MUS420 Lecture 7A
Digital Waveguide Modeling of Horns

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Outline

- Horn Modeling (Trumpet)
- Piecewise Conical Bore Modeling
- Truncated Infinite Impulse Response (TIIR) Filters
Horn Modeling

Bore Profile Reconstruction from Measured Trumpet Reflectance

- Inverse scattering applied to pulse-reflectometry data to fit piecewise-cylindrical model (like LPC model)
- Bore profile reconstruction is reasonable up to bell
- The bell is not physically equivalent to a piecewise-cylindrical acoustic tube, due to
  - complex radiation impedance,
  - conversion to higher order transverse modes
Trumpet-Bell Impulse Response Computed from Estimated Piecewise-Cylindrical Model

Bell Reflection Impulse Response

- From pulse reflectometry on trumpet with no mouthpiece
- Bore profile is reconstructed, smoothed, and segmented
- Impulse response of “bell segment” = “ideal filter”
- At $f_s = 44.1$ kHz, filter length is $\approx 400$ to $600$ samples
• A length 400 FIR bell filter is too expensive!

• Convert to IIR? Hard because
  – Phase (resonance tunings) must be preserved
  – Magnitude (resonance Q) must be preserved
  – Rise time $\approx 150$ samples
  – Phase-sensitive IIR design methods perform poorly
FIR to IIR Conversion Attempts

Bell Impulse Response (dB) Before Truncation

- 561 samples gives cut-off around -60 dB relative to maximum
- This length 561 FIR filter can be reduced to a lower-order IIR filter by minimizing some norm of the impulse-response error
• Hankel norm minimization should always work in theory

**Hankel Norm Method**

**Eigenvalues of Hankel Matrix (dB)**
Largest Eigenvalues of Hankel Matrix (dB)
- Order 15 is a “sweet spot” in the eigenvalues plot
- Hankel Norm is the only phase-sensitive IIR error norm we know which can always be reliably minimized in principle
• Norm is sensitive to linear magnitude error, not dB
• This bell filter is too “bright” and fit is generally poor
• Initial time-domain match is reasonable, but it can’t “hold on” until the main reflection
• Numerical failure is a likely (in Matlab/PentiumII doubles)
Order 8 Hankel-Norm IIR Fit to Length 561 FIR
(Evidence of Numerical Failure in Previous Example)

- Halving the order actually looks better ("can’t happen")
- Error plot indicates numerical troubles here as well
• An order $P$ IIR filter is made using $P$th eigenvector of the $561 \times 561$ Hankel matrix (condition number = 51751075)

• Numerical failure occurs at the higher orders we need

• Slow rise time of impulse response causes “numerical stress” on all phase-sensitive IIR design methods when the IIR order is much less than the rise time
Order 10 Steiglitz-McBride $L_2$ Fit to a Length 561 FIR Filter Model

- All poles concentrated at low frequencies
- Little attention to high frequencies
- Internal “equation-error” weighting
- Numerical ill-conditioning warning printed by Matlab
SM-10 Group Delay Fit

Group Delay Fit

Normalized Frequency

Group Delay (samples)

--- Ideal approximation

--- approximation

Normalized Frequency

Group Delay (samples)
SM-10 Phase Delay Fit

Phase Delay Fit

Normalized Frequency

Phase Delay (samples)

Ideal
Approximation
SM-10 Amplitude Response Fit

Magnitude Fit over Entire Nyquist Band

- - - - Ideal
- - - - Approximation

Normalized Frequency

Magnitude (dB)

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<th>Magnitude</th>
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<td>0.45</td>
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<tr>
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</table>
Another Measured Trumpet Bell Reflectance

Measured Trombone Bell Reflectance
Idea!

- Break up impulse response into *exponential* or *polynomial* segments
- Exponential and polynomial impulse-responses can be designed using *Truncated IIR (TIIR) Filters*
Four-Exponential Fit to Estimated Trumpet-Bell Filter (Exp-4)

Exp-4 Impulse Response Fit

Exp-4 Phase Response Fit
Exp-4 Amplitude Response Fit

![Magnitude Fit over Entire Nyquist Band](image)

**Magntitude** (dB)

- Ideal
- Approximation

Exp-4 Low-Frequency Zoom

![Magnitude Fit](image)

**Magnitude** (dB)

- Ideal
- Approximation

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Exp-4 Group Delay Fit

Exp-4 Phase Delay Fit
Exp-4 Impulse Response Fit (Repeated)

Exp-4 Slope Fit
Two Exponentials Connected by a Cubic Spline
Measured Trumpet Data (Exp2-S3)

Exp2-S3 Slope Fit
Exp2-S3 Impulse Response Fit

Exp2-S3 Phase Response Fit
Exp2-S3 Amplitude Response Fit

Exp2-S3 Low-Frequency Zoom
Exp2-S3 Group Delay Fit

Exp2-S3 Phase Delay Fit
Exp2-S3 Impulse Response Fit

Exp2-S3 Slope Fit
Two Exponentials Followed by a 6th-Order IIR Filter Designed by Steiglitz McBride Algorithm (Exp2-SM6)
Exp2-SM6 Impulse Response Fit

Exp2-SM6 Slope Fit
Exp2-SM6 Phase Response Fit

Normalized Frequency vs. Phase Fit

- Ideal
- Approximation
Exp2-SM6 Amplitude Response Fit

Magnitude Fit over Entire Nyquist Band

- - - Ideal
- - - Approximation

Exp2-SM6 Low-Frequency Zoom

Magnitude Fit

- - - Ideal
- - - Approximation
Exp2-SM6 Group Delay Fit

Exp2-SM6 Phase Delay Fit
Results for Measured Trumpet Data Using Two Offset Exponentials and Two Biquads

Slope Fit

- Bell model filter complexity comparable to order 8+ IIR
- Offset exponentials were fit using `fmins()` in Matlab
- Two biquads were fit as a single fourth-order filter using the Steiglitz-McBride algorithm (`stmcb()` in Matlab)
Measured Trumpet Bell Impulse Response

TIIR Trumpet Bell Impulse Response
Measured Trumpet Bell Amplitude Response

TIIR Trumpet Bell Amplitude Response
Trumpet Bell Phase Delay Fit

Input Impedance of Complete Bore + Bell Model
Comparison to Measurements

The next two pages of plots compare the measured impulse response with that produced by the final digital waveguide model consisting of a trumpet bore + bell (but no mouthpiece).

- Comparison 1: two offset exponentials and two biquads to model the bell impulse response
- Comparison 2: two offset exponentials and three biquads to model the bell impulse response
Measured Impulse Response

Synthesized Impulse Response, Order 4 Tail
Piecewise Conical Acoustic Tube Modeling

Simple Example: Cylinder with Conical Cap

Physical Outline of Cylinder and Cone:

Digital Waveguide Model (DWM) for Pressure Waves:

Reduced DWM for Maximum Computational Efficiency:

where

\[ R(z) = \left( \frac{1}{99} \right) \left( \frac{1 + z^{-1}}{1 - \frac{101}{99} z^{-1}} \right) \]

\[ T(z) = \left( \frac{100}{99} \right) \left( \frac{1 - z^{-1}}{1 - \frac{101}{99} z^{-1}} \right) = 1 + R(z) \]
• **Problem:** Reflection filter $R(z)$ and transmission filter $T(z)$ are *unstable* (pole at $z = 101/99$)

• Overall system is passive $\Rightarrow$ unstable pole is *canceled*

**Implementation Idea**

Apply TIIR “alternate and reset” idea to the unstable conical subsystem

• Cone is not truly FIR $\Rightarrow$ $t_{60}$ replaces FIR length

• When cylinder is closed-ended, cone traveling-wave components increase without bound $\Rightarrow$ must switch out and reset the entire cone assembly (scattering-junction filter $R(z)$ and cone’s entire delay line)

• According to simulations thus far, cylinder waves are well behaved and do not need to be reset (no general proof yet)

**Basic Principle**

*Periodically reset any subsystem containing a canceled unstable pole at intervals greater than or equal to the $t_{60}$ for that subsystem*
Interesting Paradox at DC

**DC Steady State: Closed-End Cylinder**

\[ p(t) = p^+(t) + p^-(t) = x \quad \text{(constant)} \]

\[ p(t) = x \quad \text{(const)} \]

- \( R(1) = -1 \) (dc response of reflection filter inverts)
- \( T(1) = 0 \) (dc does not transmit through the junction)
- Physically obvious dc solution (constant pressure offset) is not possible in either the cone or the cylinder model!
- Simulated impulse responses agree with the literature
- A final constant dc offset is observed in the simulations
Solution to Paradox

• It turns out the reflection transfer function looking into the cone from the cylinder has *two poles* and *two zeros* at dc.

• The dc poles and zeros *cancel* and leave a dc cone reflectance equal to $+1$ (the physically obvious answer).

• We can’t just set the reflection filter to its dc equivalent to figure out the dc behavior of the overall model.

• Instead, a more careful limit must be taken.

In the $s$ plane, the conical cap pressure reflectance, seen from the cylinder, can be derived to be

$$H(s) \triangleq \frac{1 + R(s)(1 + 2st_x)}{2st_x - 1 - R(s)}$$

where $t_x$ is the time (in seconds) to propagate across the cone, and

$$R(s) = -e^{-2st_x}$$

is the reflectance of the cone at its entrance. We have

$$\lim_{s \to 0} R(s) = -1$$
$$\lim_{s \to 0} H(s) = +1$$
Truncated Infinite Impulse Response (TIIR) Digital Filters

An FIR filter can be constructed as the difference of two IIR filters:

\[
\text{Infinite Impulse Response (IIR)} - \text{Delayed and Scaled IIR} = \text{Truncated IIR} = \text{FIR}
\]

General FIR filter

- Coefficients: \(\{h_0, \ldots, h_N\}\)
- Implementation (convolution):

\[
y(n) = (h \ast x)(n) = \sum_{m=0}^{N} h_m x(n - m)
\]

- Transfer function:

\[
H_{\text{FIR}}(z) \triangleq h_0 + h_1 z^{-1} + \ldots + h_N z^{-N} \equiv z^{-N} C(z),
\]
where $C(z)$ is the $N$-th degree polynomial in $z$ formed by the $h_k$

**General $P$-th order IIR filter**

- Difference equation

\[
y(n) = - \sum_{k=1}^{P} a_k y(n - k) + \sum_{\ell=0}^{P} b_\ell x(n - \ell)
\]

- Transfer function

\[
H_{\text{IIR}}(z) \triangleq \frac{b_0 + b_1 z^{-1} + \ldots + b_P z^{-P}}{1 + a_1 z^{-1} + \ldots + a_P z^{-P}} \\
\triangleq \frac{b_0 z^P + b_1 z^{P-1} + \ldots + b_P}{z^P + a_1 z^{P-1} + \ldots + a_P} \\
\triangleq \frac{B(z)}{A(z)} \\
\triangleq h_0 + h_1 z^{-1} + h_2 z^{-2} + \ldots,
\]

where

\[
A(z) \triangleq z^P + a_1 z^{P-1} + \ldots + a_P \quad \text{(monic)}
\]

\[
B(z) \triangleq b_0 z^P + b_1 z^{P-1} + \ldots + b_P
\]
Consider an FIR filter having a truncated geometric sequence \( \{h_0, h_0p, \ldots, h_0p^N\} \) as an impulse response. This filter has the same impulse response for the first \( N + 1 \) terms as the one-pole IIR filter with transfer function

\[
H_{\text{IIR}}(z) = \frac{h_0}{1 - pz^{-1}}.
\]

Subtracting off the tail of the impulse response gives

\[
H_{\text{FIR}}(z) = h_0 + h_0pz^{-1} + \cdots + h_0p^Nz^{-N} \\
= \left\{h_0 + h_0pz^{-1} + \cdots \right\} \\
- \left\{h_0p^{N+1}z^{-(N+1)} + h_0p^{(N+2)}z^{-(N+2)} + \cdots \right\} \\
= \frac{h_0}{1 - pz^{-1}} - p^{N+1}z^{-(N+1)} \frac{h_0}{1 - pz^{-1}} \\
= h_0 \frac{1 - p^{N+1}z^{-(N+1)}}{1 - pz^{-1}}
\]

The time-domain recursion for this filter is

\[
y[n] = \sum_{k=0}^{N} h_0p^kx[n - k] \\
= py[n - 1] + h_0 \left( x[n] - p^{N+1}x[n - (N + 1)] \right)
\]
Complexity Notes

- Direct FIR filter implementation requires $N + 1$ multiplies and $N$ adds
- TIIR implementation requires 3 multiplies and 2 adds, independent of $N$
- No savings in memory

Note that there is a pole-zero cancellation in the TIIR transfer function

$$H(z) = h_0 \frac{1 - p^{N+1}z^{-(N+1)}}{1 - pz^{-1}} = h_0 + h_0pz^{-1} + \cdots + h_0p^Nz^{-N}$$

- If $|p| < 1$, no problem since the canceled pole is stable
- If $|p| \geq 1$, imperfect pole-zero cancellation due to numerical rounding leads to exponentially growing round-off error

**Basic Idea:** Since the overall TIIR filter is FIR($N$), alternate between two instances of each unstable one-pole, starting each new one from the zero state $N$ samples before it is actually used. (Apparently first suggested by T. Fam at Asilomar-'87 for the case of distinct poles.)
Extension to Higher-Order TIIR Sequences

We can extend this idea from the one-pole case to any rational filter \( H(z) = B(z)/A(z) \). The general procedure is to find the “tail filter” \( H'_{\text{IIR}}(z) \) and subtract it off:

\[
H_{\text{FIR}}(z) = H_{\text{IIR}}(z) - H'_{\text{IIR}}(z)
\]

Multiply \( H_{\text{IIR}}(z) \) by \( z^N \) to obtain

\[
z^N H_{\text{IIR}}(z) = h_0 z^N + \cdots + h_{N-1} z + h_N
\]

\[+ h_{N+1} z^{-1} + h_{N+2} z^{-2} + \cdots\]

\[\Delta = C(z) + H'_{\text{IIR}}(z)\]

\[= \frac{z^N B(z)}{A(z)} \equiv C(z) + \frac{B'(z)}{A(z)}\]

- \( B'(z) \) is the unique remainder after dividing \( z^N B(z) \) by \( A(z) \) using “synthetic division”

\[(z^N B(z) \equiv B'(z) \pmod{A(z)})\]

- We may assume \( \text{Deg} \{B'(z)\} = \text{Deg} \{A(z)\} - 1 \)

- \( B'(z) \) gives us our desired “tail filter” for forming \( H_{\text{FIR}} = H_{\text{IIR}} - H'_{\text{IIR}} \):

\[
H'_{\text{IIR}}(z) = \frac{B'(z)}{A(z)}
\]
Higher-Order TIIR Filters

We have

\[
H_{\text{FIR}}(z) = H_{\text{IIR}}(z) - z^{-N} H'_{\text{IIR}}(z)
\]

\[
= \frac{B(z) - z^{-N} B'(z)}{A(z)}
\]

The corresponding difference equation is

\[
y[n] = - \sum_{k=1}^{P} a_k y[n-k] + \sum_{\ell=0}^{P} b_\ell x[n-\ell]
\]

\[- \sum_{m=0}^{P-1} b'_m x[n-m-(N+1)]
\]

Since the denominators of \( H_{\text{IIR}}(z) \) and \( H'_{\text{IIR}}(z) \) are the same, the \textit{dynamics} (poles) can be shared:
Complexity and Storage-Cost

\[ H_{\text{FIR}}(z) = \frac{B(z) - z^{-N}B'(z)}{A(z)} \]

\( N = \) FIR order and let \( P = A(z) \) order (\(#\)poles)

- The computational cost of the general truncated \( P \)-th order IIR system is \( 3P + 1 \) multiplies and \( 3P - 2 \) adds, independent of \( N \)
- Net computational savings is achieved when \( N > 3P \)

Storage Requirements

- \( P \) output samples for the IIR feedback dynamics \( A(z) \)
- \( N \) input samples of the FIR filter (main delay line)
- \( P \) input samples for \( B(z) \) (normally in delay line)
- \( P \) input samples for \( B'(z) \) (also possibly in delay line)

Thus, we need a total of at least \( N + P \) input delay samples, of which only \( 2P \) are accessed, and \( P \) output delay samples. This is between \( P \) and \( 2P \) more than a direct FIR implementation.
Example

We wish to truncate the impulse response of

\[ H^+(z) = \frac{B^+(z)}{A^+(z)} = \frac{1}{1 - 1.9z^{-1} + 0.98z^{-2}} \]

after \( N = 300 \) samples to obtain a length 301 FIR filter \( H^+_{\text{FIR}}(z) \)

Steps:

1. Perform synthetic division on \( z^{300}B^+(z) \) by \( A(z) \) to obtain the remainder

   \[ B'^+(z) = -0.162126z + 0.139770 \]

2. Form the TIIR filter as

   \[ H^+_{\text{FIR}}(z) = \sum_{k=0}^{N} h^+_k z^{-k} = \frac{B^+(z) - z^{-N} B'^+(z)}{A^+(z)} \]

   \[ = \frac{1 + 0.162126 z^{-299} - 0.139770 z^{-300}}{1 - 1.9z^{-1} + 0.98z^{-2}} \]
Impulse Response of TIIR Implementation Without Resets

Amplitude vs. Time (Samples)

Amplitude (dB) vs. Time (Samples)
• At time $n = 301$, the tail of the response is subtracted off, and the impulse-response magnitude drops by about 115 dB

• Due to quantization errors, there is a residual response

• Poles are all stable, so error decays
Impulse Response of TIIR Implementation With Resets
• Again, impulse-response tail is subtracted off at time $n = 301$, giving around 115 dB attenuation

• Additionally, state variables are cleared every 300 samples

• Residual response completely canceled at time $n = 600$

• System has truly finite memory
Unstable Example

To form a linear phase TIIR filter based on the previous example, we need also the “flipped” impulse response generated by

\[
H_{\text{FIR}}^-(z) = \frac{-0.139770z^2 + 0.162126z - z^{-300}}{0.98z^2 - 1.9z + 1}
= \frac{-0.142622z^2 + 0.165435z - 1.020408z^{-300}}{z^2 - 1.938776z + 1.020408}
\]

where the last equation is normalized by 0.98 to make the denominator monic.

This system has two unstable hidden modes.
• Tail is canceled with about 125 dB attenuation
• Due to the unstable canceled poles, quantization noise grows without bound
• By time 1500 samples, the quantization noise dominates
• (Arithmetic = double-precision floating point with single-precision state variables)
Impulse Response of TIIR Implementation With Resets
• State-variable resets zero-out the quantization noise before it becomes significant

• Overall system has truly finite memory
Algorithm for performing synthetic division to generate the tail-canceling polynomial $B'(z)$:

```c
int i, j;
double *w = (double *)malloc((P + 1) * sizeof(double));
/*** load the numerator coefficients for $B(z)$ ***/
for (i = 0; i < P + 1; i++) {
    w[i] = b[i];
}
/*** do synthetic division ***/
for (i = 0; i <= N; i++) {
    factor = w[0];
    for (j = 0; j < P; j++) {
        w[j] = w[j + 1] + factor * a[j];
    }
    w[P] = 0;
/*** The remainder after the i-th step is in w[0..(P-1)] ***/
}
/*** copy the result to the output array ***/
for (i = 0; i < P; i++) {
    bb[i] = w[i];
}
```
A One-Pole (Almost) TIIR Filter

- Generates truncated exponentials or constants
- Filter complexity on average $\approx$ one pole
- Shared delay line
- Shared dynamics

\[ 1P = \frac{1}{1 - pz^{-1}} \]
Offset Exponentials

Use two one-pole TIIRs, to make an offset exponential:

\[ h(n) = \begin{cases} 
  ae^{cn} + b, & n = 0, 1, 2, \ldots, N - 1 \\
  0, & \text{otherwise}
\end{cases} \]

- The constant portion \( b \) requires only one multiply (by \( b \)) since the pole for this TIIR filter is at \( z = 1 \)
- Resets for pure integrators are needed less often than for growing exponentials
- Using a cascade of digital integrators, any polynomial impulse response is possible
- A cubic-spline impulse response requires four integrators

\[ \delta(n) \quad \text{constant ramp quadratic cubic} \]

\[ y(n) = b_0 + b_1 n + b_2 n^2 + b_3 n^3 \]