Basic Finite Differences

• Simplest Ordinary Differential Equation (ODE):

\[ v(t) = L \frac{di}{dt} \quad ("f = ma") \]

\[ \leftrightarrow \quad V(s) = Ls I(s) - i(0) \]

– Voltage \( v \) across and current \( i \) through inductor \( L \),
or
– Force \( v \) on and velocity \( i \) of mass \( L \)

• We know how to sample \( i(t) \) and \( v(t) \) to obtain

\( i_n = i(nT) \) and \( v_n = v(nT), n = 0, 1, 2, \ldots \)

• How do we “simulate” the inductor (or mass) to produce samples of \( v(t) \) from samples of \( i(t) \) or vice versa?

• In continuous time, the differentiator is a one-zero filter:

\[ R_L(s) = \frac{V(s)}{I(s)} = ms \]

assuming \( i(0) = 0 \)

• We thus need to \textit{digitize} this continuous-time filter to obtain a \textit{digital filter}
First-Order Digitization of Derivatives

Differentiation can be “digitized” in a variety of ways:

- Backward Euler (BE):
  \[ s \leftarrow \frac{1 - z^{-1}}{T} \quad \mathcal{O}(T) \text{ accurate} \]

- Forward Euler (FE):
  \[ s \leftarrow \frac{z - 1}{T} \quad \mathcal{O}(T) \text{ accurate} \]

- Trapezoidal Rule (Bilinear Transform):
  \[ s \leftarrow \frac{2}{T} \frac{1 - z^{-1}}{1 + z^{-1}} \quad \mathcal{O}(T^2) \text{ accurate} \]

These are all first-order filters that approximate a time-derivative.

We can also use higher-order filters by doing a digital filter design to approximate the frequency response

\[ H(j\omega) = j\omega \]

Accuracy depends on filter order and error criterion used.

Digitizing Differential Operators

Backward Euler Method:

\[
\begin{align*}
\frac{di}{dt} & \approx \frac{i_n - i_{n-1}}{T} \\
\Rightarrow \quad v_n & = \frac{L}{T}(i_n - i_{n-1})
\end{align*}
\]

- Causal
- Explicit
- 1/2 sample time delay

Centered Difference:

\[
\begin{align*}
\frac{di}{dt} & \approx \frac{i_{n+1} - i_{n-1}}{2T} \\
\Rightarrow \quad v_n & = \frac{L}{2T}(i_{n+1} - i_{n-1})
\end{align*}
\]

- No time delay
- Possibly implicit
Trapezoidal rule for numerical integration:

\[ v_n = -v_{n-1} + \frac{2L}{T} (i_n - i_{n-1}) \]

Derivation:

\[ i(t) = \frac{1}{L} \left( \int_0^t v(t')dt' \right) + i_0 \]

so that

\[ i(nT) = \frac{1}{L} \left( \int_0^{nT} v(t')dt' \right) + i_0 \]

\[ = \frac{1}{L} \left( \int_0^{(n-1)T} v(t')dt' \right) + i_0 + \frac{1}{L} \int_{(n-1)T}^{nT} v(t')dt' \]

\[ = i((n-1)T) + \frac{1}{L} \int_{(n-1)T}^{nT} v(t')dt' \]

\[ \approx i((n-1)T) + \frac{T}{2L} (v((n-1)T) + v(nT)) \]

The trapezoidal rule is equivalent to the bilinear transform method for converting an analog filter to a digital filter.

Accuracy

Suppose we take the backward-Euler approximation

\[ v_n = (L/T)(i_n - i_{n-1}), \]

and expand \( i_{n-1} \) in Taylor series about \( i_n \). This yields:

\[ v_n = (L/T) \left( i_n - \left( i_n - T \frac{di}{dt}\big|_{nT} + O(T^2) \right) \right) \]

\[ = \frac{L}{T} \frac{di}{dt}\big|_{nT} + O(T) \]

We say that the backward difference approximation has an error of order \( T \).

For the trapezoid rule,

\[ v_n = L \frac{di}{dt}\big|_{nT} + O(T^2) \]

so it is second-order accurate in \( T \).

- In general, the more accurate a difference scheme, the more information from neighboring grid points it will require.
- Higher order digital filters give better approximations to differential operators.
- In audio, we are often concerned with the frequency response approximation.
Frequency Domain Interpretations

The Laplace transform of $v = L di/dt$ gives

$$V(s) = Ls I(s)$$

assuming zero initial conditions, where $s = \sigma + j\omega$ is the complex frequency variable.

Taking $z$ transforms of the sequences $v_n$ and $i_n$ in the backward-Euler scheme yields:

$$V(z) = L \frac{1 - z^{-1}}{T} I(z)$$

Thus,

- Backward Euler conformal map:
  $$s \rightarrow \frac{1 - z^{-1}}{T}$$

- Forward Euler conformal map:
  $$s \rightarrow \frac{z - 1}{T}$$

- Centered Difference conformal map:
  $$s \rightarrow \frac{z - z^{-1}}{T}$$

- Bilinear Transform conformal map:
  $$s \rightarrow \frac{2(1 - z^{-1})}{T(1 + z^{-1})}$$

Distributed Example: 1-D wave equation, solution by FDA approach

Suppose we want to simulate one direction in an acoustic space in which the air is described by the second-order wave equation

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}$$

where $u(x, t)$ is particle velocity of the air relative to equilibrium.

- This is the familiar 1-D wave equation, with wave speed given by
  $$c = \sqrt{\frac{\gamma P_0}{\rho_0}}$$

  where
  - $\gamma = 1.4$ for air ("adiabatic gas constant"),
  - $P_0$ is ambient pressure, and
  - $\rho_0$ is mass density.

- The same equation holds also for pressure $p(x, t)$ and density $\rho(x, t)$, all with the same wave speed $c$.

Let’s “digitize” this wave equation to create a finite difference scheme (FDS).
Second-Order Finite Difference Scheme

The simplest, and traditional way of discretizing the 1-D wave equation is by replacing the second derivatives by second order differences:

\[
\frac{\partial^2 u}{\partial t^2} \bigg|_{x=k\Delta,t=nT} \approx \frac{u^n_{k-1} - 2u^n_k + u^{n+1}_k}{T^2},
\]

\[
\frac{\partial^2 u}{\partial x^2} \bigg|_{x=k\Delta,t=nT} \approx \frac{u^n_{k-1} - 2u^n_k + u^{n+1}_{k+1}}{\Delta^2},
\]

where \( u^n_k \) is defined as \( u(k\Delta, nT) \). Here we have sampled the time-space plane in a uniform grid, with a timestep of \( T \) and a space step of \( \Delta \). The \( u^n_k \) are the grid variables here. Now, through substitution, the wave equation becomes:

\[
u^n_{k-1} - 2u^n_k + u^{n+1}_k = c^2T^2\Delta^2(u^n_{k-1} - 2u^n_k + u^n_{k+1})
\]

- Note that if we choose \( T/\Delta = 1/c \), the equation reduces further to:

\[
u^{n+1}_k = u^n_{k-1} + u^{n+1}_{k+1} - u^n_k
\]

Let’s examine this recursion on the time/space grid, assuming for the moment no boundary conditions:

Time-Space Grid of Second-Order FDS

\[
u^{n+1}_k = u^n_{k-1} + u^{n+1}_{k+1} - u^n_k
\]

- Grid variable at “current” point depends on value at two previous time steps (a second order scheme in time). We thus need to specify initial data for all \( m \) at times \( n = 0 \) and \( n = 1 \).
- Grid variable at “current” point depends on values at adjacent locations on the string (at previous time).
- Difference scheme is explicit (thus parallelizable); that is, each grid variable at time \( n + 1 \) depends only on grid variables at previous time instants. This is a very desirable property.
A Peek at Stability of Finite Difference Schemes

Let's look again at the difference scheme we derived for the 1-D wave eq, with the special time/space step $c = T/\Delta$:

$$u_{k}^{n+1} = u_{k-1}^{n} + u_{k+1}^{n} - u_{k}^{n-1}$$

The velocity sample $u(k, n)$ is a two-dimensional sequence with a time index and a spatial coordinate index.

Suppose we now take the DTFT with respect to the spatial index $k$:

$$\sum_{k=-\infty}^{\infty} u_{k}^{n+1} e^{-j\omega k \Delta} = \sum_{k=-\infty}^{\infty} (u_{k-1}^{n} + u_{k+1}^{n} - u_{k}^{n-1}) e^{-j\omega k \Delta}$$

or

$$U_{n+1}(\omega) = (e^{-j\omega \Delta} + e^{j\omega \Delta})U_{n}(\omega) - U_{n-1}(\omega)$$

where here $U_{n}(\omega)$ is the spatial spectrum of the solution at time $n$, and $\omega$ is the spatial frequency variable. We can also write this in vector form as:

$$\begin{bmatrix} U_{n+1}(\omega) \\ U_{n}(\omega) \end{bmatrix} = \begin{bmatrix} 2 \cos \omega \Delta & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} U_{n}(\omega) \\ U_{n-1}(\omega) \end{bmatrix}$$

Note that the state of the system is completely determined by $U_{n}(\cdot)$ and $U_{n-1}(\cdot)$.

Von Neumann Analysis

The matrix

$$A \Delta = \begin{bmatrix} 2 \cos \omega \Delta & -1 \\ 1 & 0 \end{bmatrix}$$

can be called the state transition matrix corresponding to the state-space description determined by the choice of state vector

$$x(n) \Delta = \begin{bmatrix} U_{n}(\omega) \\ U_{n-1}(\omega) \end{bmatrix}$$

and the state update can be written more simply in vector form as $x(n + 1) = Ax(n)$. Note that the state-space description is indexed by frequency $\omega$, regarded as fixed.

- From linear systems theory, we know that such a system will be asymptotically stable if the eigenvalues $\lambda$ of the matrix $A$ are both less than 1 in magnitude.
- It is easy to show that the eigenvalues of $A$ are $\lambda_+ = e^{j\omega \Delta}$ and $\lambda_- = e^{-j\omega \Delta}$. Thus, $|\lambda_\pm(\omega)| = 1, \forall \omega$.
- While we are not guaranteed asymptotic stability, $|\lambda(\omega)| = 1$ does imply that, in some sense, our solution is not getting larger with time at any spatial frequency. This can be defined as marginal stability.
• Note that we should expect the eigenvalues to have unit modulus, because the wave equation we started with corresponds to a lossless medium (an ideal gas). The original PDEs were derived without any loss mechanisms.

• A lossless discrete-time simulation can be highly desirable, particularly as a modeling starting point.

• This kind of “Von Neumann analysis” can be applied to any constant-coefficient FDS which is linear in its spatial directions.

Problems with FDS

• Convergence: Since the approximations to the second derivatives we used were second order accurate (in $T$ and $\Delta$), the scheme as a whole is accurate as $O(T^2, \Delta^2)$.

• Making an FDS more accurate (i.e., converge faster) generally requires a recursion involving more grid variables.

• An FDS for a higher order PDE also generally involves more grid variables.

• From a signal processing point of view, a more accurate simulation of an LTI medium is obtained by increasing the order of the filter.

• Note that an optimal filter design yields FDS coefficients which may be translated back to differential equation coefficients (which may or may not have physical meaning).

• Stability becomes more difficult to ensure in general (need to check eigenvalue magnitudes). The addition of boundary conditions makes this even more difficult.
• A good finite difference scheme may not be explicit, and hence may require matrix inversions (generally sparse).

For example, the dependence diagram below represents an implicit scheme: We cannot calculate the grid variables at the current timestep as weighted sums of grid variables at previous instants.

```
   current point
   n+1  n  n-1  time
   m-1  m  m+1
   spatial step
```

---

**More General Differential Equations**

A more general linear constant coefficient differential equation can be written as:

\[
\sum_{k=0}^{N} a_k \frac{d^k v}{dt^k} = \sum_{k=0}^{M} b_k \frac{d^k i}{dt^k}
\]

or, in the frequency domain (0 initial conditions):

\[
\sum_{k=0}^{N} a_k s^k V(s) = \sum_{k=0}^{M} b_k s^k I(s)
\]

We can define a transfer-function relationship as follows:

\[
Z(s) = \frac{V(s)}{I(s)} = \frac{b_0 + b_1 s + b_2 s^2 + \cdots + b_M s^M}{1 + a_1 s + a_2 s^2 + \cdots + a_N s^N}
\]

where we have normalized \(a_0 \neq 0\) to 1. Note that \(Z(s)\) is a rational function of \(s\) of order \(\max(N, M)\).

If \(i(t)\) and \(v(t)\) are measured at the same point, then \(Z(s)\) is a driving point impedance, as depicted below:

```
   i(t)  
   Z(s)  
   v(t)  
```

If the circuit (or mechanical system) is physically passive, then \(Z(s)\) must be positive real.
References

- **Numerical Sound Synthesis**  
  Stefan Bilbao, Wiley 2009

- **Finite difference schemes and von Neumann analysis:**  
  *Finite Difference Schemes and Partial Differential Equations*  
  J. C. Strikwerda, Wadsworth and Brooks 1989

- **Appendix D** of the text contains a terse introductory summary of Strikwerda from a signal processing point of view.