = m\ddot{x}(t) \]

Hooke’s law for ideal springs:

\[ f_k(t) = kx(t) \]

Newton’s third law of motion:

\[ f_m(t) + f_k(t) = 0 \]
\[ \Rightarrow m\ddot{x}(t) + kx(t) = 0 \]

We have thus derived a second-order differential equation governing the motion of the mass and spring. (Note that
\(x(t)\) is both the position of the mass and compression of the spring at time \(t\).

Taking the Laplace transform of both sides of this differential equation gives

\[
0 = \mathcal{L}_s\{m\ddot{x} + kx\} \\
= m\mathcal{L}_s\{\ddot{x}\} + k\mathcal{L}_s\{x\} \quad \text{(linearity)} \\
= m[s\mathcal{L}_s\{\dot{x}\} - \dot{x}(0)] + kX(s) \quad \text{(differentiation theorem)} \\
= m\{s[sX(s) - x(0)] - \dot{x}(0)\} + kX(s) \quad \text{(diff. thm again)} \\
= ms^2X(s) - msx(0) - m\dot{x}(0) + kX(s)
\]

Let \(x(0) = x_0\) and \(\dot{x}(0) = v_0\) for simplicity.

Solving for \(X(s)\) gives

\[
X(s) = \frac{sx_0 + v_0}{s^2 + \frac{k}{m}} \quad \overset{\text{\(\Delta\)}}{=} \quad \frac{r}{s + j\omega_0} + \frac{\bar{r}}{s - j\omega_0}, \quad \omega_0 \overset{\text{\(\Delta\)}}{=} \sqrt{\frac{k}{m}},
\]

\[
r = \frac{x_0}{2} + j\frac{v_0}{2\omega_0} \quad \overset{\text{\(\Delta\)}}{=} \quad R_re^{j\theta_r}, \quad \text{with}
\]

\[
R_r \overset{\text{\(\Delta\)}}{=} \sqrt{\frac{v_0^2 + \omega_0^2x_0^2}{2\omega_0}}, \quad \theta_r \overset{\text{\(\Delta\)}}{=} \tan^{-1}\left(\frac{v_0}{\omega_0x_0}\right)
\]

denoting the modulus and angle of the pole residue \(r\), respectively.
Mass-Spring Oscillator Analysis, Continued

We can quickly verify that

\[ e^{-at}u(t) \leftrightarrow \frac{1}{s + a} \]

where \( u(t) \) is the Heaviside unit step function which steps from 0 to 1 at time 0.

By linearity, the solution for the motion of the mass is

\[
x(t) = re^{-j\omega_0 t} + \bar{r}e^{j\omega_0 t} = 2\text{Re} \left\{ re^{-j\omega_0 t} \right\} = 2R_r \cos(\omega_0 t - \theta_r)
\]
\[
= \frac{\sqrt{v_0^2 + \omega_0^2 x_0^2}}{\omega_0} \cos \left[ \omega_0 t - \tan^{-1} \left( \frac{v_0}{\omega_0 x_0} \right) \right]
\]

If the initial velocity is zero \((v_0 = 0)\), the above formula reduces to \( x(t) = x_0 \cos(\omega_0 t) \) and the mass simply oscillates sinusoidally at frequency \( \omega_0 = \sqrt{k/m} \), starting from its initial position \( x_0 \). If instead the initial position is \( x_0 = 0 \), we obtain

\[
x(t) = \frac{v_0}{\omega_0} \sin(\omega_0 t)
\]
\[ \Rightarrow \ v(t) = v_0 \cos(\omega_0 t). \]