MUS420 Lecture
Digital Waveguide Modeling of Horns

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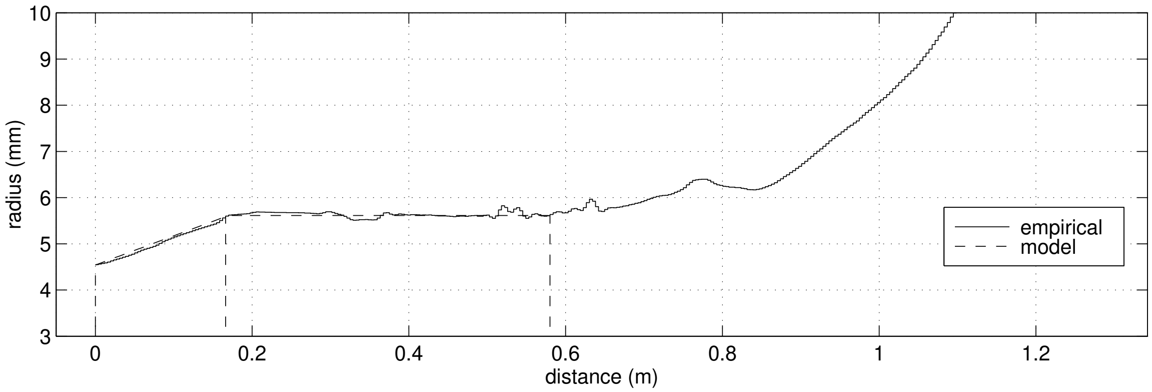
Outline

- Horn Modeling (Trumpet)
- Piecewise Conical Bore Modeling
- Truncated Infinite Impulse Response (TIIR) Filters

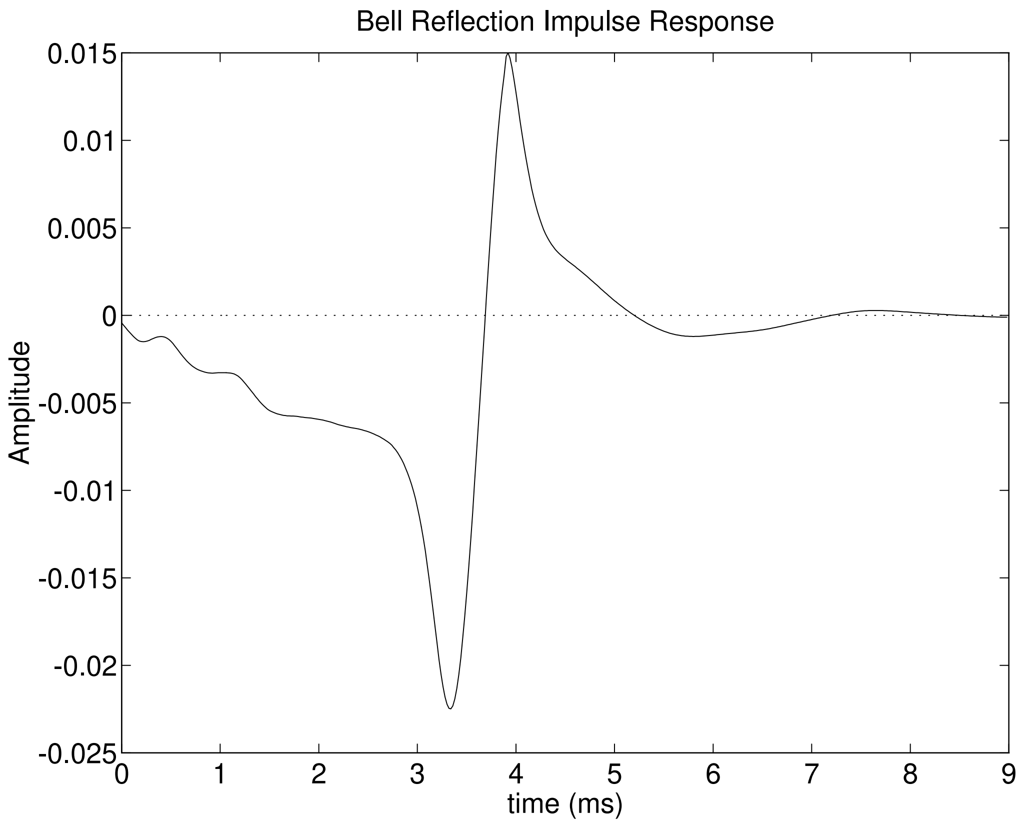
Horn Modeling

Bore Profile Reconstruction from Measured Trumpet Reflectance



- Inverse scattering applied to pulse-reflectometry data to fit piecewise-cylindrical model (like LPC model)
- Bore profile reconstruction is reasonable up to bell
- The bell is not physically equivalent to a piecewise-cylindrical acoustic tube, due to
  - complex radiation impedance,
  - conversion to higher order transverse modes

Trumpet-Bell Impulse Response Computed from Estimated Piecewise-Cylindrical Model

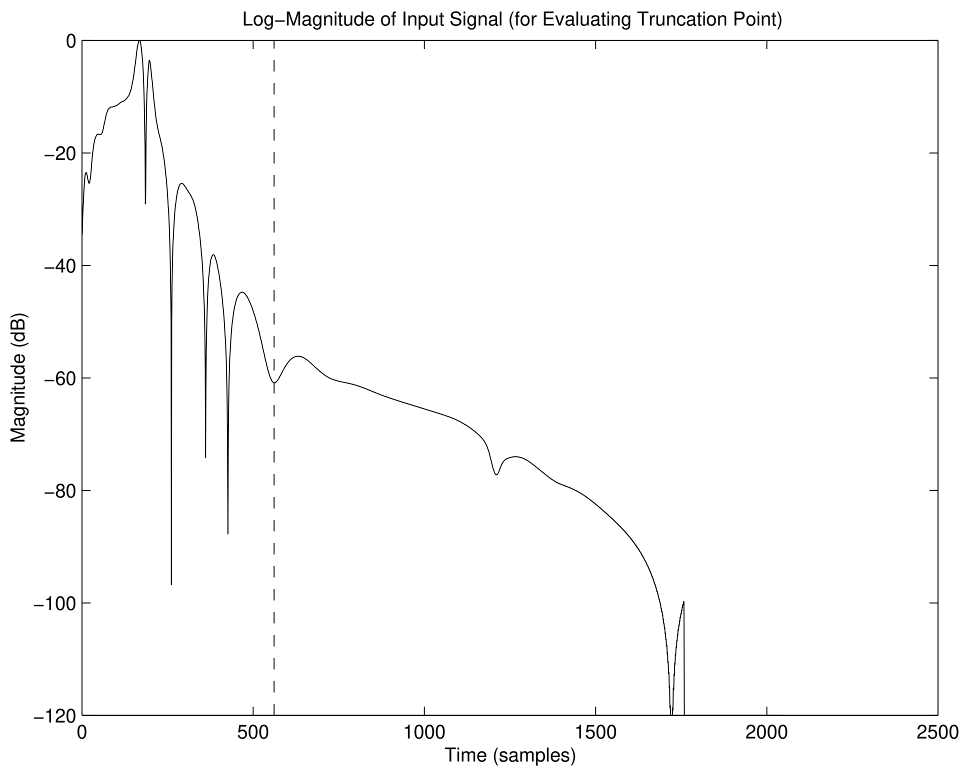


- From pulse reflectometry on trumpet with no mouthpiece
- Bore profile is reconstructed, smoothed, and segmented
- Impulse response of "bell segment" = "ideal filter"
- At $f_s = 44.1$ kHz, filter length is $\approx$ 400 to 600 samples

- A length 400 FIR bell filter is too expensive!
- Convert to IIR? Hard because
  - Phase (resonance tunings) must be preserved
  - Magnitude (resonance Q) must be preserved
  - Rise time $\approx$ 150 samples
  - Phase-sensitive IIR design methods perform poorly

FIR to IIR Conversion Attempts

Bell Impulse Response (dB) Before Truncation

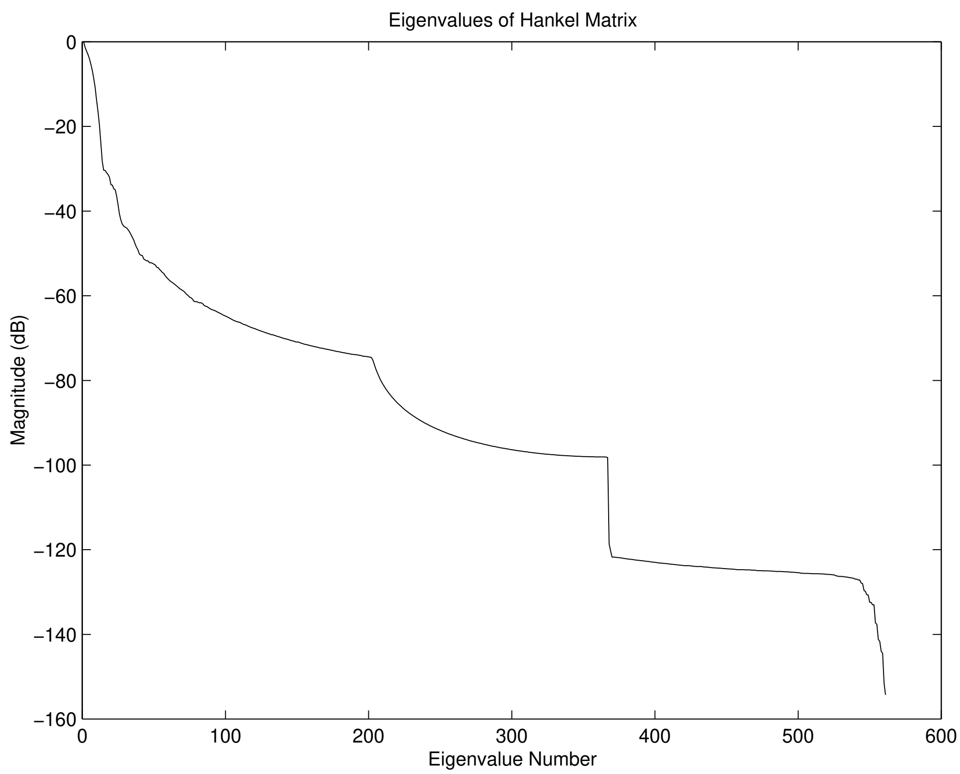


- 561 samples gives cut-off around -60 dB relative to maximum
- This length 561 FIR filter can be reduced to a lower-order IIR filter by minimizing some norm of the impulse-response error

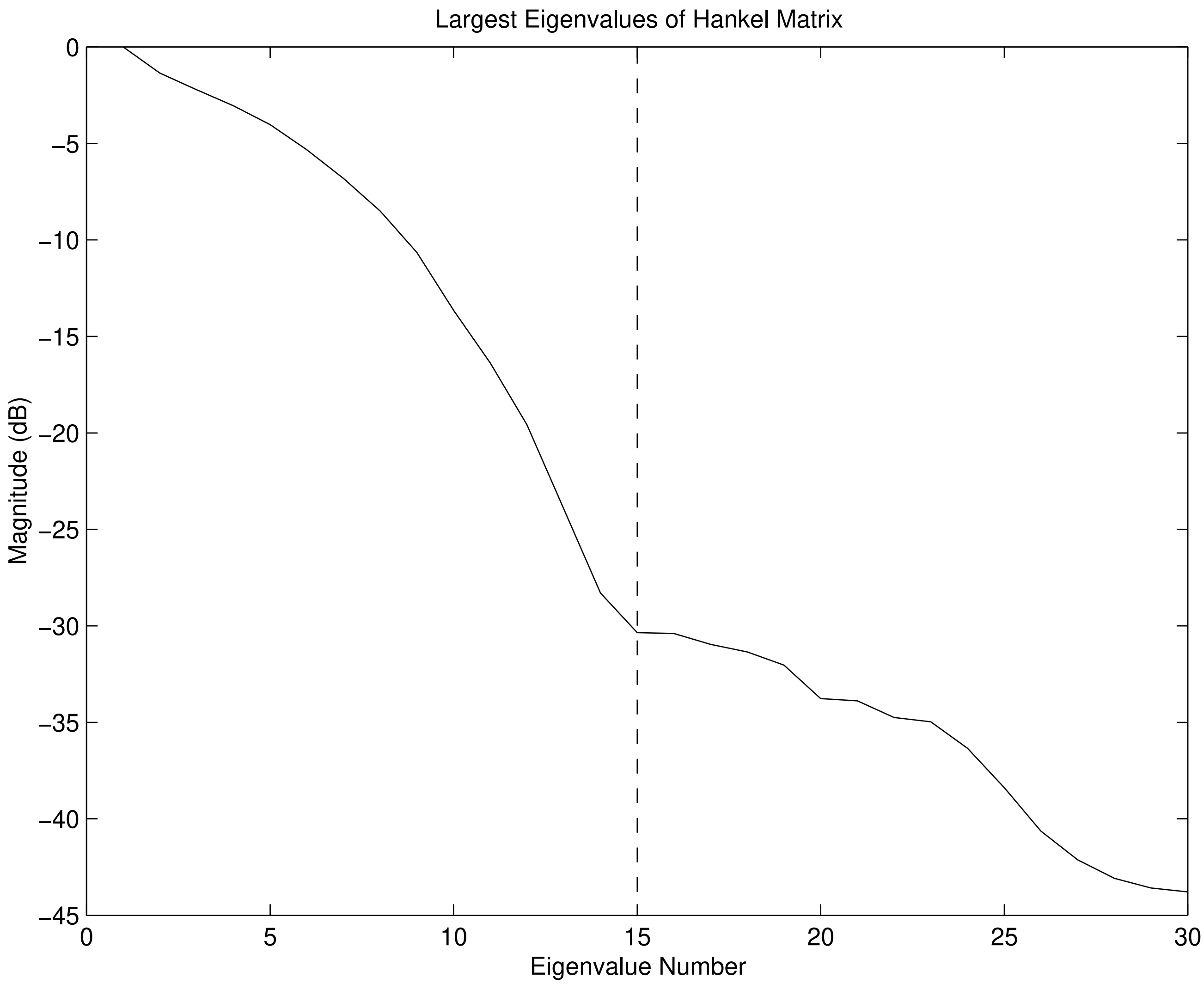
- Hankel norm minimization should always work in theory

Hankel Norm Method

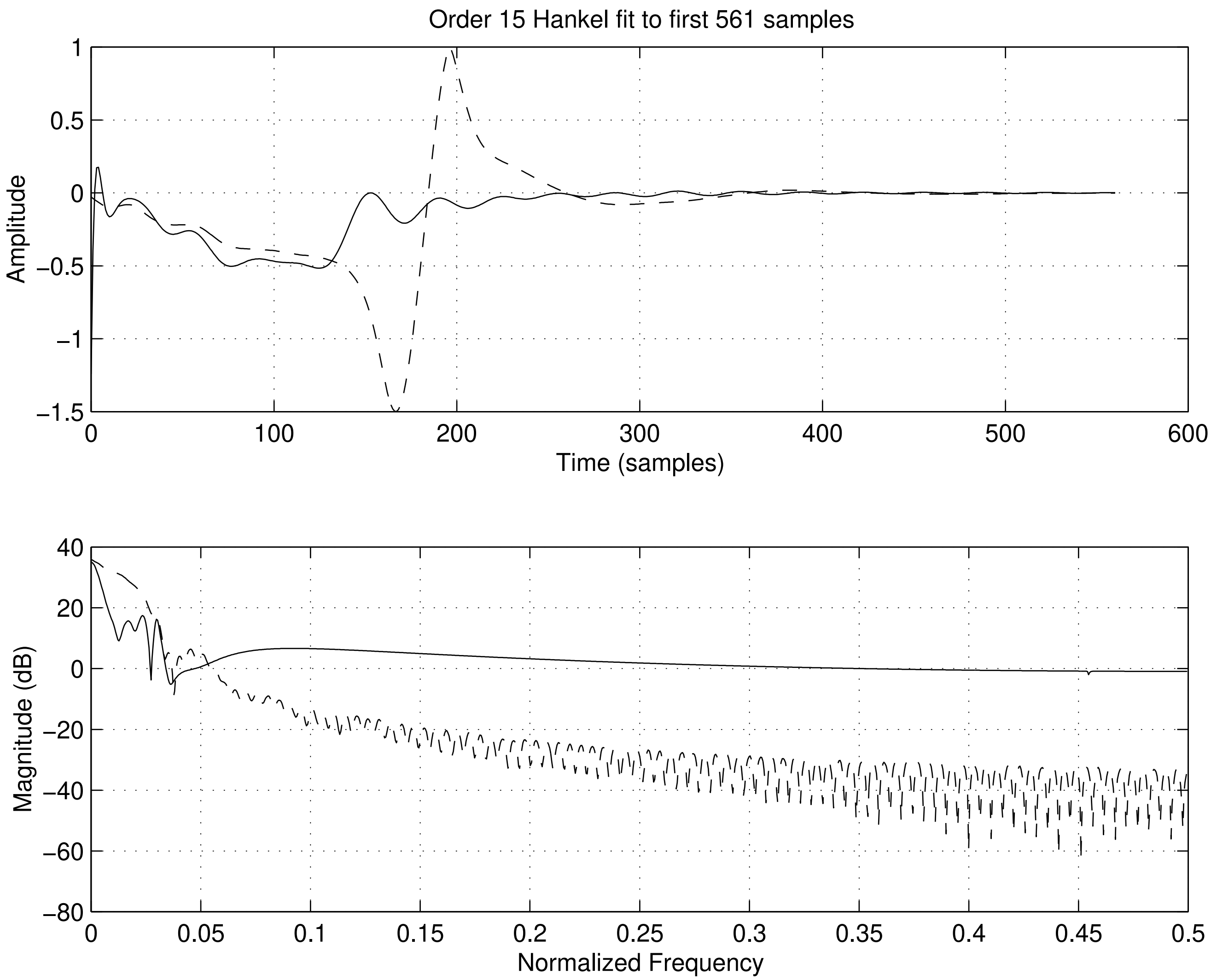
Eigenvalues of Hankel Matrix (dB)



Largest Eigenvalues of Hankel Matrix (dB)



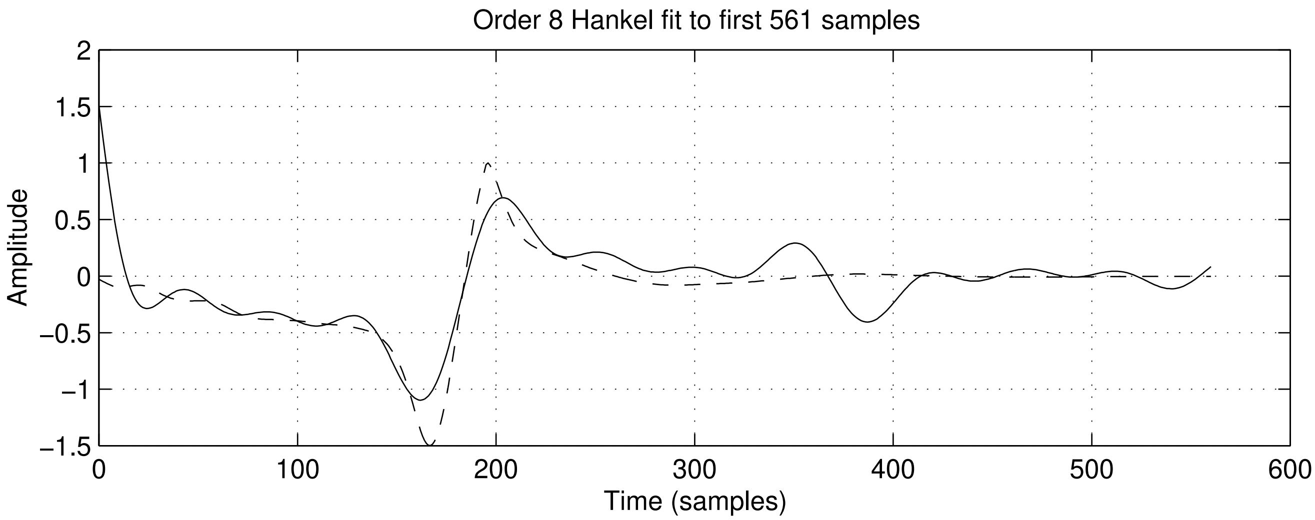
Order 15 Hankel-Norm IIR Fit to Length 561 FIR Measured Trumpet-Bell Reflectance

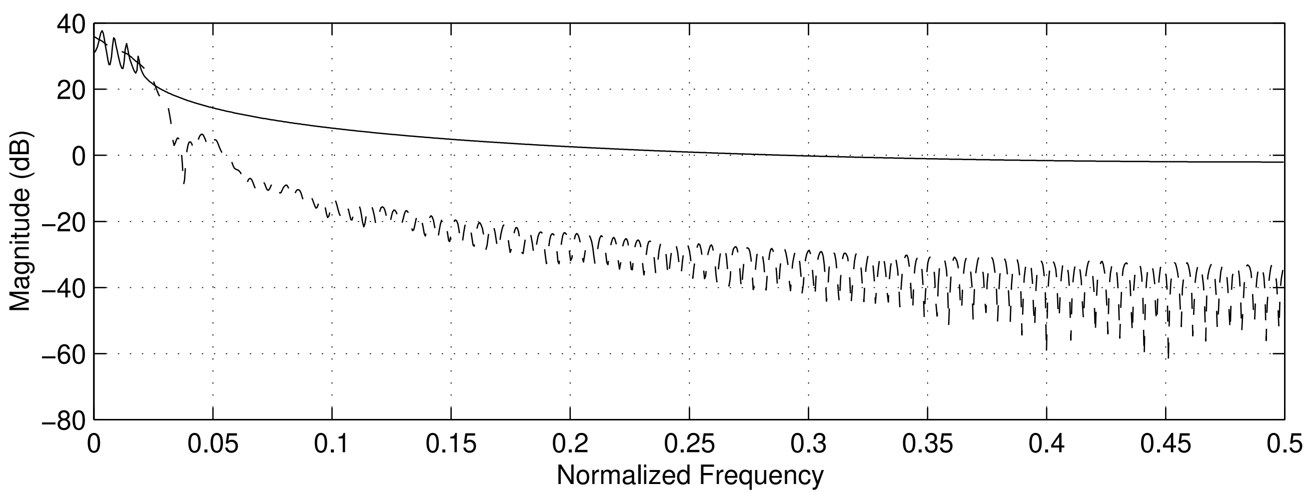


- Order 15 is a "sweet spot" in the eigenvalues plot
- Hankel Norm is the *only* phase-sensitive IIR error norm we know which can always be reliably minimized in principle

- Norm is sensitive to *linear* magnitude error, not dB
- This bell filter is too "bright" and fit is generally poor
- Initial time-domain match is reasonable, but it can't "hold on" until the main reflection
- Numerical failure is a likely (in Matlab/PentiumII doubles)

Order 8 Hankel-Norm IIR Fit to Length 561 FIR (Evidence of Numerical Failure in Previous Example)

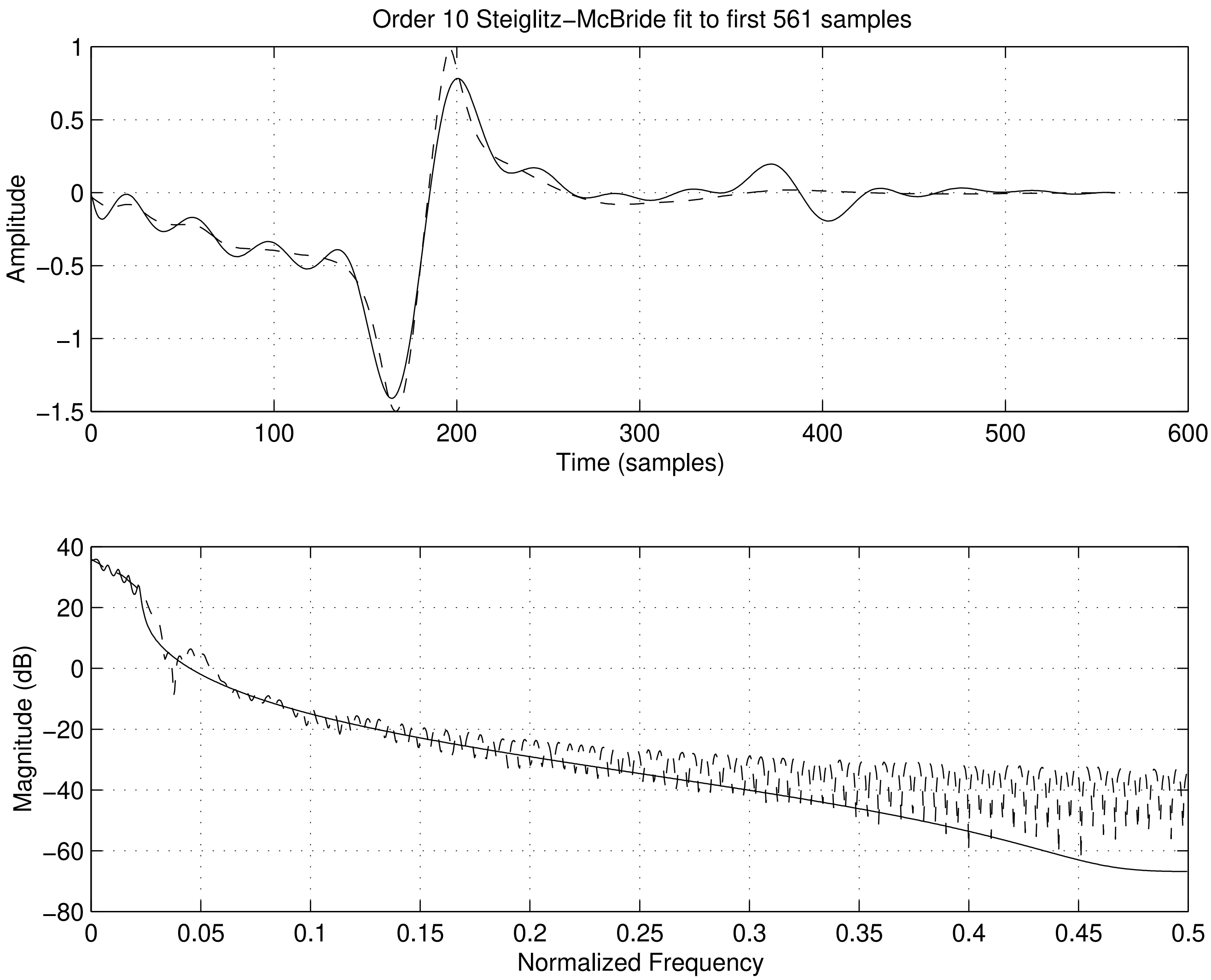




- Halving the order actually looks better ("can't happen")
- Error plot indicates numerical troubles here as well

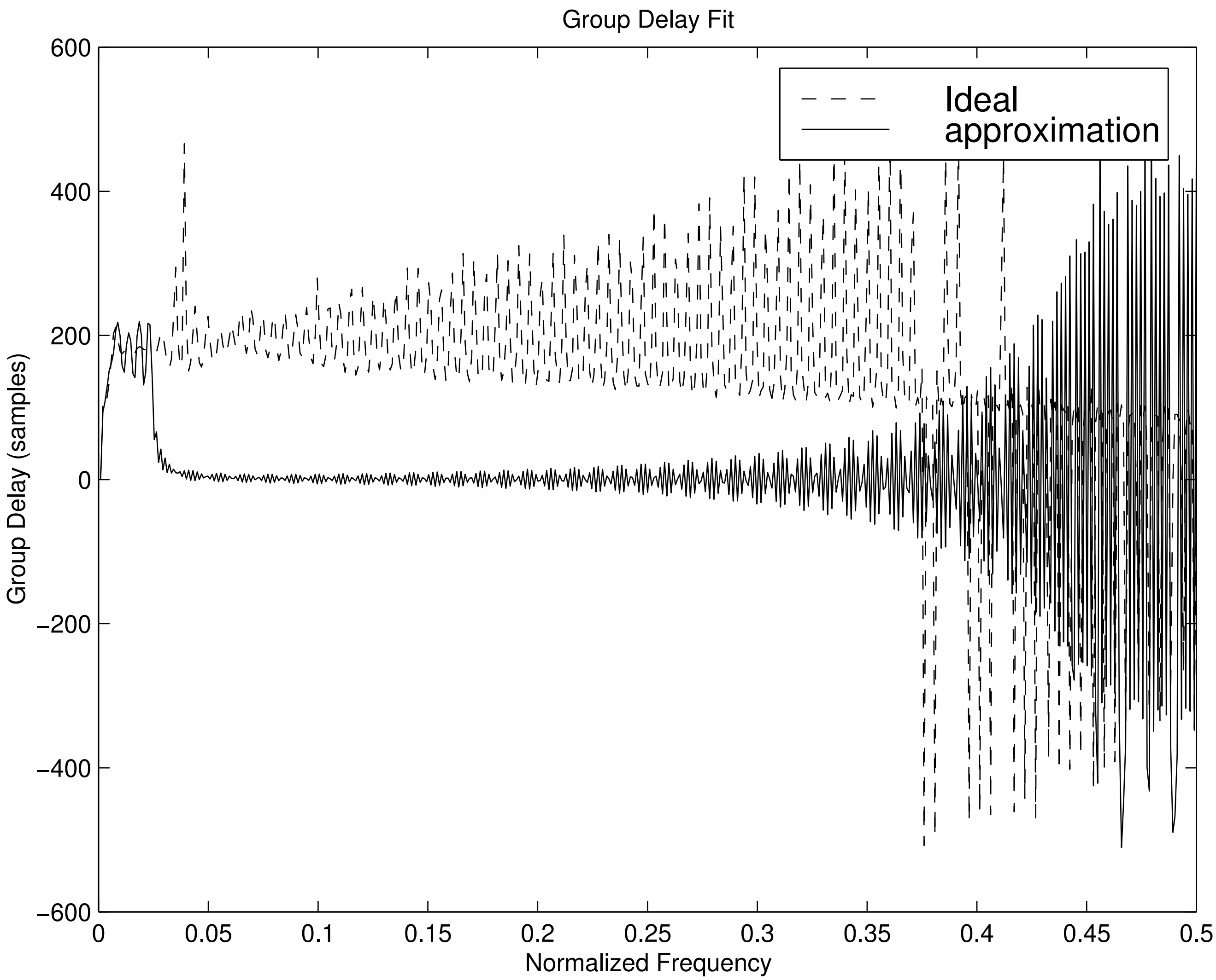
- An order P IIR filter is made using Pth eigenvector of the 561×561 Hankel matrix (condition number = 51751075)
- Numerical failure occurs at the higher orders we need
- Slow rise time of impulse response causes “numerical stress” on all phase-sensitive IIR design methods when the IIR order is much less than the rise time

Order 10 Steiglitz-McBride L_2 Fit to a Length 561 FIR Filter Model

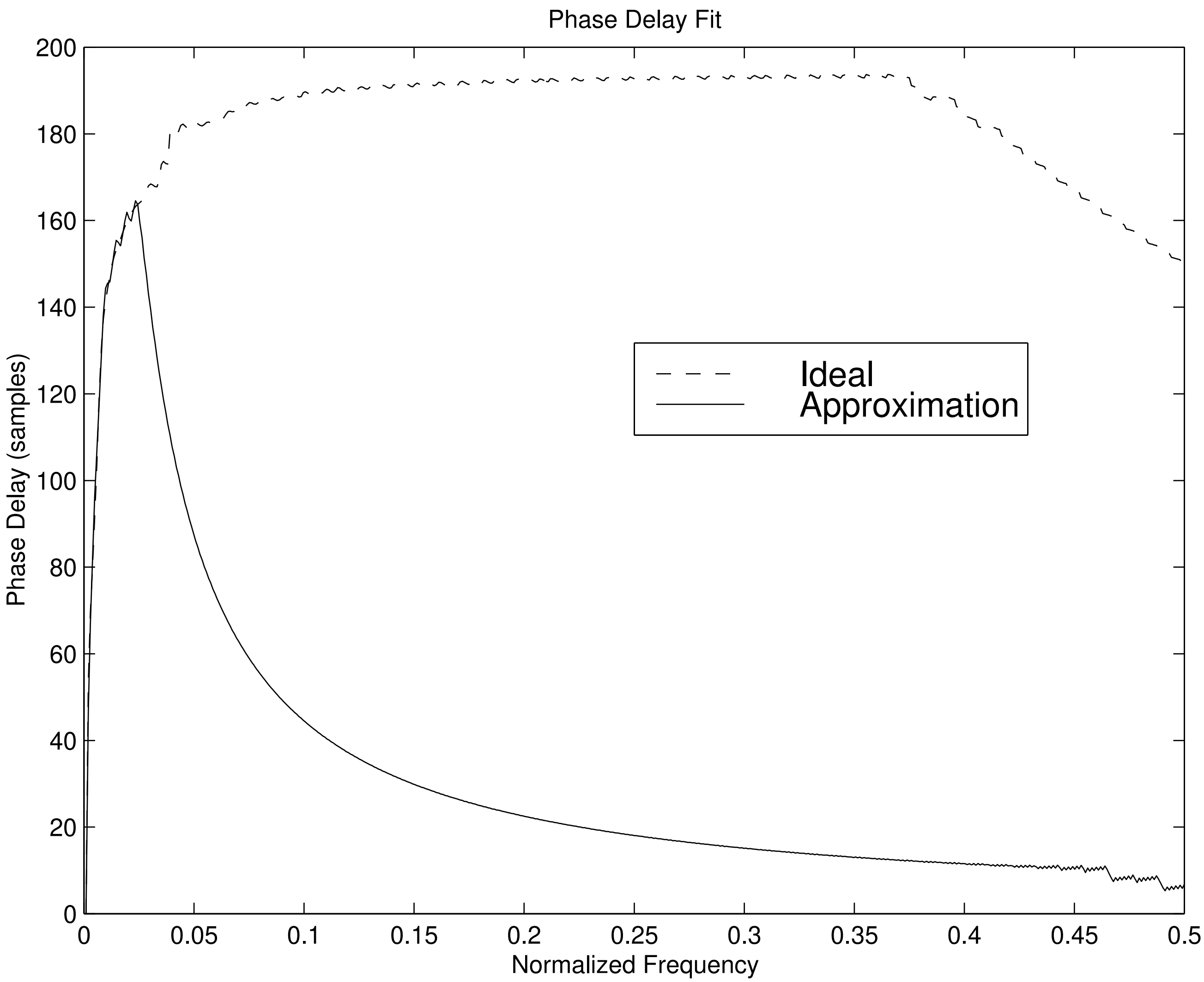


- All poles concentrated at low frequencies
- Little attention to high frequencies
- Internal "equation-error" weighting
- Numerical ill-conditioning warning printed by Matlab

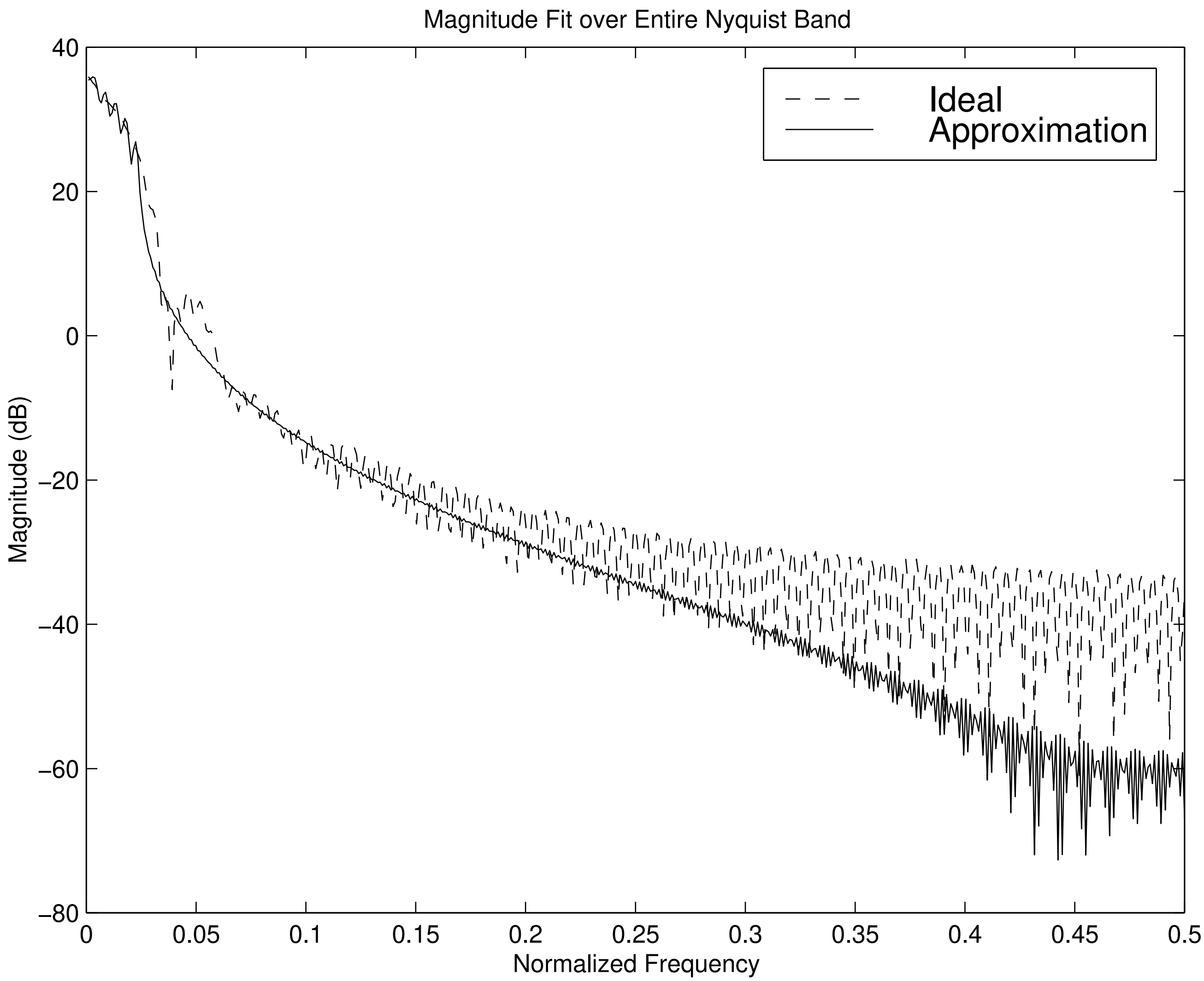
SM-10 Group Delay Fit



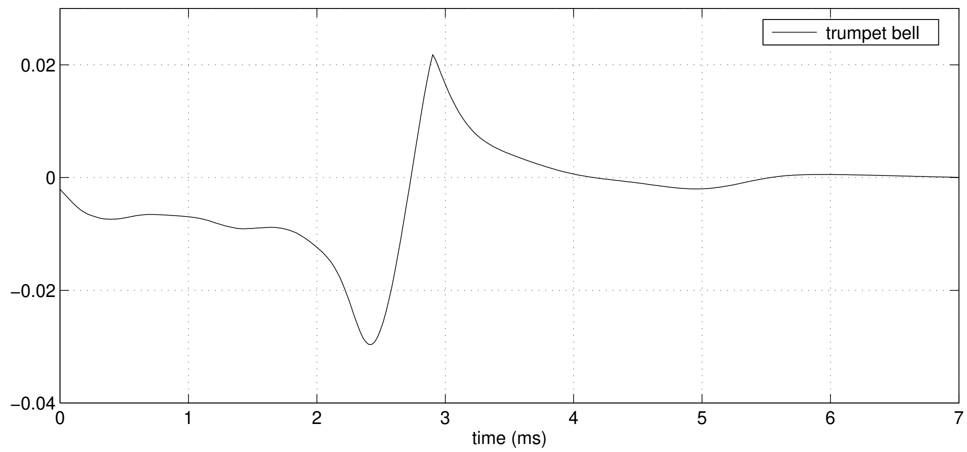
SM-10 Phase Delay Fit



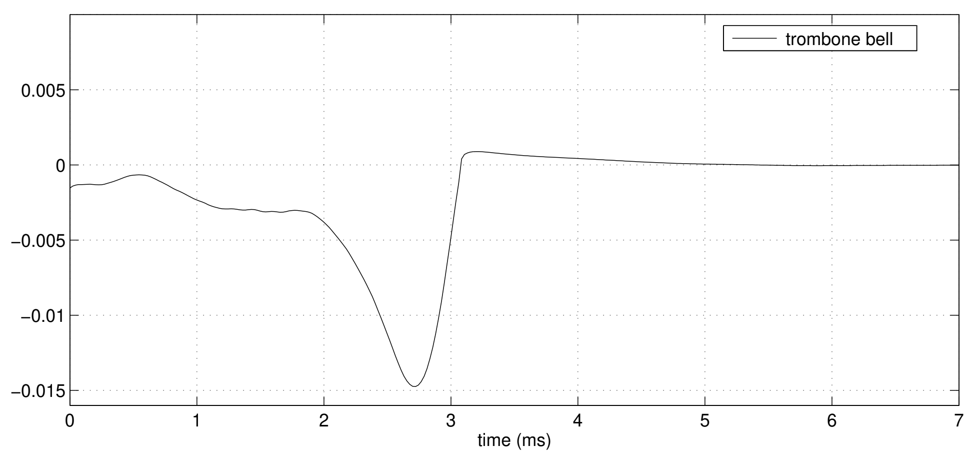
SM-10 Amplitude Response Fit



Another Measured Trumpet Bell Reflectance

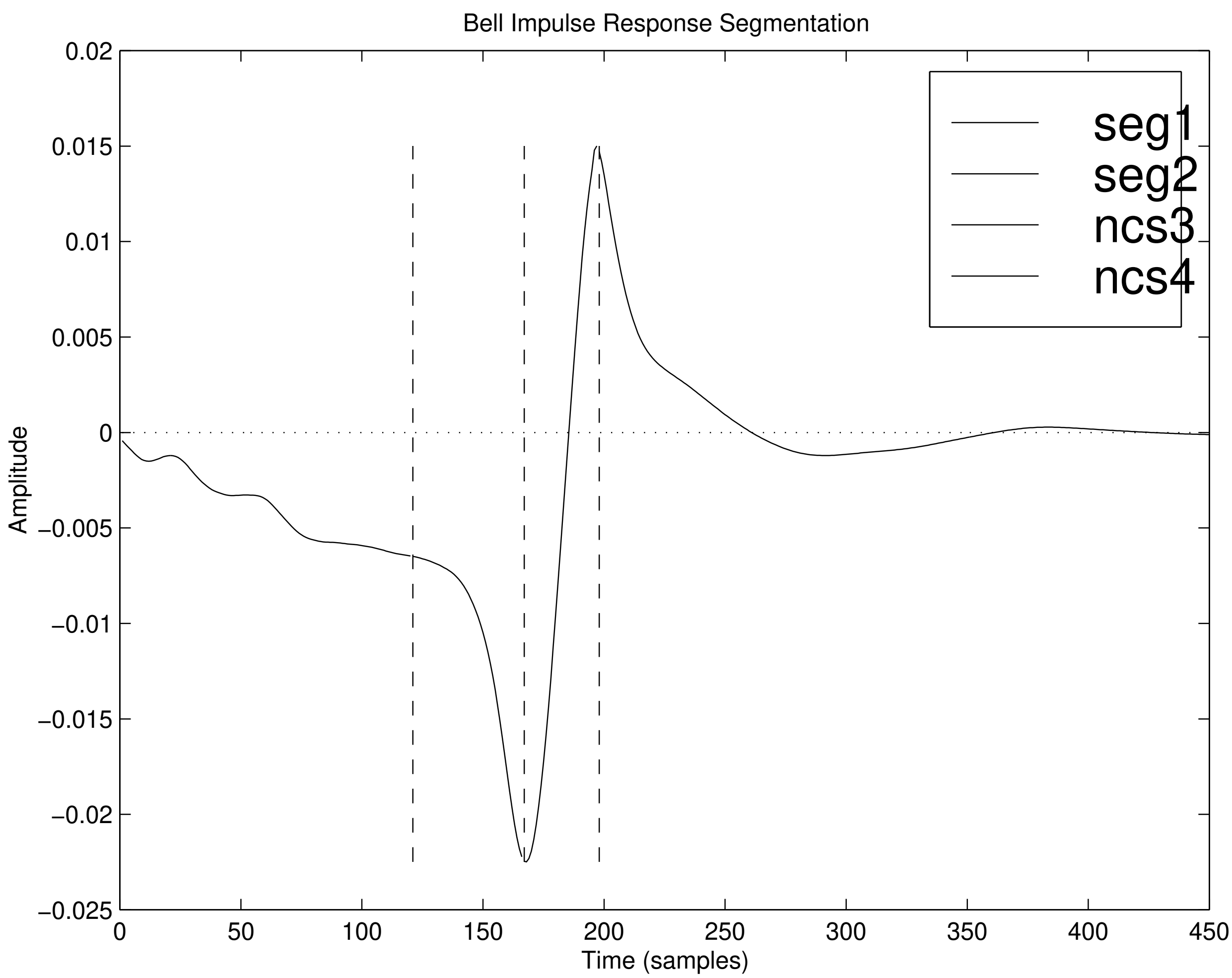


Measured Trombone Bell Reflectance

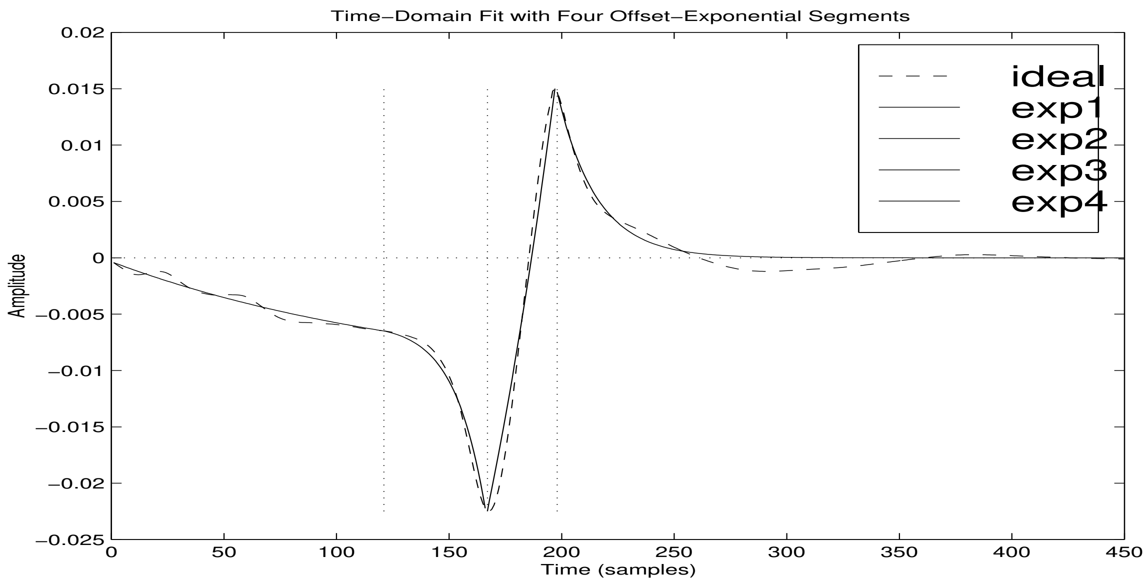


Idea!

- Break up impulse response into *exponential* or *polynomial segments*
- Exponential and polynomial impulse-responses can be designed using *Truncated IIR (TIIR) Filters*

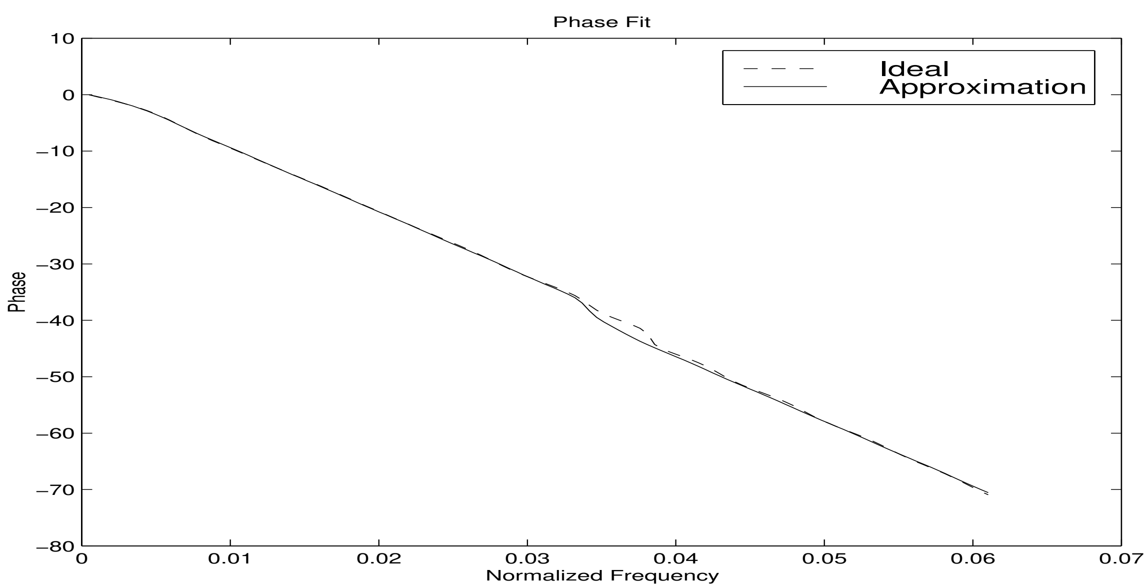


Four-Exponential Fit to Estimated Trumpet-Bell Filter (Exp-4)

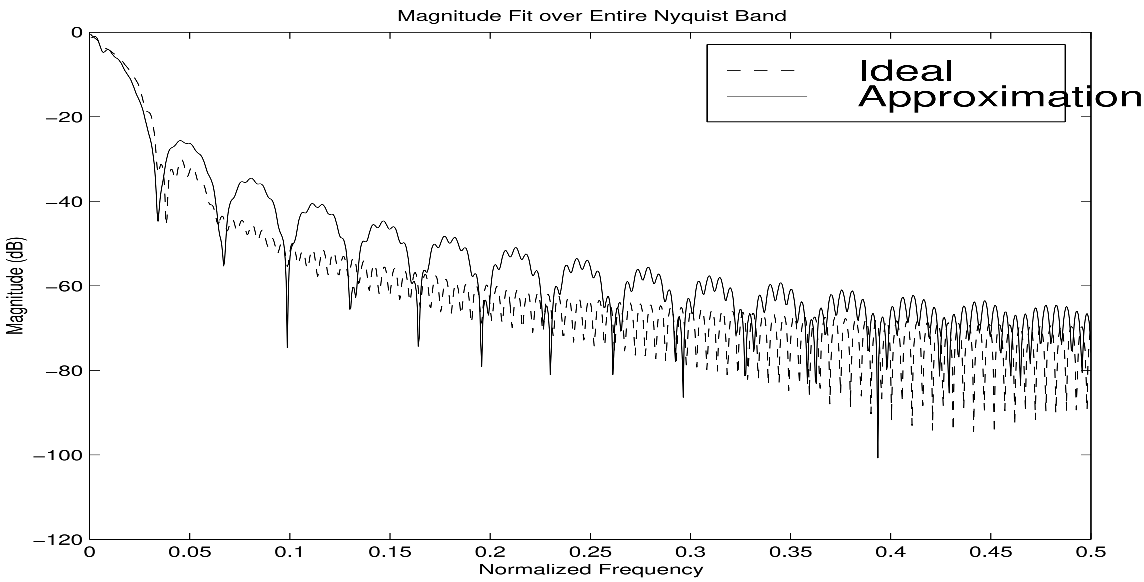


Exp-4 Impulse Response Fit

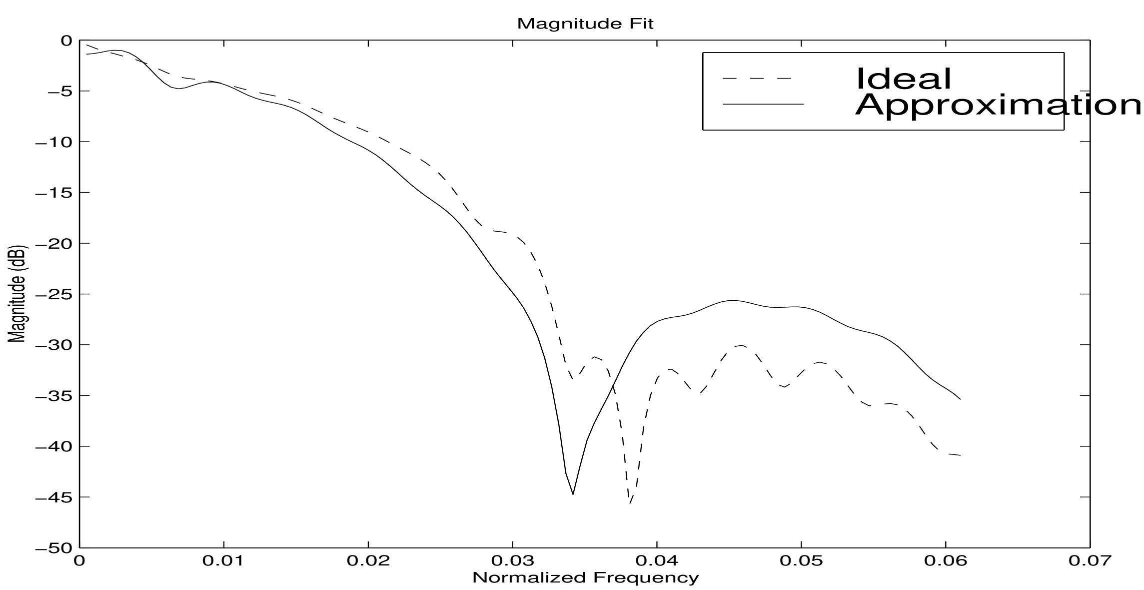
Exp-4 Phase Response Fit



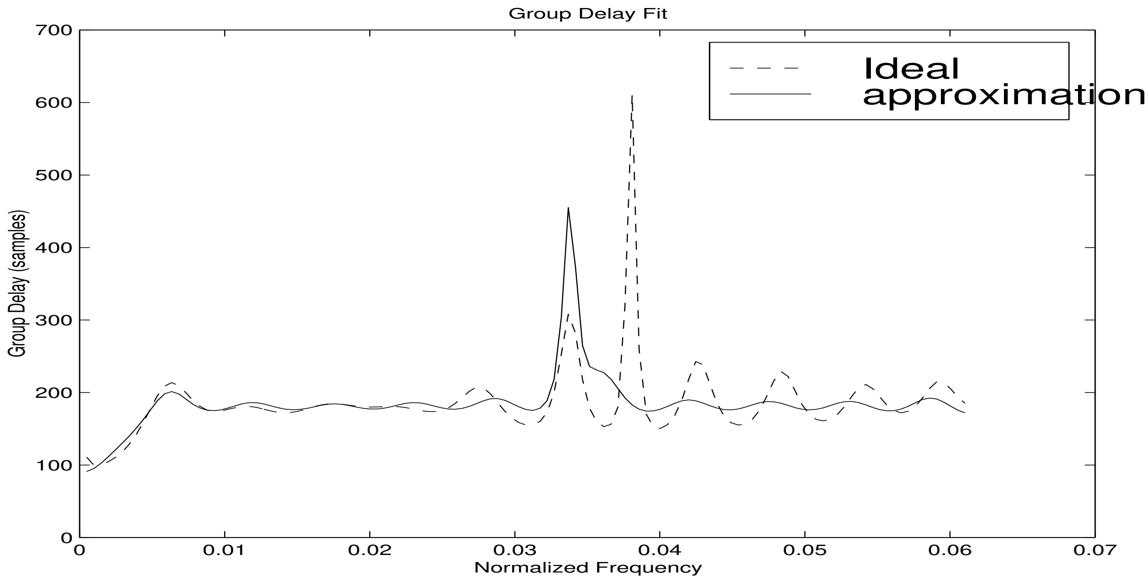
Exp-4 Amplitude Response Fit



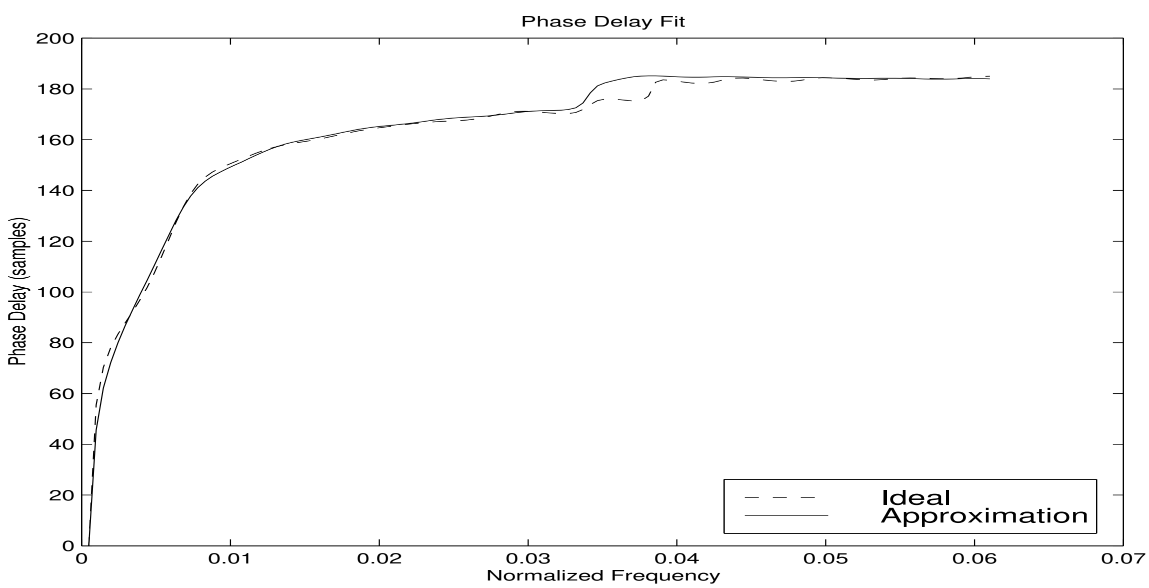
Exp-4 Low-Frequency Zoom



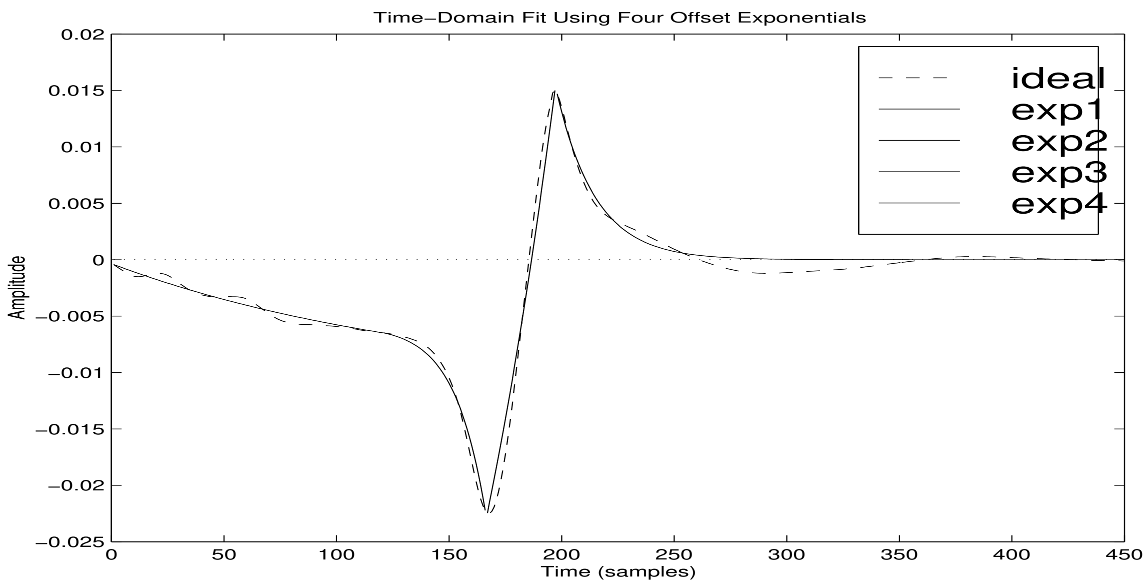
Exp-4 Group Delay Fit



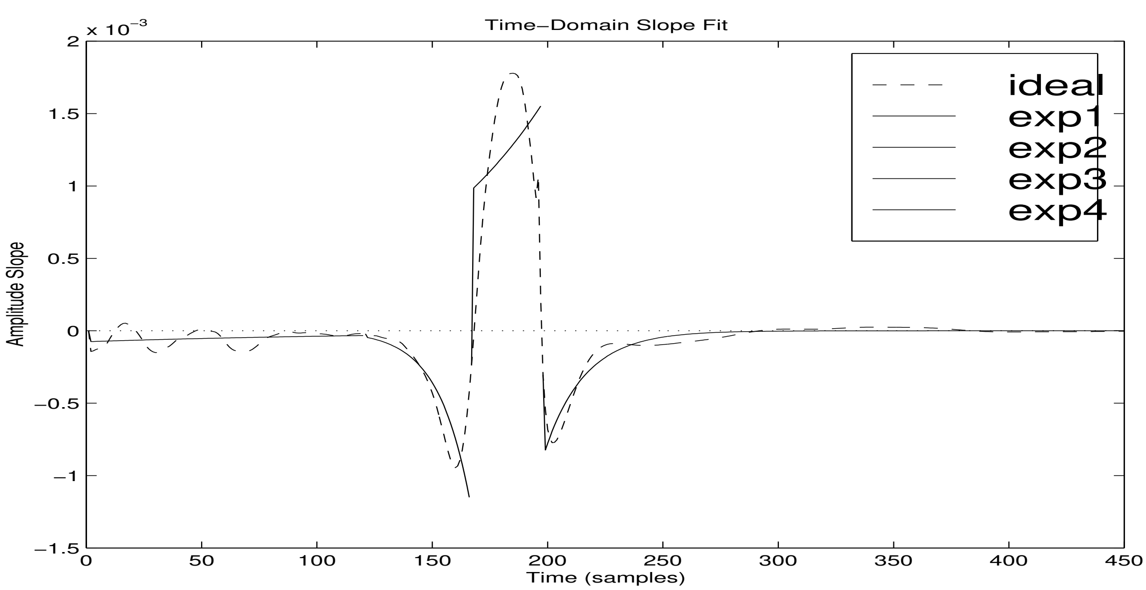
Exp-4 Phase Delay Fit



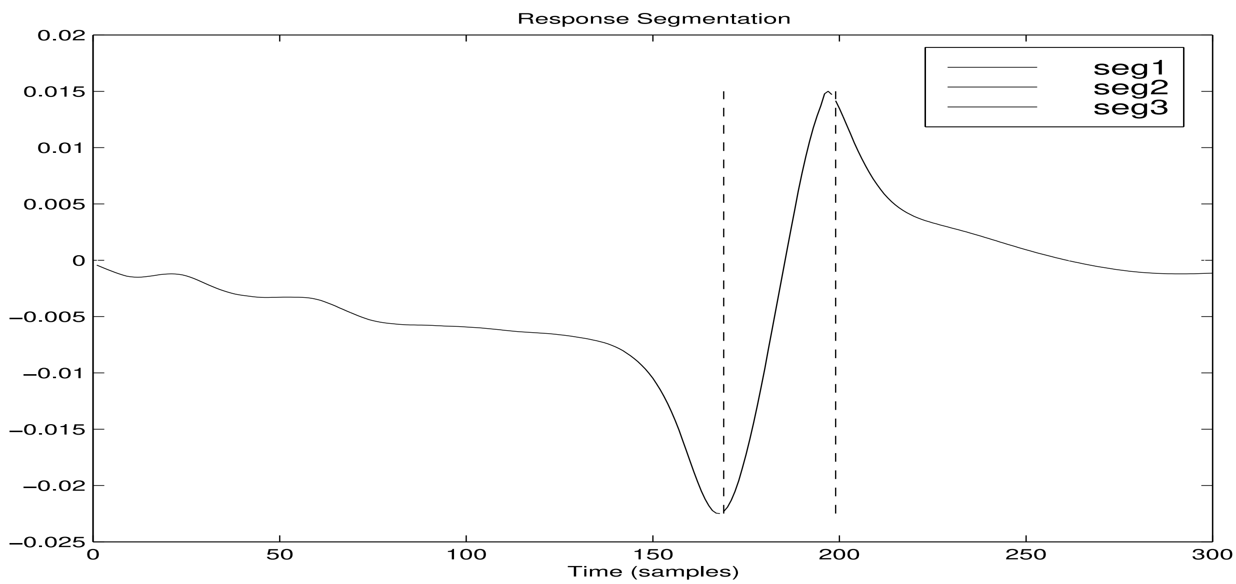
Exp-4 Impulse Response Fit (Repeated)



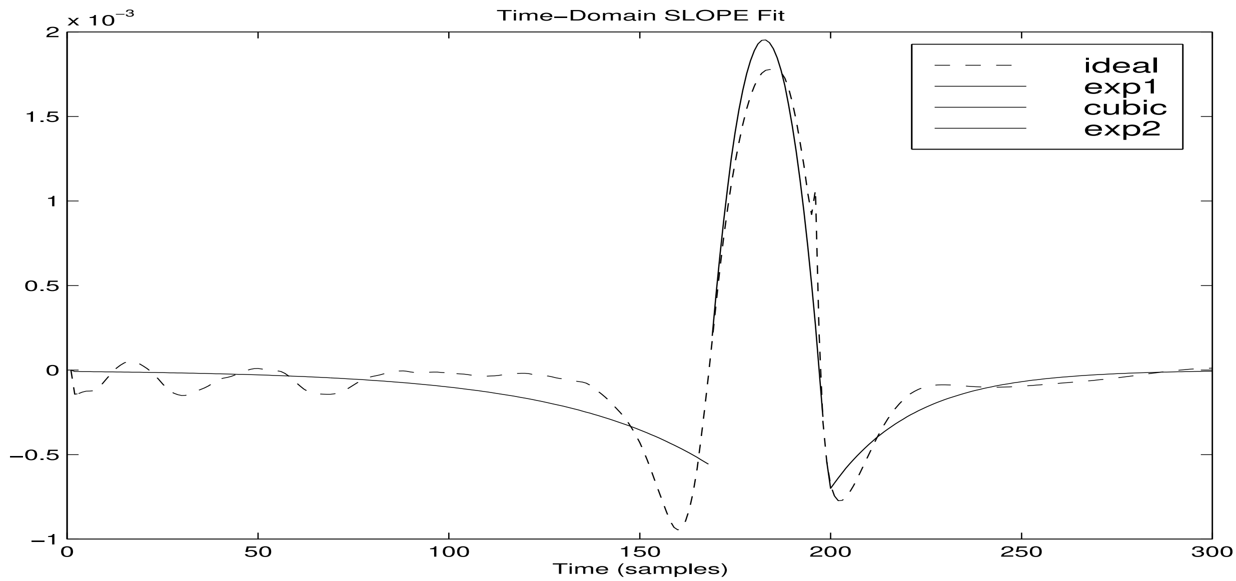
Exp-4 Slope Fit



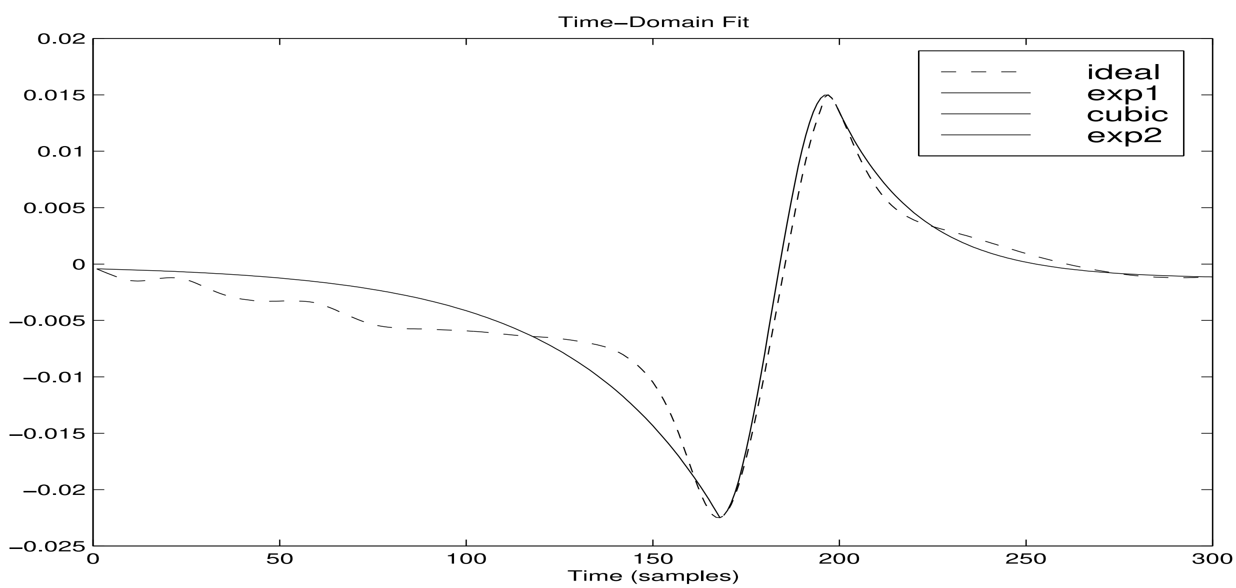
Two Exponentials Connected by a Cubic Spline Measured Trumpet Data (Exp2-S3)



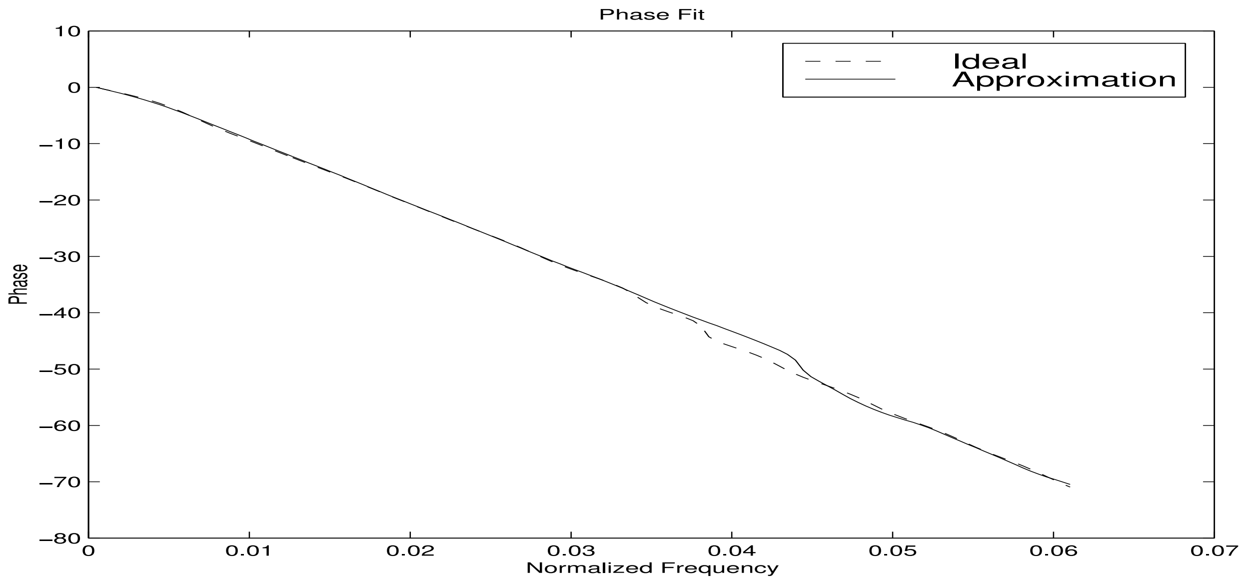
Exp2-S3 Slope Fit



Exp2-S3 Impulse Response Fit

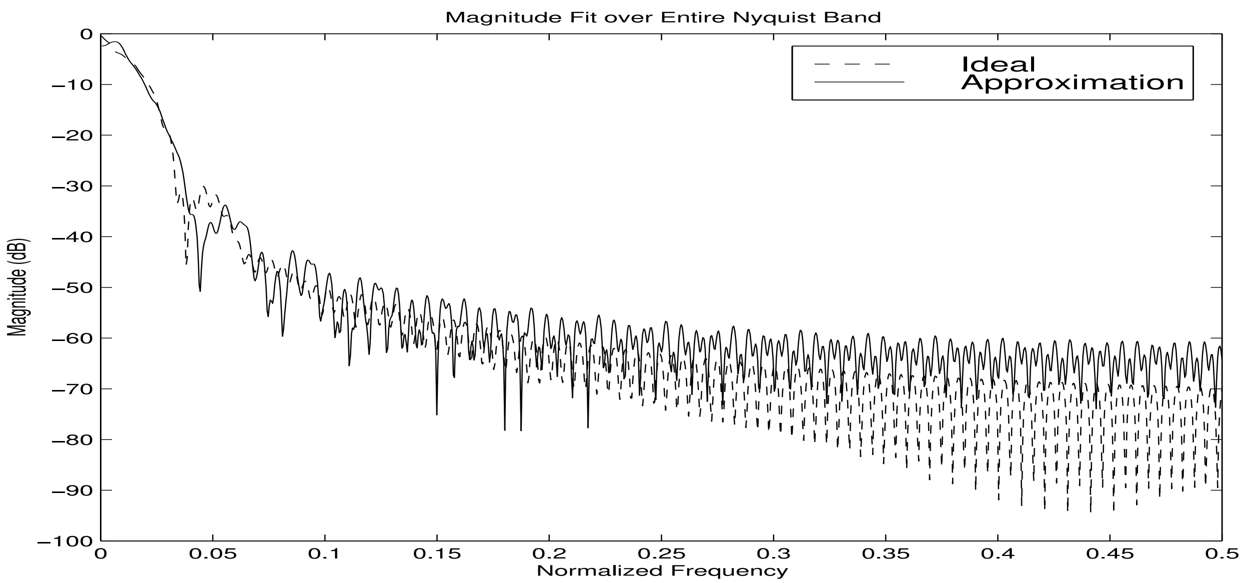


Exp2-S3 Phase Response Fit

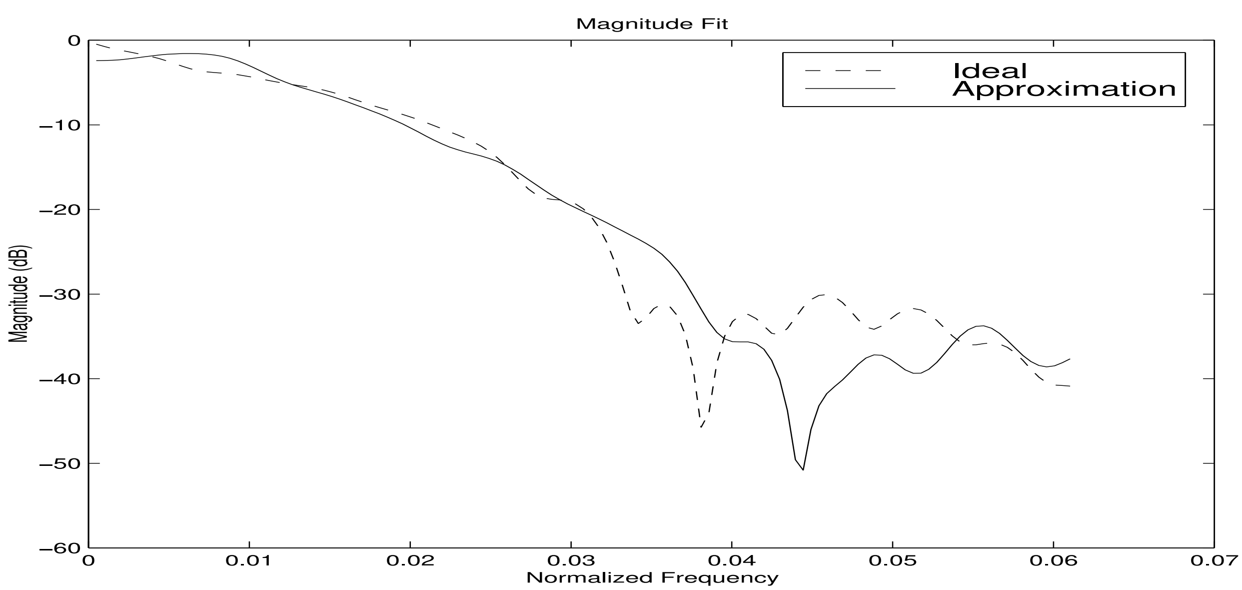


Phase Fit
Ideal
Approximation
Phase
Normalized Frequency
10
0
−10
−20
−30
−40
−50
−60
−70
−80
0
0.01
0.02
0.03
0.04
0.05
0.06
0.07

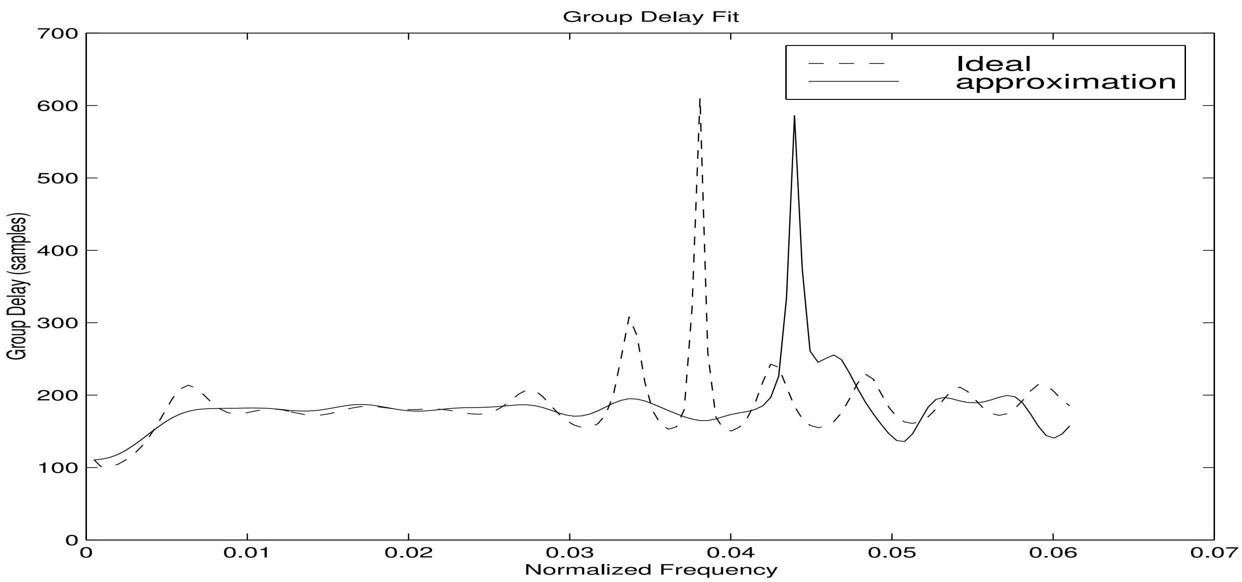
Exp2-S3 Amplitude Response Fit



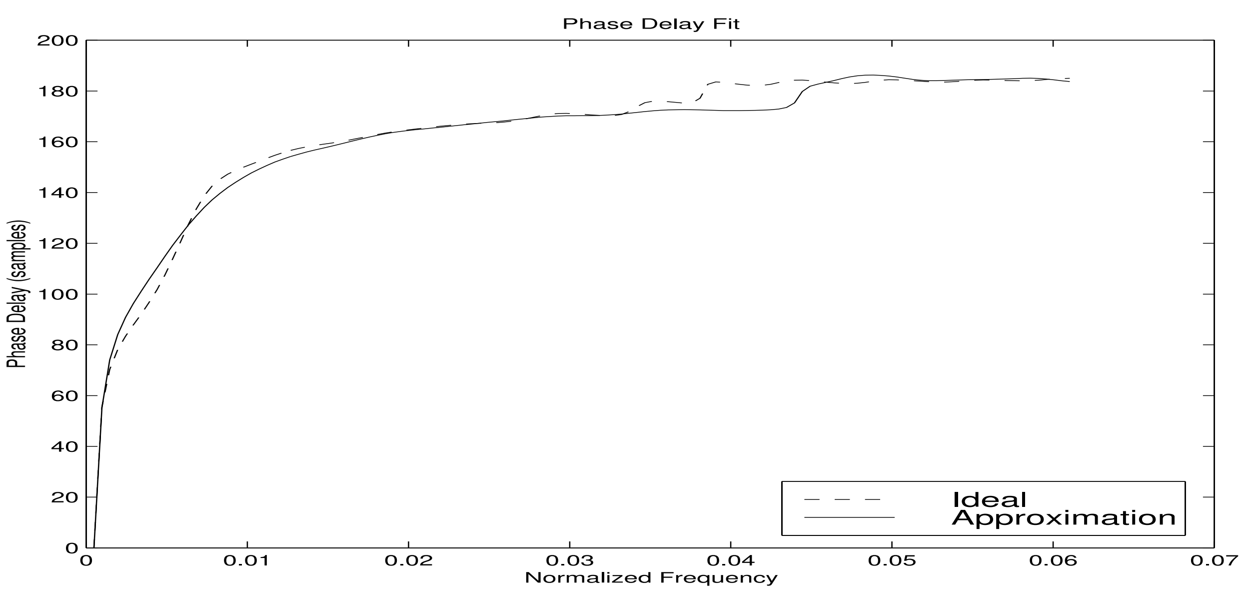
Exp2-S3 Low-Frequency Zoom



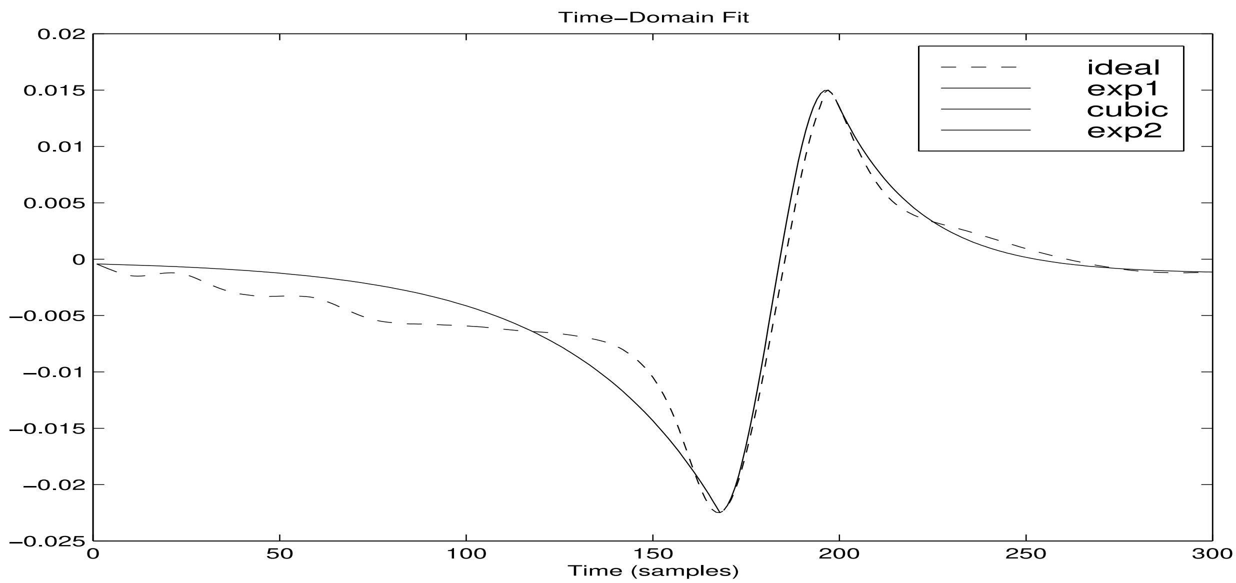
Exp2-S3 Group Delay Fit



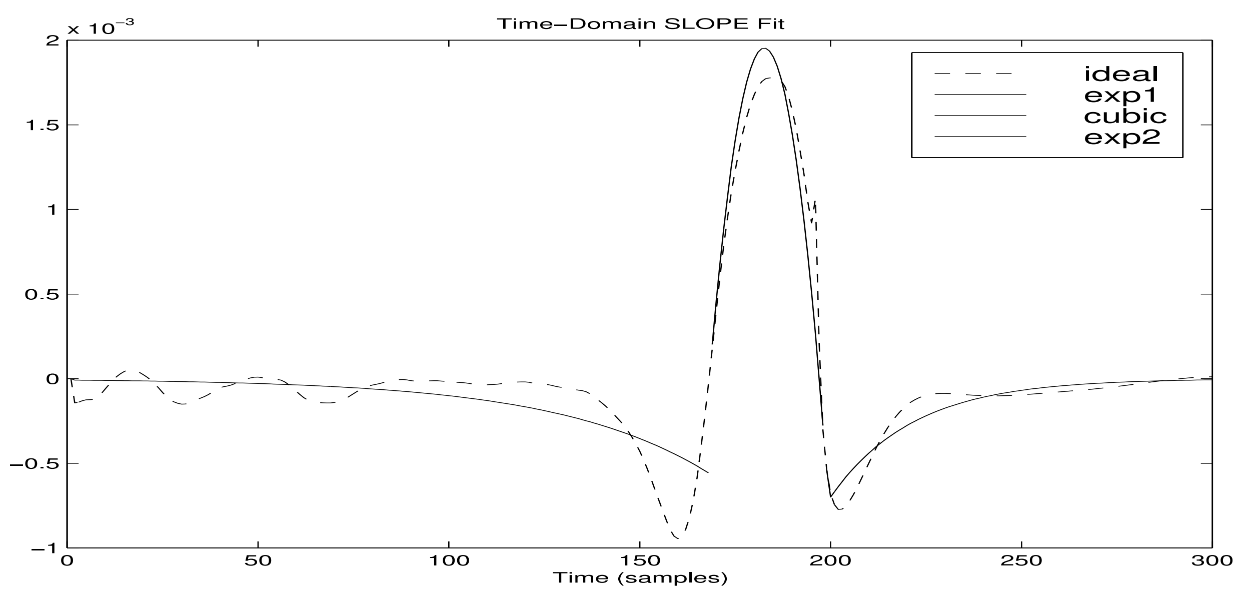
Exp2-S3 Phase Delay Fit



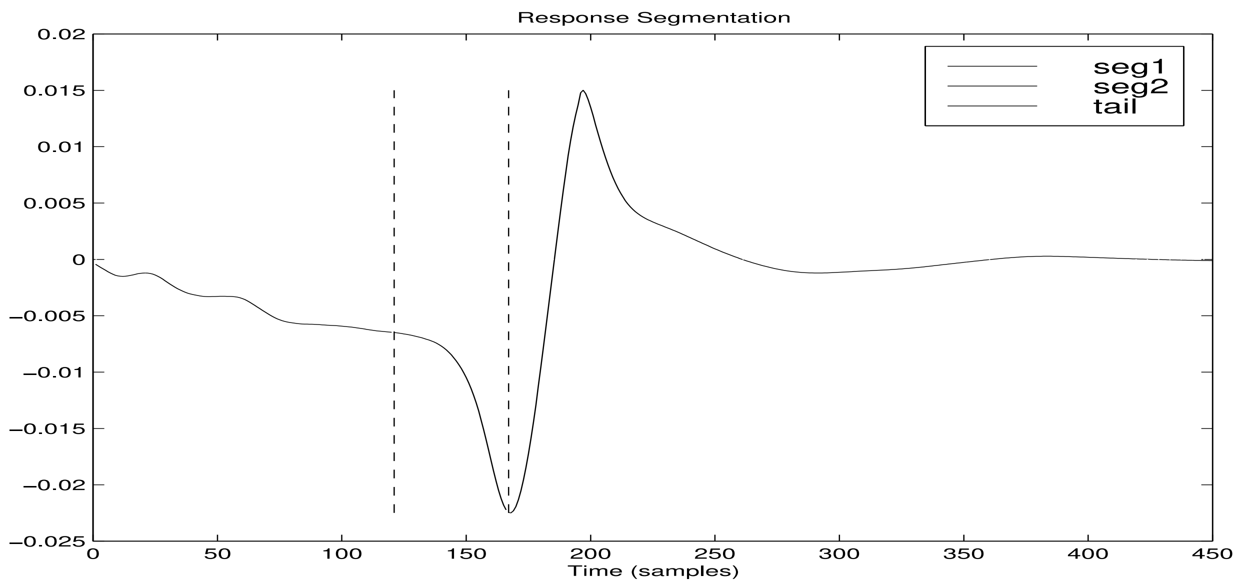
Exp2-S3 Impulse Response Fit



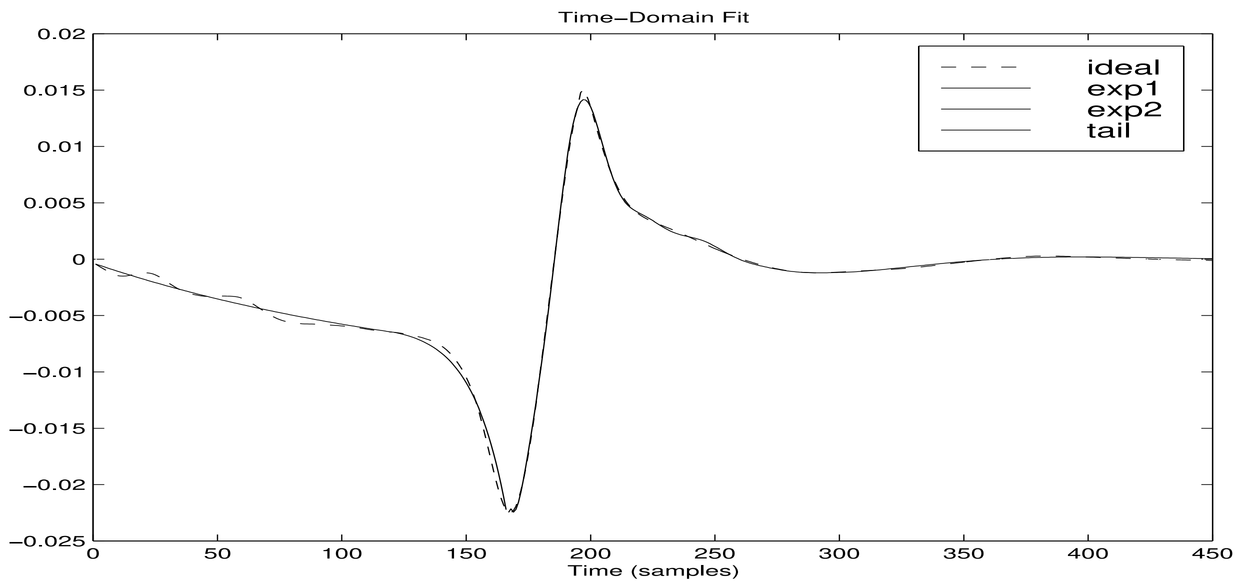
Exp2-S3 Slope Fit



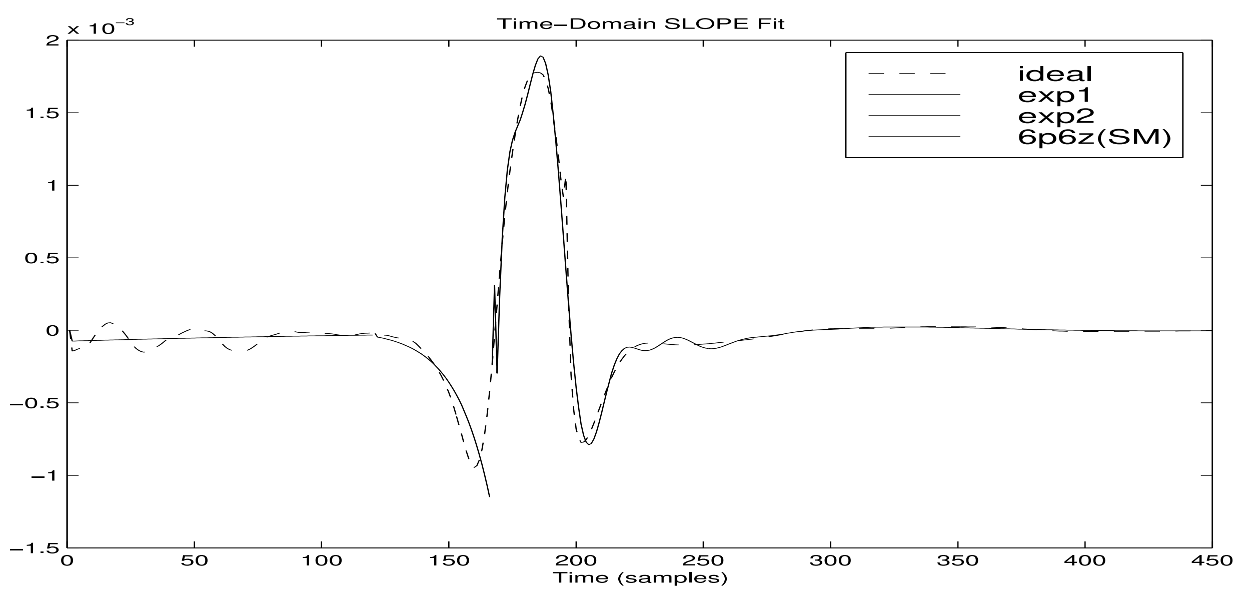
Two Exponentials Followed by a 6th-Order IIR Filter Designed by Steiglitz McBride Algorithm (Exp2-SM6)



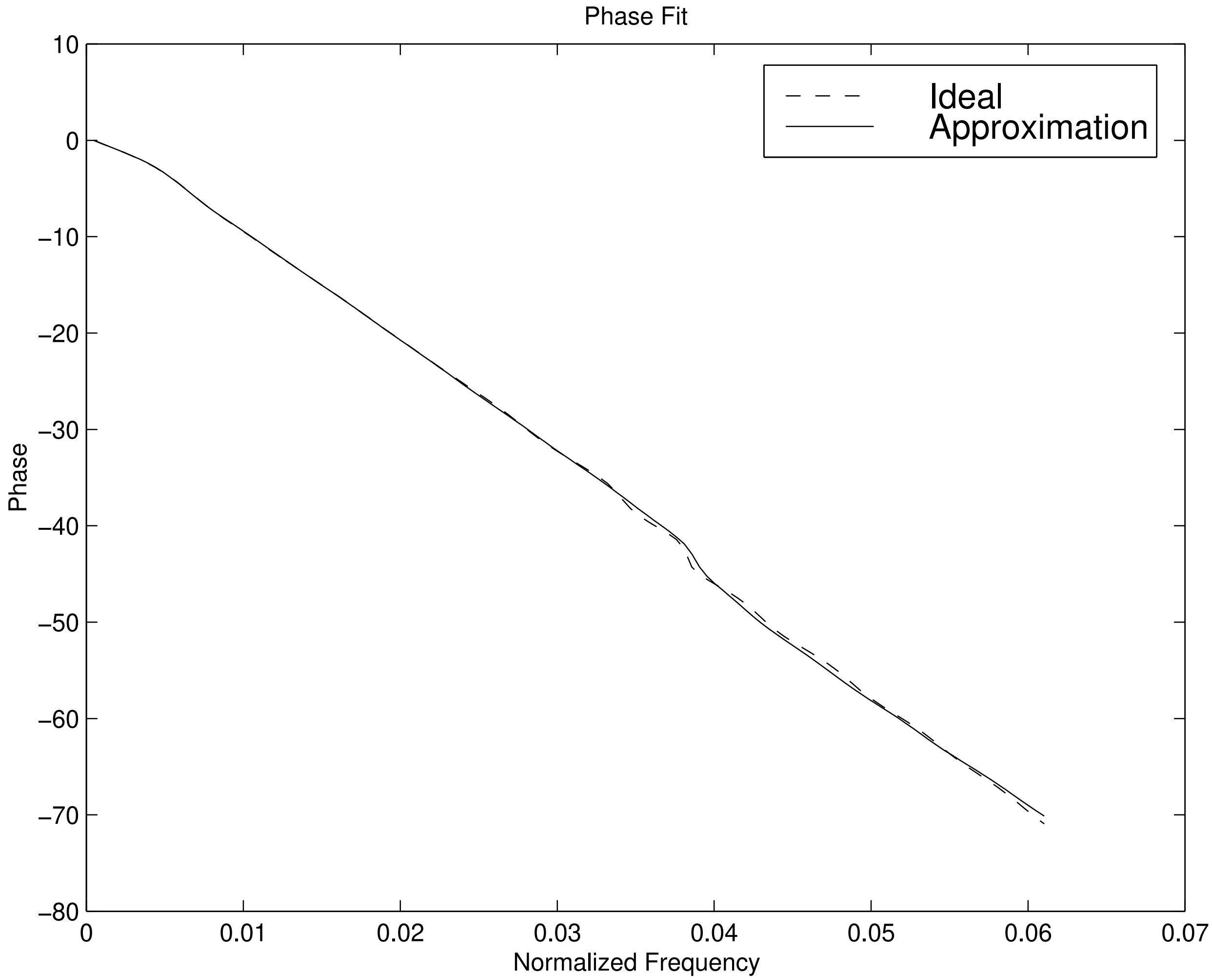
Exp2-SM6 Impulse Response Fit



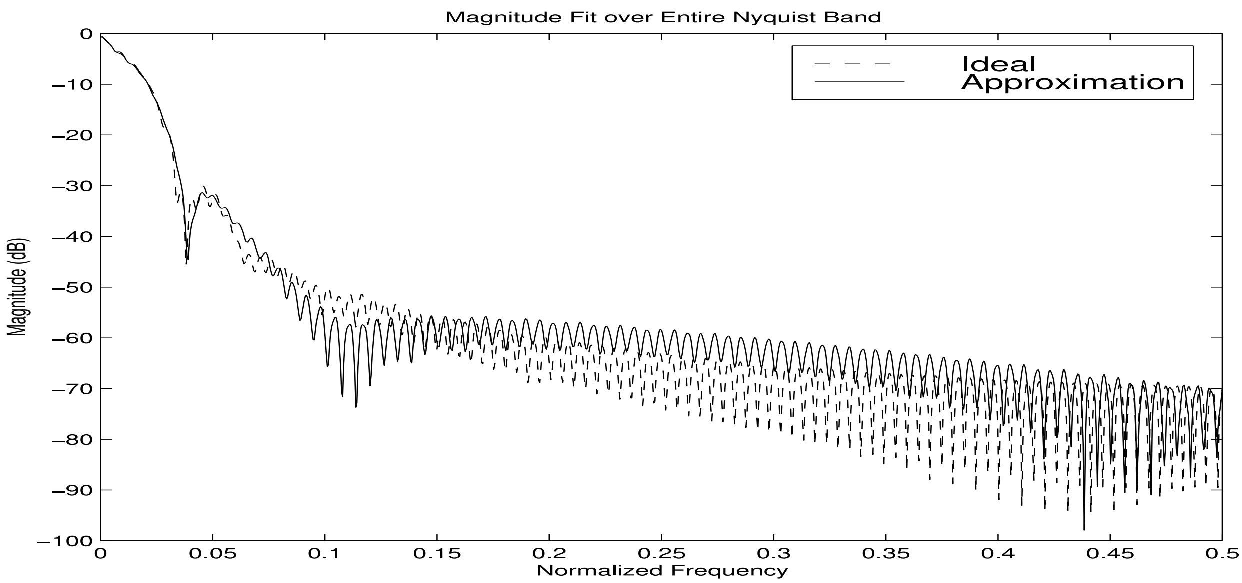
Exp2-SM6 Slope Fit



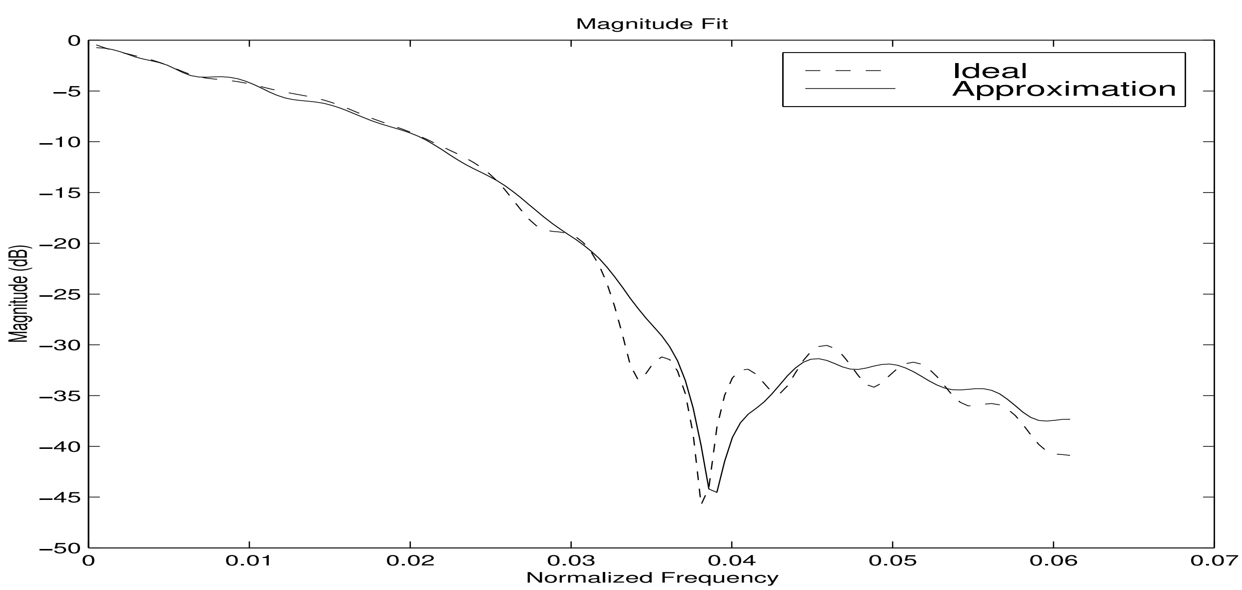
Exp2-SM6 Phase Response Fit



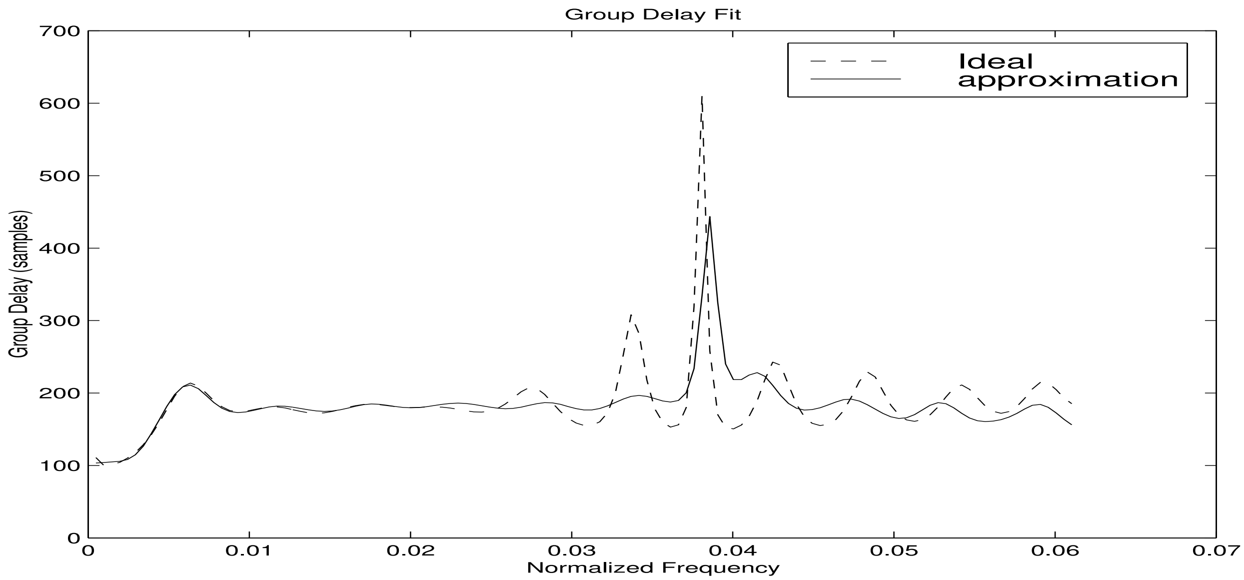
Exp2-SM6 Amplitude Response Fit



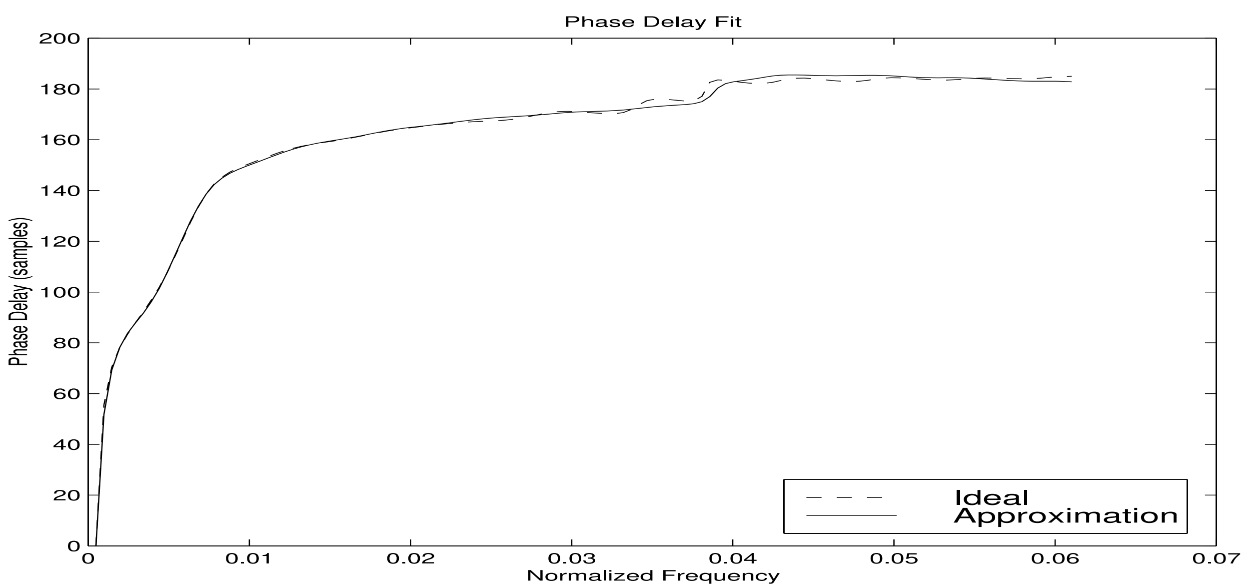
Exp2-SM6 Low-Frequency Zoom



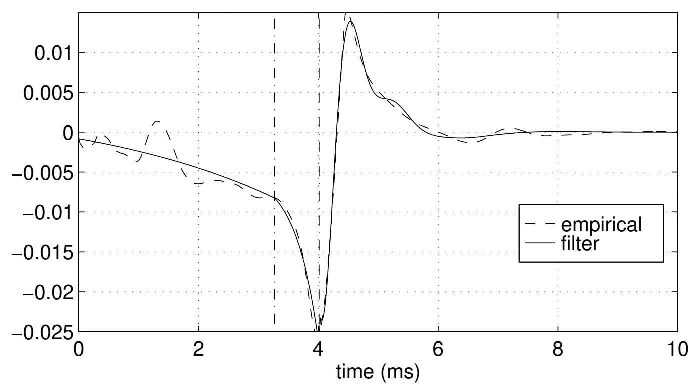
Exp2-SM6 Group Delay Fit

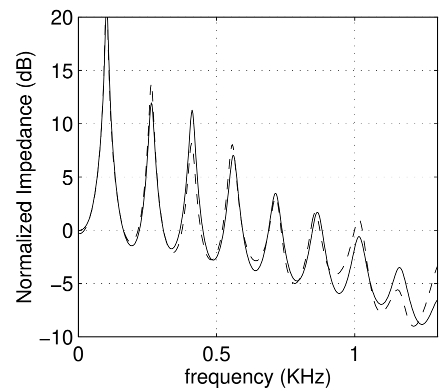


Exp2-SM6 Phase Delay Fit



Results for Measured Trumpet Data Using Two Offset Exponentials and Two Biquads

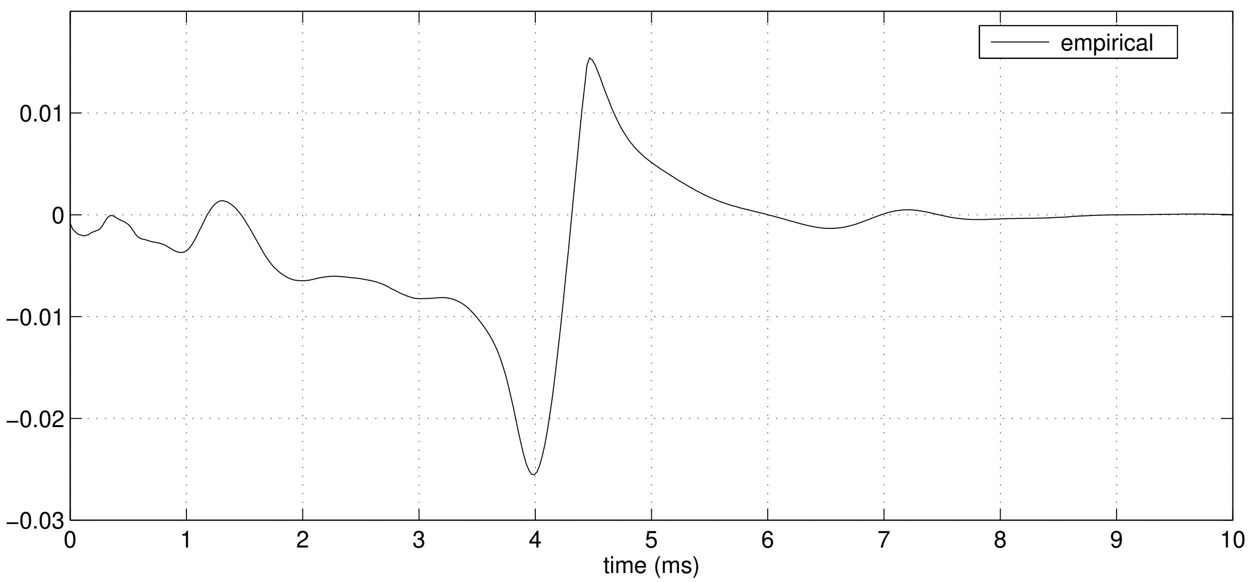




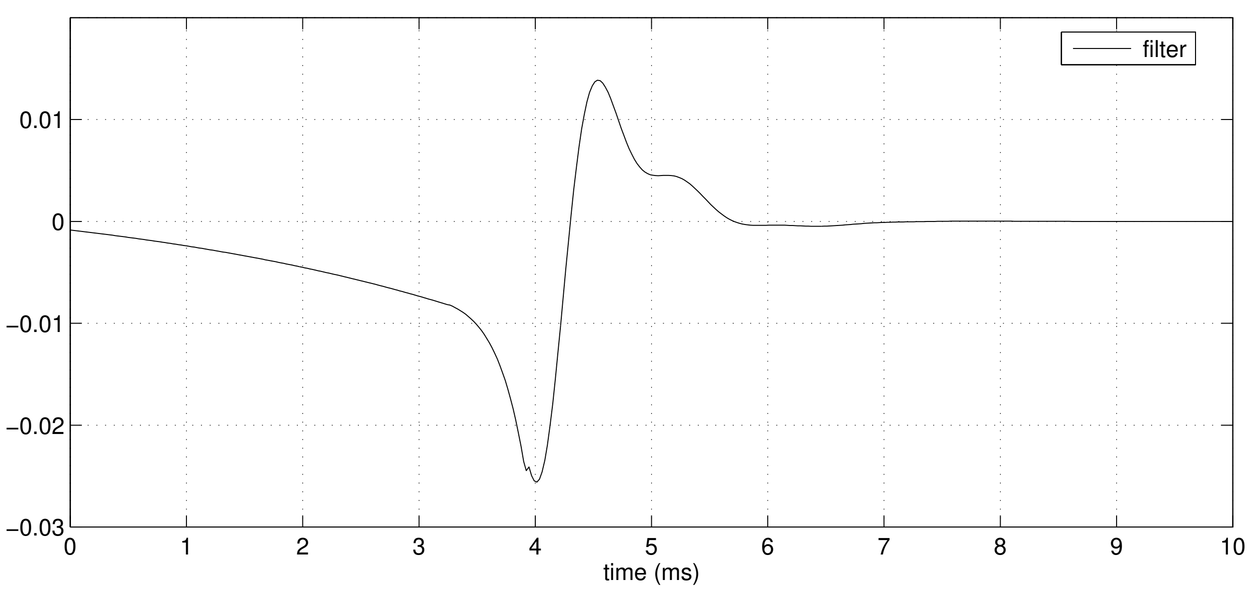
Slope Fit

- Bell model filter complexity comparable to order 8+ IIR
- Offset exponentials were fit using `fmins()` in Matlab
- Two biquads were fit as a single fourth-order filter using the Steiglitz-McBride algorithm (`stmcb()` in Matlab)

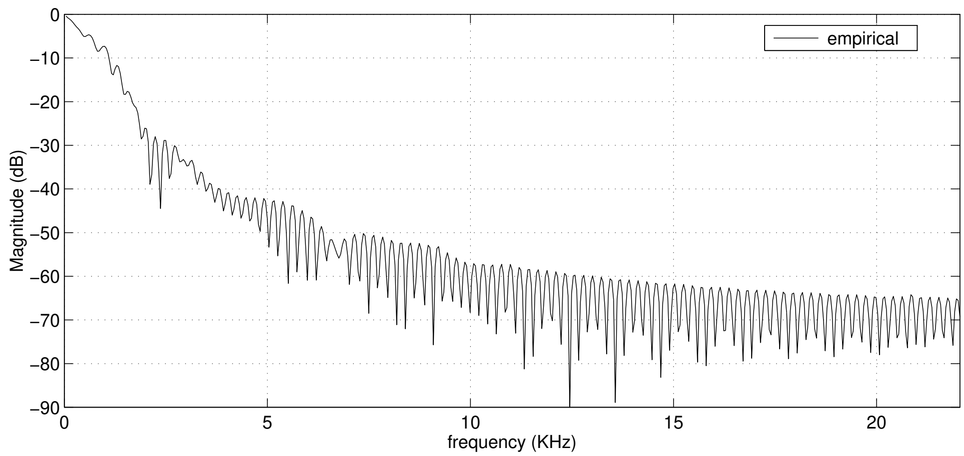
Measured Trumpet Bell Impulse Response



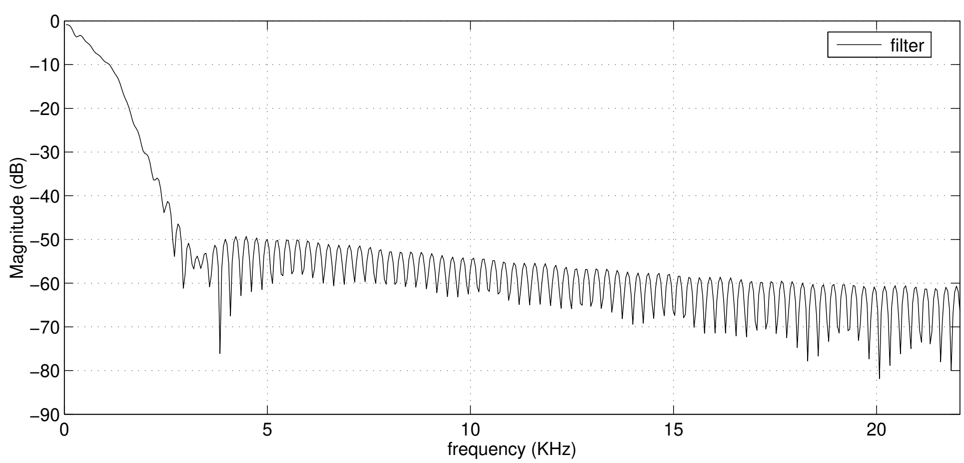
TIIR Trumpet Bell Impulse Response



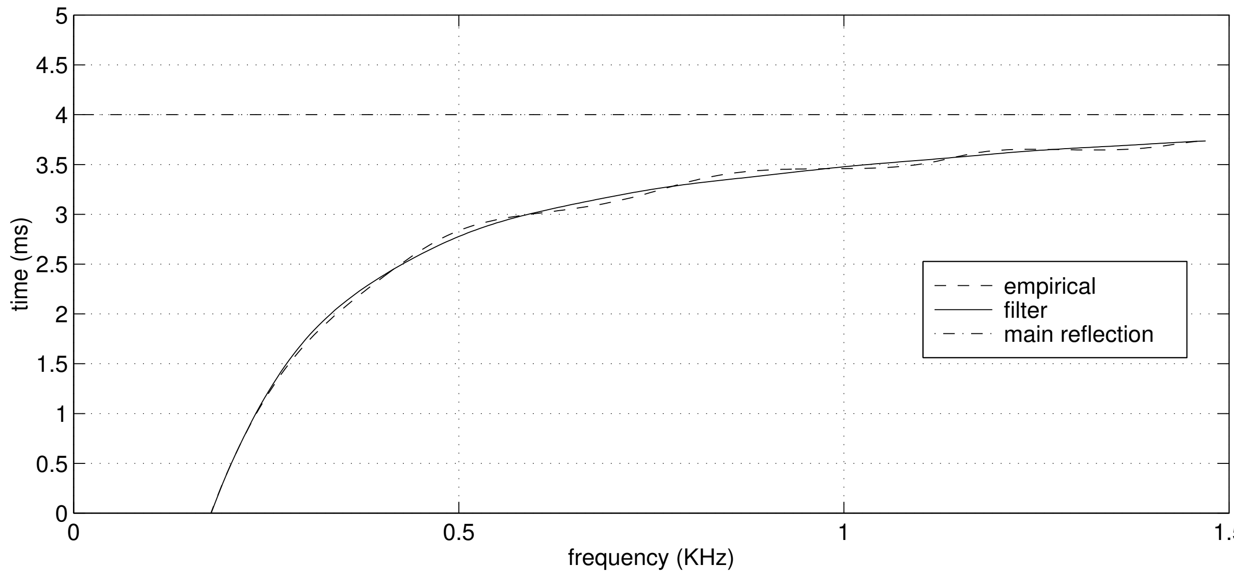
Measured Trumpet Bell Amplitude Response



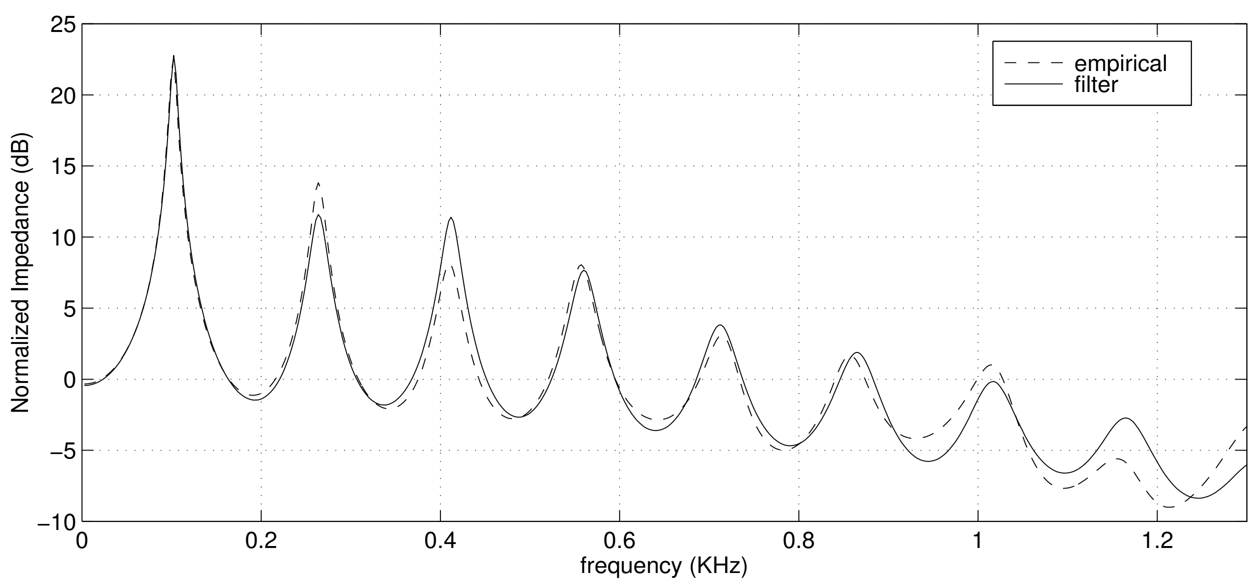
TIIR Trumpet Bell Amplitude Response



Trumpet Bell Phase Delay Fit



Input Impedance of Complete Bore + Bell Model

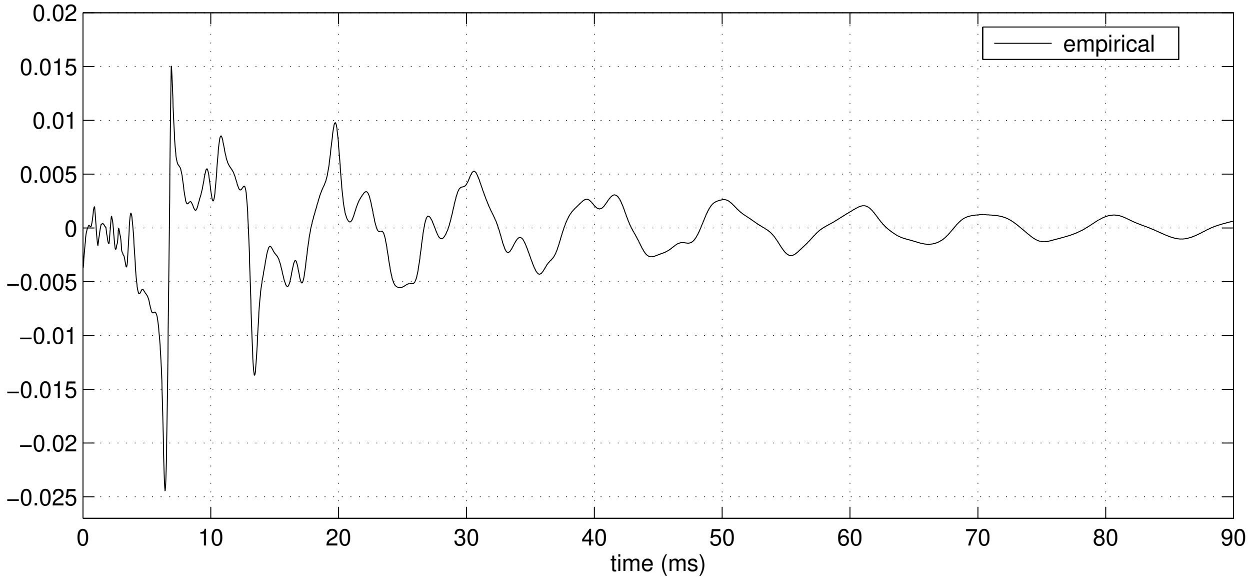


Comparison to Measurements

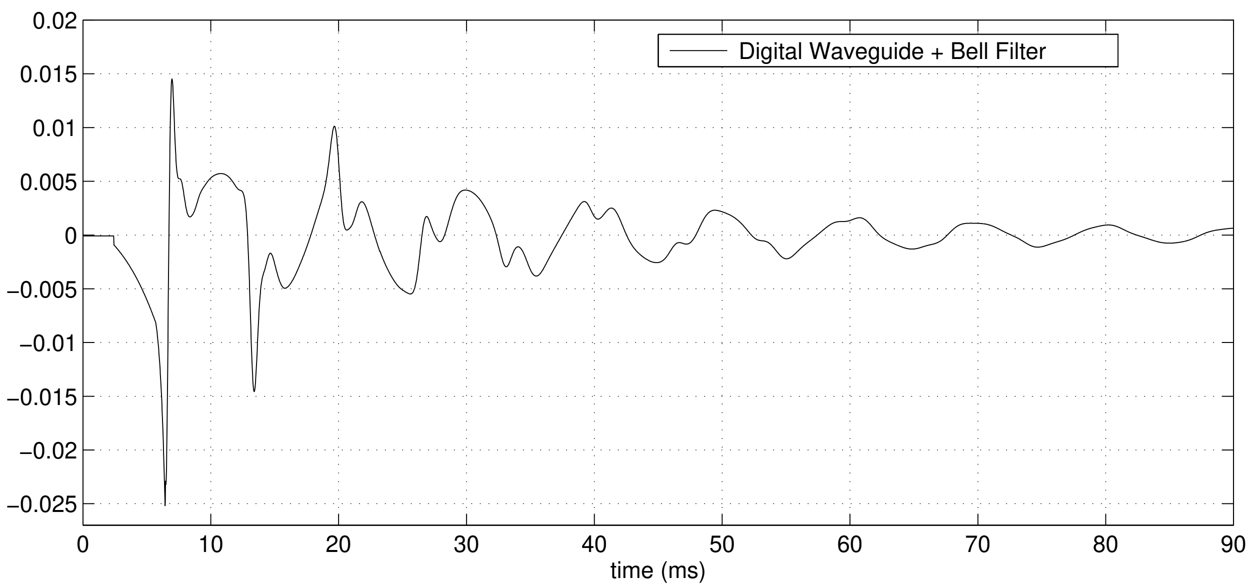
The next two pages of plots compare the *measured impulse response* with that produced by the final digital waveguide model consisting of a trumpet bore + bell (but no mouthpiece).

- Comparison 1: two offset exponentials and two biquads to model the bell impulse response
- Comparison 2: two offset exponentials and three biquads to model the bell impulse response

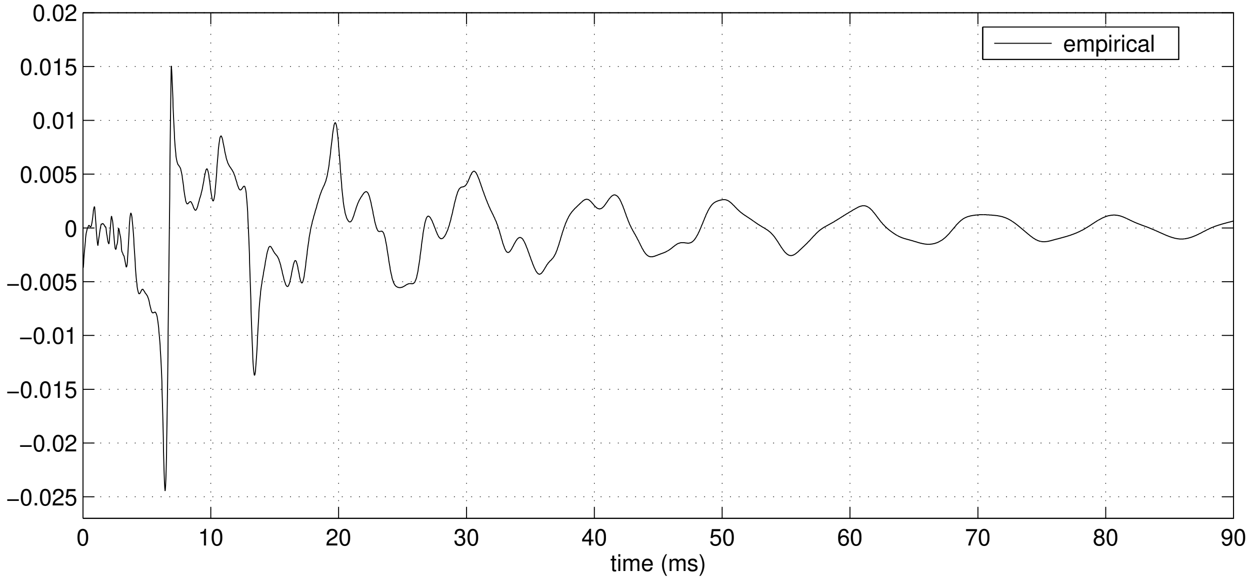
Measured Impulse Response



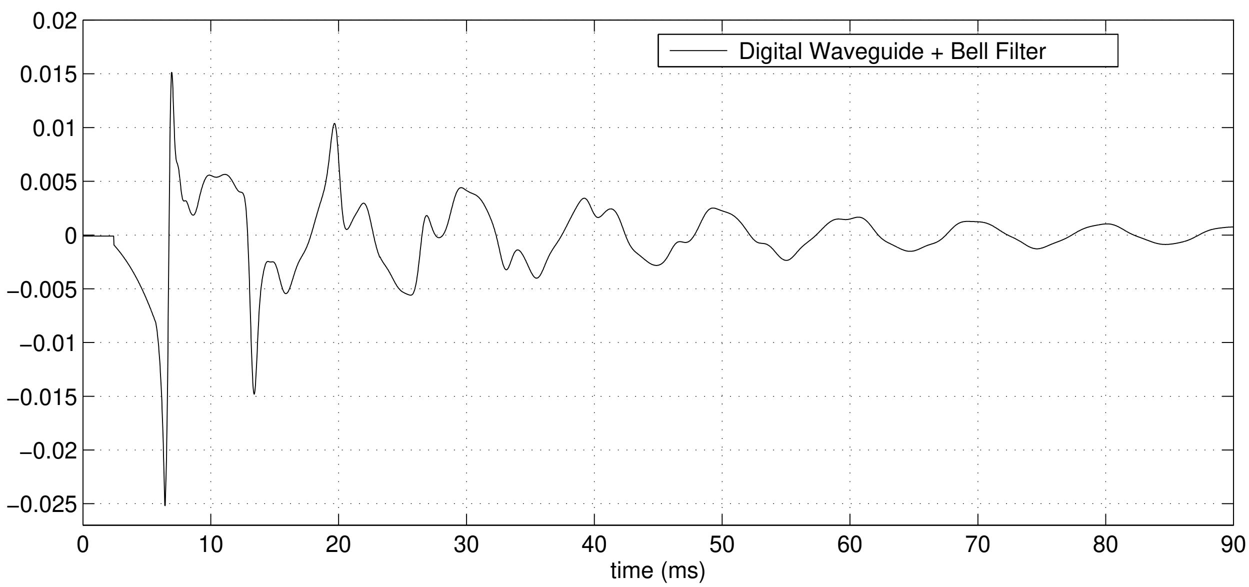
Synthesized Impulse Response, Order 4 Tail



Measured Impulse Response



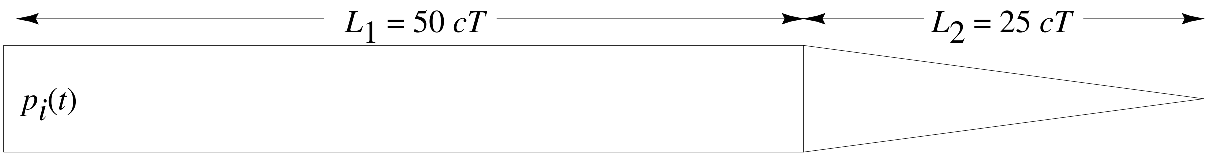
Synthesized Impulse Response, Order 6 Tail



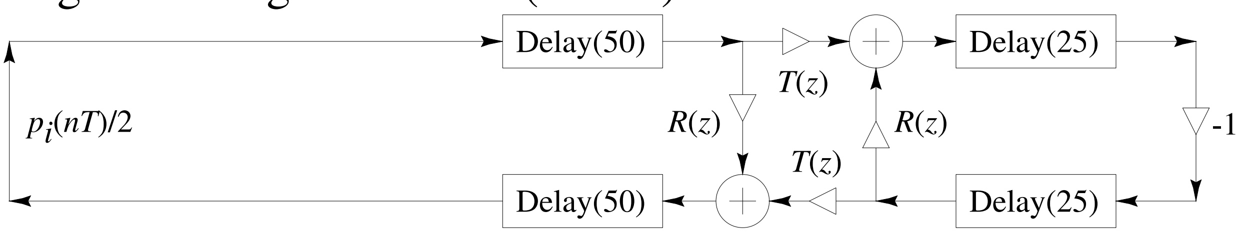
Piecewise Conical Acoustic Tube Modeling

Simple Example: Cylinder with Conical Cap

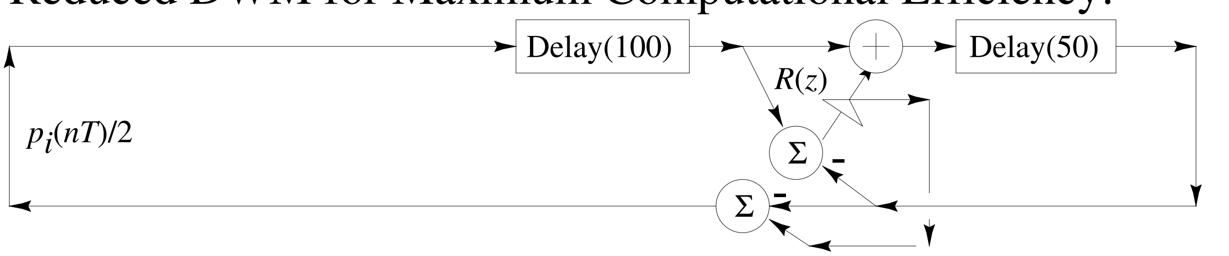
Physical Outline of Cylinder and Cone:



Digital Waveguide Model (DWM) for Pressure Waves:



Reduced DWM for Maximum Computational Efficiency:



where

$$
\begin{aligned}
R(z) &= \left(\frac{1}{99}\right)\left(\frac{1+z^{-1}}{1-\frac{101}{99}z^{-1}}\right) \\
T(z) &= \left(\frac{100}{99}\right)\left(\frac{1-z^{-1}}{1-\frac{101}{99}z^{-1}}\right) = 1 + R(z)
\end{aligned}
$$

- **Problem:** Reflection filter $R(z)$ and transmission filter $T(z)$ are *unstable* (pole at $z = 101/99$)
- Overall system is passive $\Rightarrow$ unstable pole is *canceled*

Implementation Idea

Apply TIIR "alternate and reset" idea
to the unstable conical subsystem

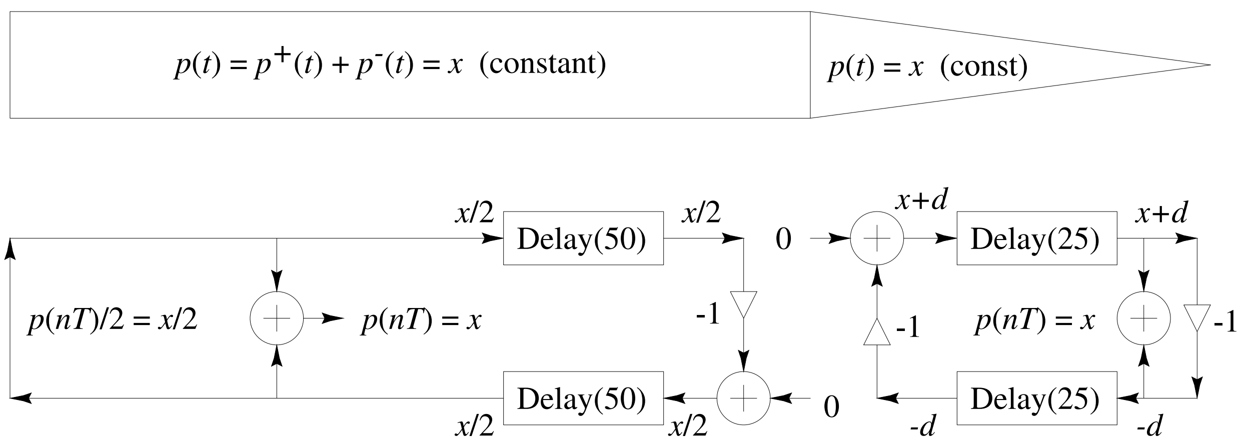
- Cone is not truly FIR $\Rightarrow$ t_{60} replaces FIR length
- When cylinder is closed-ended, cone traveling-wave components increase without bound $\Rightarrow$ must switch out and reset the entire cone assembly (scattering-junction filter $R(z)$ and cone's entire delay line)
- According to simulations thus far, cylinder waves are well behaved and do not need to be reset (no general proof yet)

Basic Principle

Periodically reset any subsystem containing a canceled unstable pole at intervals greater than or equal to the t_{60} for that subsystem

Interesting Paradox at DC

DC Steady State: Closed-End Cylinder



- $R(1) = -1$ (dc response of reflection filter inverts)
- $T(1) = 0$ (dc does not transmit through the junction)
- Physically obvious dc solution (constant pressure offset) is not possible in either the cone or the cylinder model!
- Simulated impulse responses agree with the literature
- A final constant dc offset *is* observed in the simulations

Solution to Paradox

- It turns out the reflection transfer function looking into the cone from the cylinder has *two poles* and *two zeros* at dc
- The dc poles and zeros *cancel* and leave a dc cone reflectance equal to $+1$ (the physically obvious answer)
- We can't just set the reflection filter to its dc equivalent to figure out the dc behavior of the overall model
- Instead, a more careful limit must be taken

In the s plane, the conical cap pressure reflectance, seen from the cylinder, can be derived to be

$$H(s) \triangleq \frac{1 + R(s)(1 + 2st_x)}{2st_x - 1 - R(s)}$$

where t_x is the time (in seconds) to propagate across the cone, and

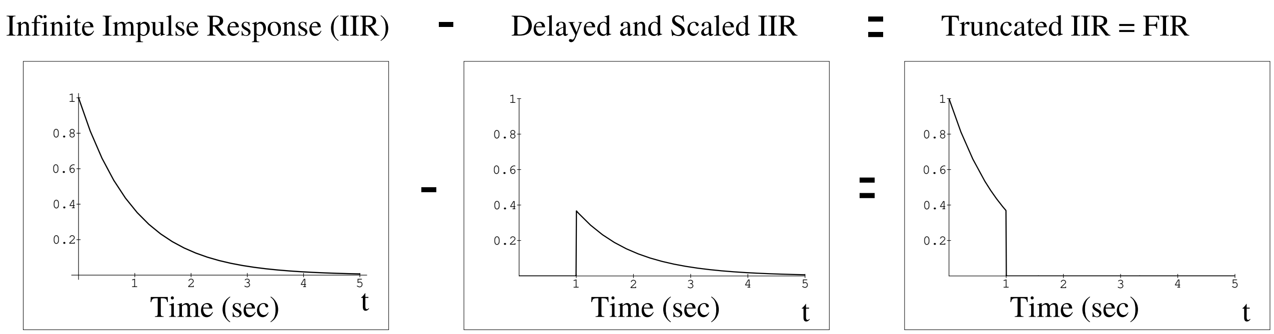
$$R(s) = -e^{-2st_x}$$

is the reflectance of the cone at its entrance. We have

$$\begin{aligned} \lim_{s\to 0} R(s) &= -1 \\ \lim_{s\to 0} H(s) &= +1 \end{aligned}$$

Truncated Infinite Impulse Response (TIIR) Digital Filters

An FIR filter can be constructed as the difference of two IIR filters:



General FIR filter

- Coefficients: $\{h_0, \ldots, h_N\}$
- Implementation (convolution):

$$y(n) = (h * x)(n) = \sum_{m=0}^{N} h_m x(n-m)$$

- Transfer function:

$$\begin{aligned} H_{\text{FIR}}(z) &\stackrel{\Delta}{=} h_0 + h_1 z^{-1} + \ldots + h_N z^{-N} \\ &\stackrel{\Delta}{=} z^{-N} C(z), \end{aligned}$$

where $C(z)$ is the N-th degree polynomial in z formed by the h_k

General P-th order IIR filter

- Difference equation

$$y(n) = -\sum_{k=1}^{P} a_k y(n-k) + \sum_{\ell=0}^{P} b_\ell x(n-\ell)$$

- Transfer function

$$\begin{aligned} H_{\mathrm{IIR}}(z) &\triangleq \frac{b_0 + b_1 z^{-1} + \ldots + b_P z^{-P}}{1 + a_1 z^{-1} + \ldots + a_P z^{-P}} \\ &\triangleq \frac{b_0 z^P + b_1 z^{P-1} + \ldots + b_P}{z^P + a_1 z^{P-1} + \ldots + a_P} \\ &\triangleq \frac{B(z)}{A(z)} \\ &\triangleq h_0 + h_1 z^{-1} + h_2 z^{-2} + \ldots, \end{aligned}$$

where

$$\begin{aligned} A(z) &\triangleq z^P + a_1 z^{P-1} + \ldots + a_P \qquad \text{(monic)} \\ B(z) &\triangleq b_0 z^P + b_1 z^{P-1} + \ldots + b_P \end{aligned}$$

TIIR Construction: A One-Pole Example

Consider an FIR filter having a truncated geometric sequence $\{h_0, h_0 p, \ldots, h_0 p^N\}$ as an impulse response. This filter has the same impulse response for the first $N+1$ terms as the one-pole IIR filter with transfer function

$$H_{\mathrm{IIR}}(z) = \frac{h_0}{1 - pz^{-1}}.$$

Subtracting off the tail of the impulse response gives

$$\begin{aligned}
H_{\mathrm{FIR}}(z) &= h_0 + h_0 p z^{-1} + \cdots + h_0 p^N z^{-N} \\
&= \left\{ h_0 + h_0 p z^{-1} + \cdots \right\} \\
&\qquad - \left\{ h_0 p^{N+1} z^{-(N+1)} + h_0 p^{(N+2)} z^{-(N+2)} + \cdots \right\} \\
&= \frac{h_0}{1 - pz^{-1}} - p^{N+1} z^{-(N+1)} \frac{h_0}{1 - pz^{-1}} \\
&= h_0 \frac{1 - p^{N+1} z^{-(N+1)}}{1 - pz^{-1}}
\end{aligned}$$

The time-domain recursion for this filter is

$$\begin{aligned}
y[n] &= \sum_{k=0}^{N} h_0 p^k x[n-k] \\
&= py[n-1] + h_0 \left(x[n] - p^{N+1} x[n-(N+1)] \right)
\end{aligned}$$

Complexity Notes

- Direct FIR filter implementation requires $N + 1$ multiplies and N adds
- TIIR implementation requires 3 multiplies and 2 adds, independent of N
- No savings in memory

Note that there is a pole-zero cancellation in the TIIR transfer function

$$H(z) = h_0 \frac{1 - p^{N+1} z^{-(N+1)}}{1 - pz^{-1}} = h_0 + h_0 p z^{-1} + \cdots + h_0 p^N z^{-N}$$

- If $|p| < 1$, no problem since the canceled pole is stable
- If $|p| \geq 1$, imperfect pole-zero cancellation due to numerical rounding leads to exponentially growing round-off error

Basic Idea: Since the overall TIIR filter is FIR(N), *alternate* between two instances of each unstable one-pole, starting each new one from the zero state N samples before it is actually used. (Apparently first suggested by T. Fam at Asilomar-'87 for the case of distinct poles.)

Extension to Higher-Order TIIR Sequences

We can extend this idea from the one-pole case to any rational filter $H(z) = B(z)/A(z)$. The general procedure is to find the "tail filter" $H'_{\mathrm{IIR}}(z)$ and subtract it off:

$$H_{\mathrm{FIR}}(z) = H_{\mathrm{IIR}}(z) - H'_{\mathrm{IIR}}(z)$$

Multiply $H_{\mathrm{IIR}}(z)$ by z^N to obtain

$$\begin{aligned} z^N H_{\mathrm{IIR}}(z) &= h_0 z^N + \cdots + h_{N-1} z + h_N \\ &\qquad + h_{N+1} z^{-1} + h_{N+2} z^{-2} + \cdots \\ &\stackrel{\Delta}{=} C(z) + H'_{\mathrm{IIR}}(z) \\ &= \frac{z^N B(z)}{A(z)} \stackrel{\Delta}{=} C(z) + \frac{B'(z)}{A(z)} \end{aligned}$$

- $B'(z)$ is the unique remainder after dividing $z^N B(z)$ by $A(z)$ using "synthetic division" ($z^N B(z) \equiv B'(z) \pmod{A(z)}$)
- We may assume $\mathrm{Deg}\left\{B'(z)\right\} = \mathrm{Deg}\left\{A(z)\right\} - 1$
- $B'(z)$ gives us our desired "tail filter" for forming $H_{\mathrm{FIR}} = H_{\mathrm{IIR}} - H'_{\mathrm{IIR}}$:

$$H'_{\mathrm{IIR}}(z) = \frac{B'(z)}{A(z)}$$

Higher-Order TIIR Filters

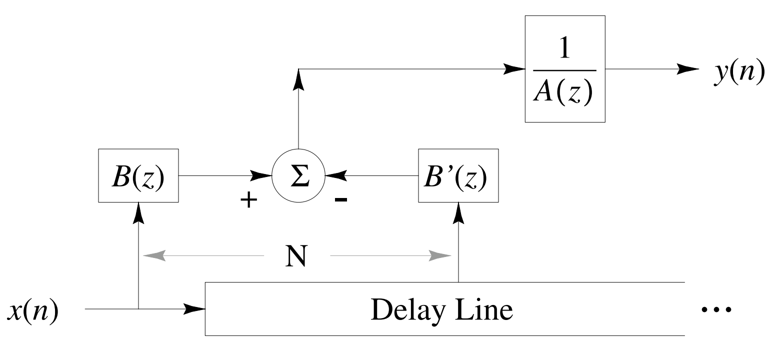
We have

$$\begin{aligned}
H_{\text{FIR}}(z) &= H_{\text{IIR}}(z) - z^{-N} H'_{\text{IIR}}(z) \\
&= \frac{B(z) - z^{-N} B'(z)}{A(z)}
\end{aligned}$$

The corresponding difference equation is

$$\begin{aligned}
y[n] &= -\sum_{k=1}^{P} a_k y[n-k] + \sum_{\ell=0}^{P} b_\ell x[n-\ell] \\
&\quad - \sum_{m=0}^{P-1} b'_m x[n-m-(N+1)]
\end{aligned}$$

Since the denominators of $H_{\text{IIR}}(z)$ and $H'_{\text{IIR}}(z)$ are the same, the *dynamics* (poles) can be shared:



Complexity and Storage-Cost

$$H_{\mathrm{FIR}}(z) = \frac{B(z) - z^{-N}B'(z)}{A(z)}$$

$N =$ FIR order and let $P = A(z)$ order (#poles)

- The computational cost of the general truncated P-th order IIR system is $3P + 1$ multiplies and $3P - 2$ adds, independent of N
- Net computational savings is achieved when $N > 3P$

Storage Requirements

- P output samples for the IIR feedback dynamics $A(z)$
- N input samples of the FIR filter (main delay line)
- P input samples for $B(z)$ (normally in delay line)
- P input samples for $B'(z)$ (also possibly in delay line)

Thus, we need a total of at least $N + P$ input delay samples, of which only $2P$ are accessed, and P output delay samples. This is between P and $2P$ more than a direct FIR implementation.

Example

We wish to truncate the impulse response of

$$H^{+}(z) = \frac{B^{+}(z)}{A^{+}(z)} = \frac{1}{1 - 1.9z^{-1} + 0.98z^{-2}}$$

after $N = 300$ samples to obtain a length 301 FIR filter $H^{+}_{\mathrm{FIR}}(z)$

Steps:

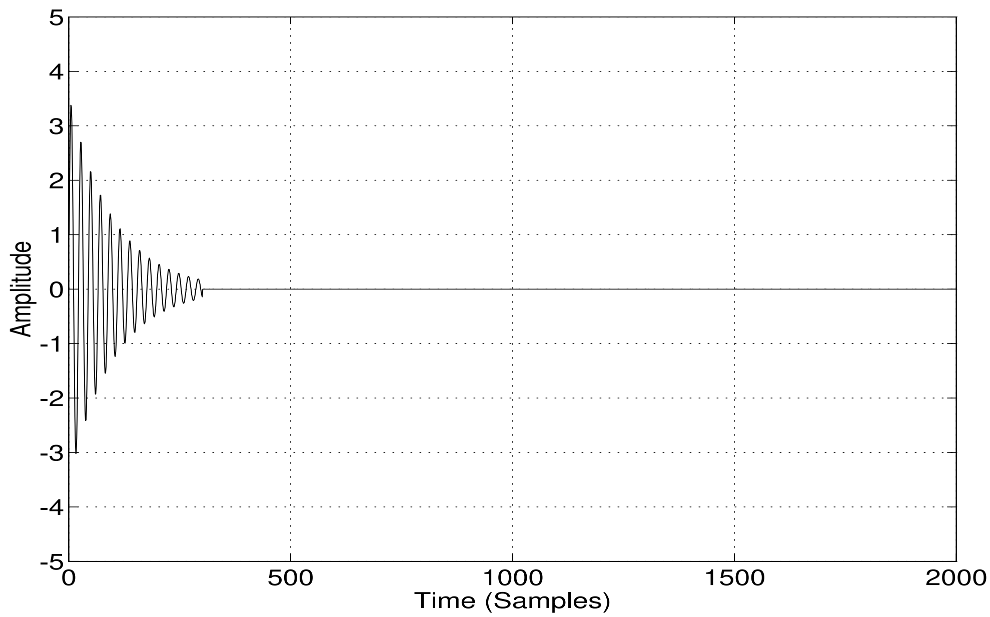
1. Perform synthetic division on $z^{300}B^{+}(z)$ by $A(z)$ to obtain the remainder

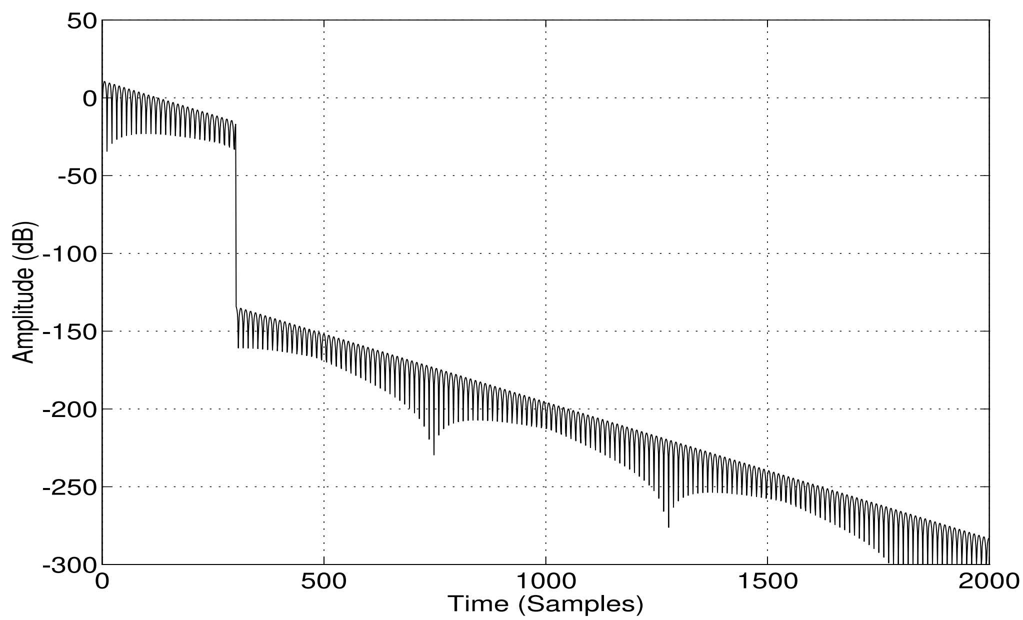
$$B'^{+}(z) = -0.162126z + 0.139770$$

2. Form the TIIR filter as

$$\begin{aligned} H^{+}_{\mathrm{FIR}}(z) &= \sum_{k=0}^{N} h_k^{+} z^{-k} = \frac{B^{+}(z) - z^{-N}B'^{+}(z)}{A^{+}(z)} \\ &= \frac{1 + 0.162126\, z^{-299} - 0.139770\, z^{-300}}{1 - 1.9z^{-1} + 0.98z^{-2}} \end{aligned}$$

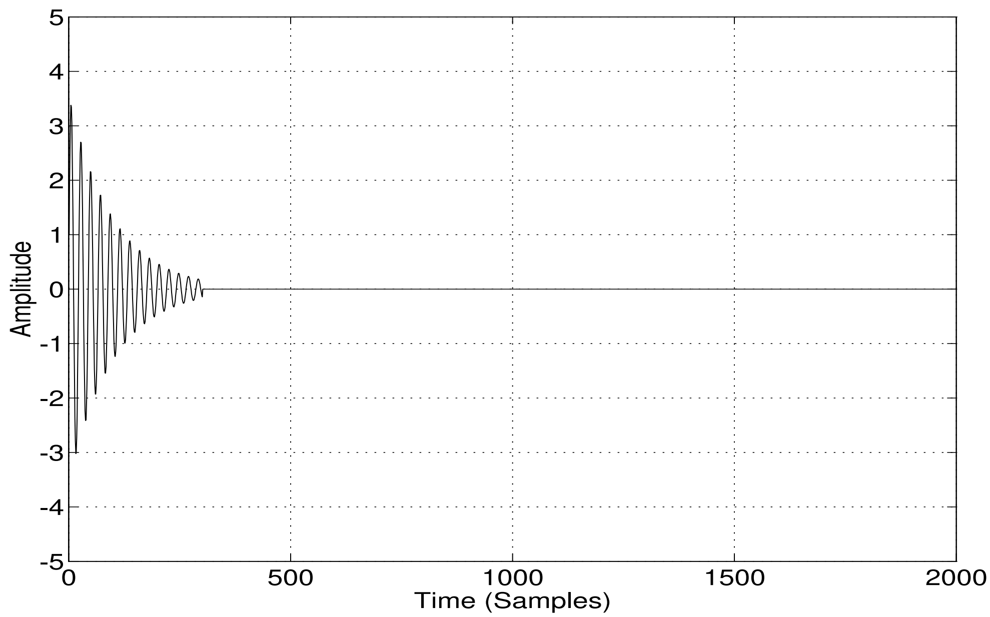
Impulse Response of TIIR Implementation Without Resets

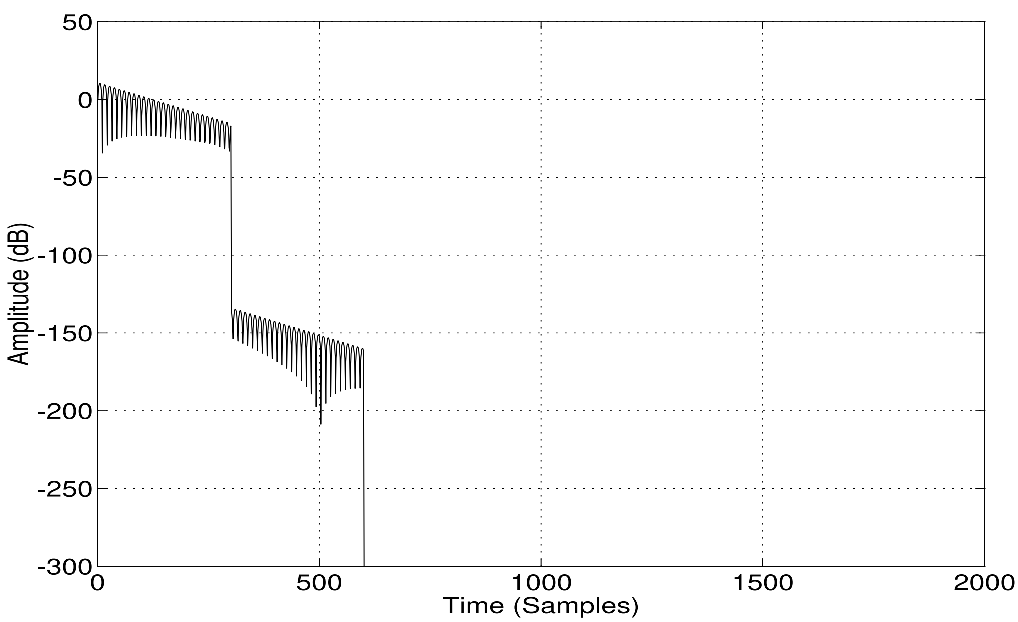




- At time $n = 301$, the tail of the response is subtracted off, and the impulse-response magnitude drops by about
  115 dB
- Due to quantization errors, there is a residual response
- Poles are all stable, so error decays

Impulse Response of TIIR Implementation With Resets





- Again, impulse-response tail is subtracted off at time $n = 301$, giving around 115 dB attenuation
- Additionally, state variables are cleared every 300 samples
- Residual response completely canceled at time $n = 600$
- System has truly finite memory

Unstable Example

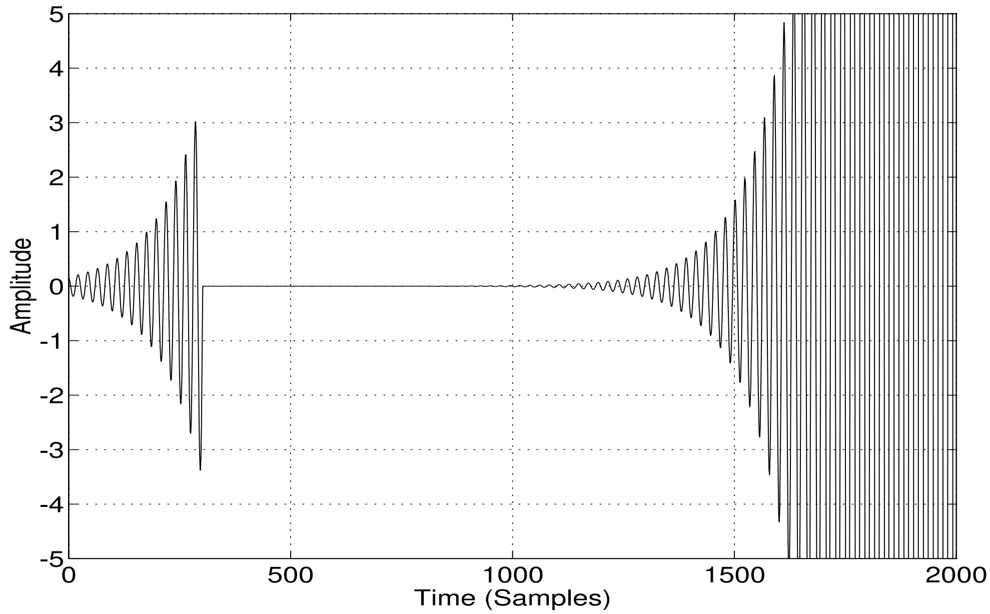
To form a linear phase TIIR filter based on the previous example, we need also the "flipped" impulse response generated by

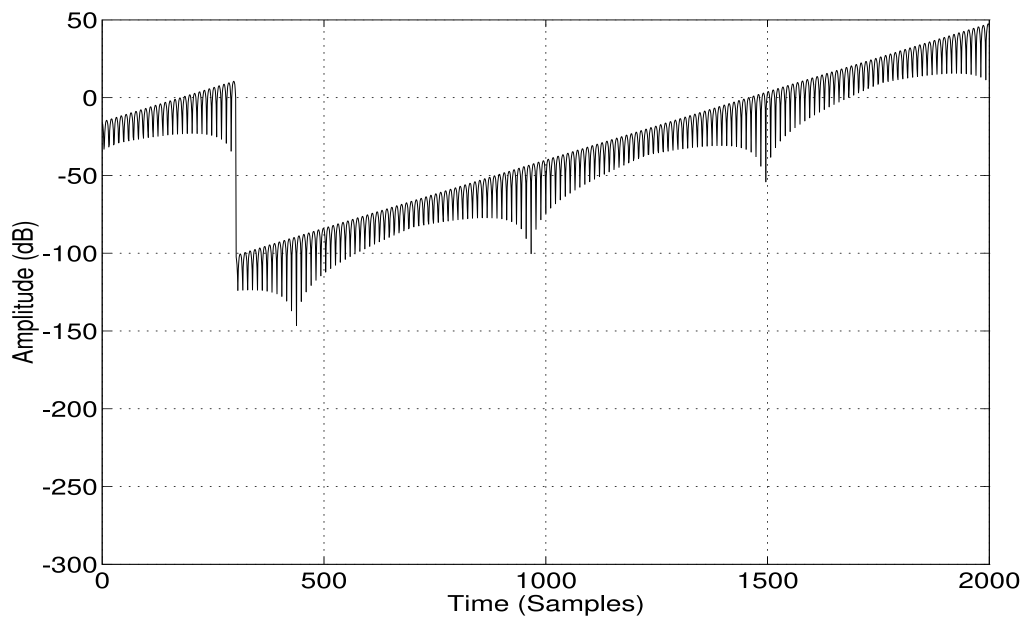
$$\begin{aligned} H^{-}_{\mathrm{FIR}}(z) &= \frac{-0.139770z^2 + 0.162126z - z^{-300}}{0.98z^2 - 1.9z + 1} \\ &= \frac{-0.142622z^2 + 0.165435z - 1.020408z^{-300}}{z^2 - 1.938776z + 1.020408} \end{aligned}$$

where the last equation is normalized by 0.98 to make the denominator monic.

This system has two unstable hidden modes.

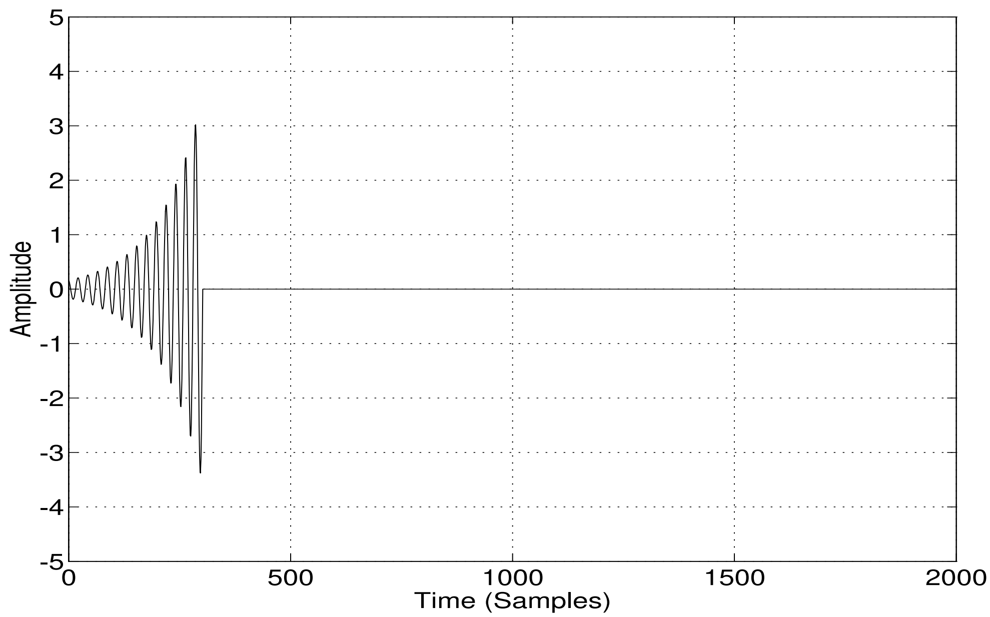
Impulse Response Without Resets

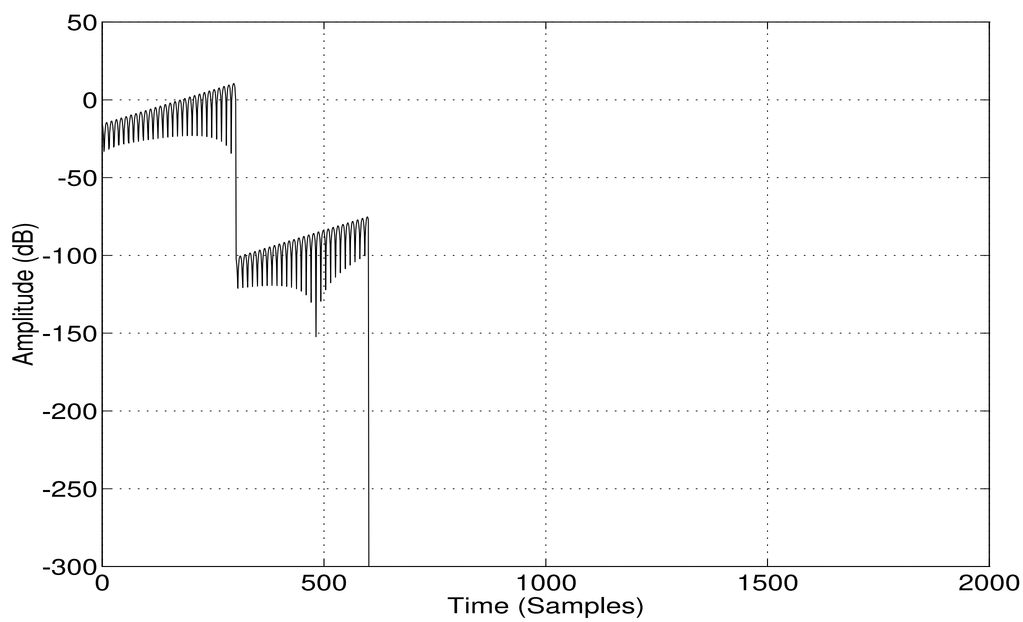




- Tail is canceled with about 125 dB attenuation
- Due to the unstable canceled poles, quantization noise grows without bound
- By time 1500 samples, the quantization noise dominates
- (Arithmetic = double-precision floating point with single-precision state variables)

Impulse Response of TIIR Implementation With Resets





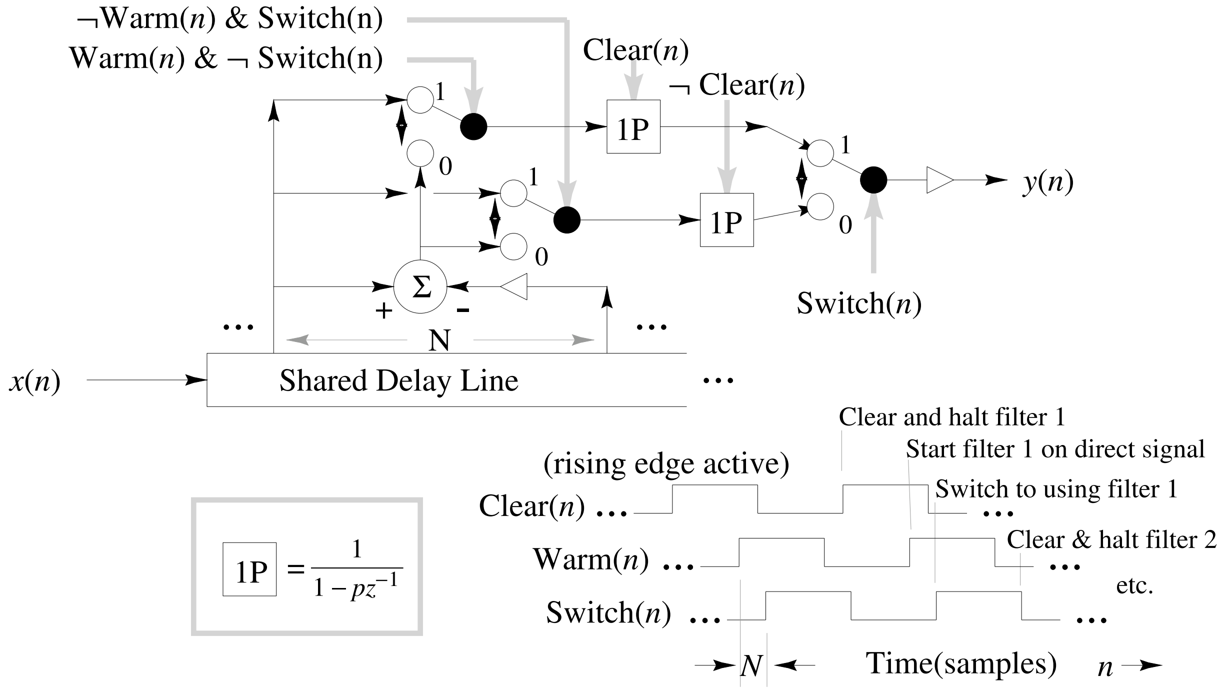
- State-variable resets zero-out the quantization noise before it becomes significant
- Overall system has truly finite memory

Synthetic Division Algorithm

Algorithm for performing synthetic division to generate the tail-canceling polynomial $B'(z)$:

```
     int i,j;
     double *w=(double *)malloc((P+1)*sizeof(double));
/*** load the numerator coefficients for B(z) ***/
     for(i=0;i<P+1;i++){
          w[i]=b[i];
     }
/*** do synthetic division ***/
     for(i=0; i<=N; i++){
          factor=w[0];
          for(j=0;j<P;j++){
               w[j]=w[j+1]+factor*a[j];
          }
          w[P]=0;
/**** The remainder after the i-th step is in w[0..(P-1)] ***
     }
/*** copy the result to the output array ***/
     for(i=0;i<P;i++) {
          bb[i]=w[i];
     }
```

A One-Pole (Almost) TIIR Filter



- Generates truncated exponentials or constants
- Filter complexity *on average* $\approx$ one pole
- Shared delay line
- Shared dynamics

Offset Exponentials

Use *two* one-pole TIIRs, to make an *offset exponential*:

$$h(n) = \begin{cases} ae^{cn} + b, & n = 0, 1, 2, \ldots, N-1 \\ 0, & \text{otherwise} \end{cases}$$

- The constant portion b requires only one multiply (by b) since the pole for this TIIR filter is at $z = 1$
- Resets for pure integrators are needed less often than for growing exponentials
- Using a *cascade* of digital integrators, any *polynomial* impulse response is possible
- A cubic-spline impulse response requires four integrators