Music 420 Lecture
Elementary Finite Different Schemes

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Outline

- White Box and Black Box Physical Modeling
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- Difference Equations (Finite Difference Schemes)
  - Backward Euler (BE)
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  - Trapezoidal Rule for Numerical Integration
  - Bilinear Transform (BLT)
- Digital Filter Design Formulation

Two Approaches to Physical Modeling

1. "White Box" Modeling:
   (a) Find the describing *differential equations* from basic physical principles
   (b) *Digitize* the differential equations to obtain *difference equations* implemented in software
2. "Black Box" Modeling:
   (a) Measure the *system response* to a representative set of input signals
   (b) Fit a *computational model* to the measured input-output set
   (c) In the Linear, Time-Invariant (LTI) case, a Multi-Input, Multi-Output (MIMO) *digital filter* will suffice

This class *blends* white- and black-box approaches:

1. LTI sections become fast, accurate digital filters
2. *Nonlinear* or *rapidly time-varying* subsystems normally get a white-box approach (reeds, hammers, bows, ...)

Ordinary Differential Equations

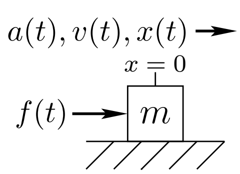
Ordinary Differential Equations (ODEs) typically result from Newton's laws of motion:

$$f(t) \;=\; m\,a(t) \quad \text{(Force = Mass times Acceleration)}$$

Acceleration $a(t)$ relates to velocity $v(t)$ and position $x(t)$ by differentiation with respect to time t:

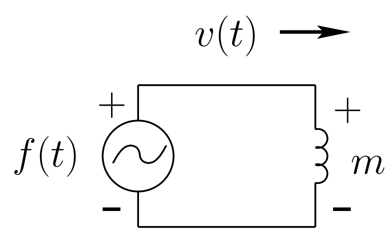
$$a(t) \;\isdef\; \dot{v}(t) \;\isdef\; \frac{d\,\dot{x}(t)}{dt} \;\isdef\; \ddot{x}(t) \;\isdef\; \frac{d^2x(t)}{dt^2}$$

Physical Diagram:



Force $f(t)$ driving mass m along frictionless surface

Equivalent Circuit for a Force-Driven Mass



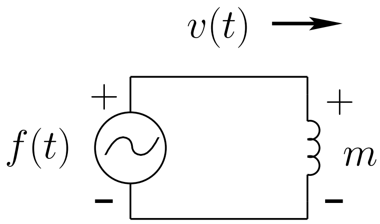
- Mass m is an *inductor* $L=m$ Henrys
- Driving force $f(t)$ is a *voltage source*
- Mass velocity $v(t)$ is the *loop current*

The ODE is obtained from the equivalent circuit by summing all "voltages" around the current loop to zero to obtain

$$\boxed{-f(t)+m\dot{v}=0}$$

The minus sign for $f(t)$ occurs because the current arrow entered the minus side of the "voltage source"

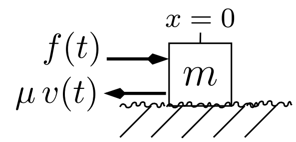
Reference Directions in Equivalent Circuits



$$-f(t) + m\dot{v} = 0$$

- "Reference directions" ($\pm$) on the voltage source and circuit elements may be chosen arbitrarily—just keep track and be consistent
- When $f(t)$ is positive, "current" is pushed from its $+$ to its $-$ terminal, *i.e.*, $v(t)$ will be positive if the rest of the circuit is just a wire or a resistor
- The "force drop" across the mass m is positive when $v(t)$ increases in the direction going from its $+$ to $-$ terminal. This can be interpreted as the inertial *reaction force* of the mass that opposed the external *applied force* (Newton's first law of motion)

ODE for a Mass Sliding with Friction

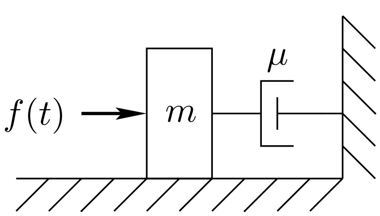


Force $f(t)$ driving mass m along surface with friction force $\mu\, v(t)$:

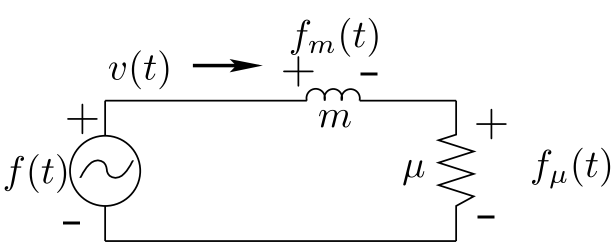
$$\begin{aligned} f(t) &= m\,\ddot{x}(t) + \mu\, v(t) \\ &= m\,\ddot{x}(t) + \mu\,\dot{x}(t) \end{aligned}$$

- Note that the friction force is positive to the *left* in this figure, *i.e.*, it is a *reaction force*
- The inertial reaction force of the mass points to the left as well (not shown, but equal to $-f(t)$)

Force-Driven Mass with Friction Diagram and Equivalent Circuit



Force driving an ideal mass and dashpot



Equivalent Circuit

$$\begin{aligned} 0 &= -f(t) + f_m + f_\mu \\ 0 &= -f(t) + m\,\dot{v}(t) + \mu\, v(t) \end{aligned}$$

Mass-Spring ODE

An *ideal spring* described by Hooke's law

$$f(t) \;=\; k\,x(t) \;=\; k\int_0^t v(\tau)\,d\tau \quad\longleftrightarrow\quad \frac{V(s)}{s}$$

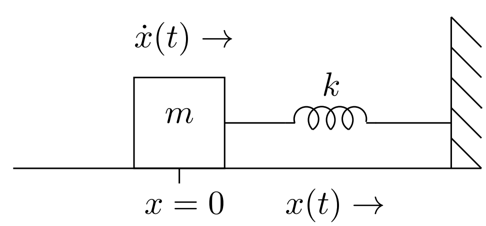
where k denotes the *spring constant*, $x(t)$ denotes the *compressive* spring displacement from rest at time t, and $f(t)$ is the force required for displacement $x(t)$

If the force on a mass is due to a spring then, as discussed later, we may write the ODE as

$$k\,x(t) + m\,\ddot{x}(t) \;=\; 0$$

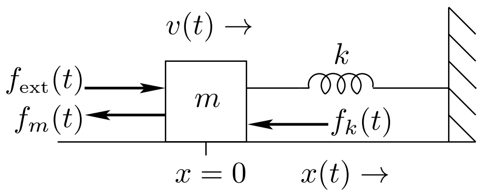
(Spring Force + Mass Inertial Force = 0)

Physical diagram:



Mass-Spring-Wall System

$$f_{\text{ext}}(t) - f_m(t) - f_k(t) = 0$$

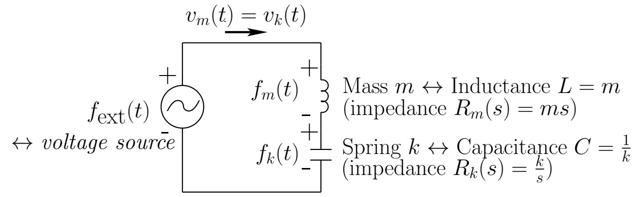


- Driving force $f_{\text{ext}}(t)$ is to the right on the mass
- Driving force + mass inertial force + spring force = 0
- Mass velocity = spring velocity
- This is a *series* combination of the spring and mass

If two physical elements are connected so that they share a *common velocity*, then they are said to be formally connected *in series*

Equivalent Circuit for Mass-Spring-Wall

The "series" nature of the connection becomes more clear when the *equivalent circuit* is considered:



- The driving force is applied to the mass such that a positive force results in a positive mass displacement and positive spring displacement (compression)
- The common mass and spring velocity appear as a single current running through the inductor and capacitor that model the mass and spring, respectively

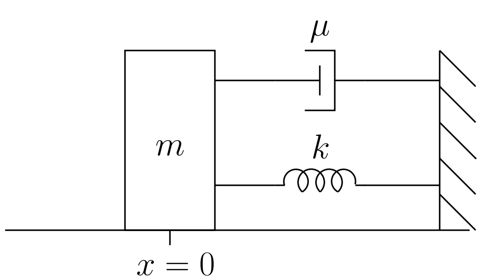
Mass-Spring-Dashpot ODE

If the mass is sliding with *friction*, then a simple ODE model is given by

$$k\,x(t) + \mu\,\dot{x}(t) + m\,\ddot{x}(t) \;=\; 0$$

(Spring + Friction + Inertial Forces = 0)

Physical diagram:



We will use such ODEs to model mass, spring, and dashpot elements, and their equivalent circuits

Difference Equations (Finite Difference Schemes)

- There are many methods for converting ODEs to difference equations
- For white-box modeling, we'll use a very simple, order-preserving methods which *replaces each derivative or integral with a first-order finite difference:*

$$\dot{x}(t) \;\triangleq\; \frac{d}{dt}x(t) \;\triangleq\; \lim_{\delta\to 0}\frac{x(t)-x(t-\delta)}{\delta}$$
$$\approx\; \frac{x(nT) - x[(n-1)T]}{T} \;\triangleq\; \hat{\dot{x}}(t)$$

  for sufficiently small T (the sampling interval)

- This is formally known as the *Backward Euler* (BE), or *backward difference* method for differentiation approximation
- In addition to BE, we'll look at Forward Euler (FE), BiLinear Transform (BLT), and a few others
- For a more advanced treatment of finite difference schemes, see **Numerical Sound Synthesis** by Stefan Bilbao (2009, Wiley)

Backward Euler Finite-Difference Equation for a Force-Driven Mass

- Newton's $f = ma$ can be written in terms of force f and velocity v or momentum $p = mv$ as

$$f(t) = m\,\dot{v}(t) = \dot{p}(t)$$

- The backward-difference substitution gives

$$f(nT) \;\approx\; m\,\frac{v(nT) - v[(n-1)T]}{T} \;\stackrel{\Delta}{=}\; m\,\hat{\dot{v}}(nT)$$

for $n = 0, 1, 2, \ldots$. Or, in a lighter notation,

$$f_n \;\approx\; m\,\frac{v_n - v_{n-1}}{T} \;\stackrel{\Delta}{=}\; m\,\hat{\dot{v}}_n, \quad n = 0, 1, 2, \ldots$$

with $v_{-1} \stackrel{\Delta}{=} 0$

- We often use a "hat" to denote *approximation:* $\hat{v} \approx v$
- In this case, $\hat{\dot{v}}_n$ is more accurately written as $\hat{\dot{v}}_{n-1/2}$
- Solving for v_n yields a *difference equation* (finite difference scheme):

$$\hat{v}_n \;=\; \hat{v}_{n-1} + \frac{T}{m} f_n, \quad n = 0, 1, 2, \ldots$$

with $\hat{v}_{-1} \stackrel{\Delta}{=} 0$

Accuracy of Backward Euler

Suppose we take the backward-difference approximation $f_n = (m/T)(v_n - v_{n-1})$, and expand v_{n-1} in Taylor series about v_n. This yields:

$$\begin{aligned} f_n &= \frac{m}{T}\left(v_n - \left(v_n - T\,\frac{dv}{dt}\bigg|_{nT} + \mathcal{O}(T^2)\right)\right) \\ &= m\frac{dv}{dt}\bigg|_{nT} + \mathcal{O}(T) \end{aligned}$$

- We say that the backward difference approximation has an error of *order* T, written $\mathcal{O}(T)$
- The order of the error tells us how fast the error approaches zero as the sampling rate $f_s = 1/T$ approaches infinity
- Backward Euler maps infinite frequency $s = \infty$ to $z = 0$ (maximally damped), while trapezoidal rule (bilinear transform) maps $s = \infty$ to $z = -1$ (no damping introduced)

Summary of Backward Euler

$$\begin{aligned} v_n &= v_{n-1} + T\,\hat{\dot{v}}_n \\ &\Longleftrightarrow \boxed{\hat{\dot{v}}_n \;=\; \frac{v_n - v_{n-1}}{T}} \\ \updownarrow &= \qquad \updownarrow \\ V(z) &= z^{-1}V(z) + T\,\hat{\dot{V}}(z) \\ \Rightarrow & \qquad \boxed{\hat{\dot{V}}(z) \;=\; \frac{1 - z^{-1}}{T} V(z)} \end{aligned}$$

Expressing BE as a *conformal map* from s to z:

$$\boxed{s \;\leftarrow\; \frac{1 - z^{-1}}{T}}$$

The ideal differentiator $H(s) = s$, which is a first-order continuous-time LTI filter, is mapped to a first-order *discrete*-time LTI filter $H(z) = (1 - z^{-1})/T$.

Delay-Free Loops

Backward-Euler numerical integrator:

$$v_n \;=\; v_{n-1} + T\,\hat{\dot{v}}_n$$

Corresponding BE digital mass model:

$$\hat{v}_n \;=\; \hat{v}_{n-1} + \frac{T}{m} f_n$$

where $\hat{v}_n$ is the nth sample of the estimated velocity, f_n is the driving force at sample n, m is the mass, and T is the sampling interval

- Note that a *delay-free loop* appears if f_n depends on v_n (*e.g.*, due to friction):

$$\hat{v}_n \;=\; \hat{v}_{n-1} + \frac{T}{m} f_n(\hat{v}_n)$$

- In such a case, the difference equation is not *computable* in this form
- Non-computable finite-differences schemes such as this are said to be *implicit*
- We can address this by using a *forward-difference* ("Forward Euler")in place of a backward difference

Forward-Euler (FE)

The *backward difference* was based on the usual left-sided limit in the definition of the time derivative:

$$\dot{x}(t) \;=\; \lim_{\delta\to 0}\frac{x(t)-x(t-\delta)}{\delta} \;\approx\; \frac{x_n - x_{n-1}}{T}$$

The *forward difference* comes from the right-sided limit:

$$\dot{x}(t) \;=\; \lim_{\delta\to 0}\frac{x(t+\delta)-x(t)}{\delta} \;\approx\; \boxed{\frac{x_{n+1} - x_{n}}{T}}$$

- As $T \to 0$, the forward and backward difference approximations approach the same limit, because $x(t)$ is assumed continuous and differentiable at t
- The forward difference gives an *explicit finite difference scheme* for the force-driven-mass problem above, even if the driving force f_n depends on current velocity v_n:

$$\hat{v}_{n+1} \;=\; \hat{v}_n + \frac{T}{m} f_n, \quad n = 0, 1, 2, \ldots$$

  with $v_0 \triangleq 0$
- We obtain the same finite-difference scheme by introducing an *ad hoc delay* in the driving force of the Backward Euler scheme to get $\hat{v}_n = \hat{v}_{n-1} + (T/m) f_{n-1}$

Centered Finite Difference

Backward Euler $[s \leftarrow (1 - z^{-1})/T)]$ has a 1/2 sample *delay* at all frequencies, while Forward Euler $[s \leftarrow (z - 1)/T)]$ has a 1/2 sample *advance*. We can eliminate this time-skew using a *centered finite difference:*

$$\hat{\dot{v}}(nT) \;=\; \frac{v_{n+1} - v_{n-1}}{2T}$$

$$\Rightarrow \quad f_n \;\approx\; \frac{m}{2T}(v_{n+1} - v_{n-1})$$

$$\Rightarrow \quad \hat{v}_{n+1} \;=\; \hat{v}_{n-1} + \frac{2T}{m} f_n$$

- No time delay or advance
- Compare the *Leapfrog integrator*
- s to z mapping is

$$s \;=\; \frac{z - z^{-1}}{2T} \;\to\; \frac{e^{j\omega T} - e^{-j\omega T}}{2T} \;=\; j\frac{\sin(\omega T)}{T} \;\approx\; j\omega$$

  at low frequencies, but note how it reaches a maximum at $\omega T = \pi/2$ and comes back down to 0 at $\omega T = \pi$

Trapezoidal Rule for Numerical Integration

The velocity $v(t)$ can be written as

$$v(t) = v(0) + \left(\int_0^t \dot{v}(\tau)d\tau\right)$$

In particular,

$$\begin{aligned} v(nT) \;&=\; v(0) + \int_0^{(n-1)T} \dot{v}(\tau)d\tau + \int_{(n-1)T}^{nT} \dot{v}(\tau)d\tau \\ &=\; v[(n-1)T] + \int_{(n-1)T}^{nT} \dot{v}(\tau)d\tau \\ &\approx\; v[(n-1)T] + T\frac{\dot{v}[(n-1)T] + \dot{v}(nT)}{2} \end{aligned}$$

- This approximation replaces a one-sample integral by the area under the *trapezoid* having vertices $(n-1, 0), (n-1, \dot{v}_{n-1}), (n, 0), (n, \dot{v}_n)$
- In other words, $\dot{v}(t)$ is approximated by a *straight line* between time $n-1$ and n
- This is a *first-order* approximation of $\dot{v}(t)$ in contrast to the *zero-order* approximation used by forward and backward Euler schemes

- We will see that the commonly used *bilinear transform* is *equivalent*
- Model is *exact* if driving force is *piecewise linear*, having a constant slope over each sampling interval
- (Backward Euler is similarly exact for a *piecewise-constant* driving force)

Bilinear Transform as Compensated BE/FE

In Newton's law $f = m\dot{v}$, look at the Backward Euler (BE) approximation of the time-derivative:

$$f(t) \;=\; m\,\dot{v} \;\approx\; m\,\frac{v(t) - v(t-T)}{T}$$

We see there is a 1/2 sample delay in the first-order difference on the right. This misaligns the force $f(t)$ and subsequent velocity by half a sample. A very simple delay compensation is to use a *two-point average* on the left:

$$\frac{f(n) + f(n-1)}{2} \;\approx\; m\,\frac{v(n) - v(n-1)}{T}$$

The extra attenuation at high frequencies due to the two-point average actually *helps*. Taking the z transform:

$$\frac{1 + z^{-1}}{2} F(z) \;\approx\; m\,\frac{1 - z^{-1}}{T} V(z)$$

or

$$F(z) \;\approx\; m\left(\frac{2}{T}\frac{1-z^{-1}}{1+z^{-1}}\right)V(z)$$

which is the *bilinear transform* of $F(s) = ms\,V(s)$:

$$\boxed{s \mapsto \frac{2}{T}\frac{1-z^{-1}}{1+z^{-1}}}$$

Frequency Warping is the Only Error

We have

$$F(z) \;\approx\; m\left(\frac{2}{T}\frac{1-z^{-1}}{1+z^{-1}}\right)V(z)$$

using the *bilinear transform* (*trapezoidal integration* in the time domain)

Let's look along the unit circle in the z plane:

$$\frac{F(e^{j\omega T})}{V(e^{j\omega T})} \;\approx\; m\left(\frac{2}{T}\frac{1-e^{-j\omega T}}{1+e^{-j\omega T}}\right) \;=\; m\,j\left(\frac{2}{T}\tan\left(\frac{\omega T}{2}\right)\right)$$

Since the exact formula is $F(e^{j\omega T})/V(e^{j\omega T}) = m\,j\omega$, we can push all of the error into a *frequency warping*:

$$\omega_d \triangleq \frac{2}{T}\tan\left(\frac{\omega_a T}{2}\right)$$

- Frequency-warping is the *only error* over the unit circle when using the bilinear transform
- What started out as different gain errors on the left and right became the correct gains at warped frequency locations
- Frequency-warping implications should also be considered in the time domain

Filter Design Approach

We've been talking about the *white-box* approach in which every first-order element (mass, spring, ...) is *explicitly modeled* by a first-order finite-difference scheme. This is especially needed for elements that are *time varying* or pushed into *nonlinear* regimes of operation.

When a system is linear and time-invariant (LTI), there is no need for such fine-grained modeling, and we can take a a *black-box* approach, in which we need only model the *frequency response* from the input(s) to output(s) of the system using a *digital filter*.

Filter Design Approach to Ideal Integrators and Differentiators

Consider the following simple cases:

- *Integrator:* $H(s) = 1/s$
  (*e.g.*, force-driven mass with a velocity output)
- *Differentiator:* $H(s) = s$
  (*e.g.*, force-driven spring with a velocity output)

The *digital filter design formulation* typically minimizes *frequency-response* error with respect to the filter coefficients

Ideal Frequency Responses

- *Ideal Digital Integrator*

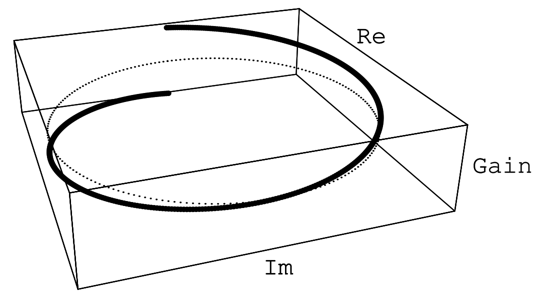
$$H(e^{j\omega T}) \;=\; \frac{1}{j\omega}, \quad \omega \in [-\pi/T, \pi/T]$$

- *Ideal Digital Differentiator:*

$$H(e^{j\omega T}) \;=\; j\omega, \quad \omega \in [-\pi/T, \pi/T]$$

- Exact match is *not possible* in *finite order*
- Minimize $\left\| H(e^{j\omega T}) - \hat{H}(e^{j\omega T}) \right\|$ where $\hat{H}$ is the digital filter frequency response and $\| E \|$ denotes some *norm* of E
- This is a *digital filter design* formulation

Ideal Differentiator Frequency Response



- Discontinuity at $z = -1 \;\Rightarrow$ *no exact solution* (polynomial approximation over the unit circle)
- Need *oversampling* and a *don't-care band* at high frequencies (e.g., 20 kHz to 22.05 kHz)
- *The frequency response can be arbitrary between the upper limit of human hearing (20kHz) and* $f_s/2$
- A small increment in oversampling factor yields a large decrease in required filter order for a given spec

Explicit and Implicit Finite Difference Schemes

Explicit:

$$y_{n+1} = x_n + f(y_n)$$

Implicit:

$$y_{n+1} = x_n + f(y_{n+1})$$

- A finite difference scheme is said to be *explicit* when it can be computed forward in time using quantities from previous time steps
- We will associate explicit finite difference schemes with *causal digital filters*
- In *implicit* finite-difference schemes, the output of the time-update (y_{n+1} above) depends on itself, so a causal recursive computation is not specified
- Implicit schemes are generally solved using
  - iterative methods (such as Newton's method) in nonlinear cases, and
  - matrix-inverse methods for linear problems
- Implicit schemes are typically used *offline* (not in real time)

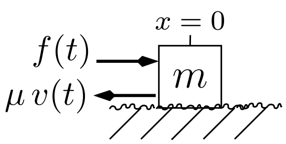
Semi-Implicit Finite Difference Schemes

- *Implicit* schemes can often be converted to *explicit* schemes (*e.g.*, for real-time usage) by limiting the number of iterations used to solve the implicit scheme
- These are called *semi-implicit finite-difference schemes*
- Iterative convergence is generally improved by working at a very high sampling rate, and by initializing each iteration to the solution for the previous sample
- See the 2009 CCRMA/EE thesis by David Yeh[1] for semi-implicit schemes for real-time computational modeling of nonlinear analog guitar effects (such as overdrive distortion)
- *Convex optimization* methods can be used to develop powerful new semi-implicit finite-difference schemes: `http://www.stanford.edu/~boyd/cvxbook/`

[1] http://ccrma.stanford.edu/~dtyeh

ODE Laplace Transform Analysis

Recall the mass m sliding on friction μ:



ODE:

$$\begin{aligned} f(t) &= m\,\ddot{x}(t) + \mu\, v(t) \\ &= m\,\ddot{x}(t) + \mu\,\dot{x}(t) \end{aligned}$$

Take the Laplace Transform of both sides and apply the *differentiation theorem* (three times):

$$\begin{aligned} F(s) &= m\,\left[s^2X(s) - s\,x(0) - \dot{x}(0)\right] + \mu\,\left[s\,X(s) - x(0)\right] \\ &= m\,s^2X(s) + \mu\,s\,X(s) \end{aligned}$$

assuming zero initial conditions $x(0) = \dot{x}(0) = 0$.

Force-to-Velocity Transfer Function
(often called the "admittance" or "mobility"):

$$H(s) \triangleq \frac{V(s)}{F(s)} = \frac{sX(s)}{F(s)} = \boxed{\frac{1}{ms + \mu}}$$

Bilinear Transform

The *bilinear transform* is a one-to-one mapping from the s plane to the z plane:

$$\begin{aligned} s &= c\frac{1 - z^{-1}}{1 + z^{-1}}, \quad c > 0, \quad c = \frac{2}{T} \quad \text{(typically)} \\ \Rightarrow\ z &= \frac{1 + s/c}{1 - s/c} \end{aligned}$$

Starting with a *continuous-time* transfer function $H_a(s)$, we obtain the *discrete-time* transfer function

$$\boxed{H_d(z) \triangleq H_a\left(c\frac{1 - z^{-1}}{1 + z^{-1}}\right)}$$

where "d" denotes "digital," and "a" denotes "analog."

Properties of the Bilinear Transform

The bilinear transform maps an s-plane transfer function $H_a(s)$ to a z-plane transfer function:

$$\boxed{H_d(z) \triangleq H_a\left(c\frac{1 - z^{-1}}{1 + z^{-1}}\right)}$$

We can observe the following *properties* of the bilinear transform:

- Analog dc ($s = 0$) maps to digital dc ($z = 1$)
- Infinite analog frequency ($s = \infty$) maps to the maximum digital frequency ($z = -1$)
- The entire $j\omega$ axis in the s plane (where $s \triangleq \sigma + j\omega$) is mapped exactly *once* around the unit circle in the z plane (rather than summing around it infinitely many times, or "aliasing" as it does in ordinary sampling)
- *Stability is preserved* (when c is real and positive)
- *Order of the transfer function is preserved*
- Choose c to map any particular finite frequency (such as a resonance frequency) from the $j\omega_a$ axis in the s plane to a particular desired location on the unit circle $e^{j\omega_d}$ in the z plane. Other frequencies are *"warped"*.

Bilinear Transform of Force-Driven Mass

We have, from $f = m\dot{v} \leftrightarrow F(s) = ms\,V(s)$,

$$V(s) = \frac{1}{ms}F(s)$$

Setting $s = (2/T)(1 - z^{-1})/(1 + z^{-1})$ according to the bilinear transform yields

$$V_d(z) = \frac{T}{2m}\frac{1 + z^{-1}}{1 - z^{-1}}F_d(z)$$

where we defined

$$\begin{aligned} F_d(z) &= F\left(\frac{2}{T}\frac{1 - z^{-1}}{1 + z^{-1}}\right) \\ V_d(z) &= V\left(\frac{2}{T}\frac{1 - z^{-1}}{1 + z^{-1}}\right) \end{aligned}$$

The resulting finite-difference scheme is then

$$v_d(n) - v_d(n-1) = \frac{T}{2m}\left[f_d(n) + f_d(n-1)\right]$$

i.e.,

$$v_d(n) = v_d(n-1) + \frac{T}{2m}\left[f_d(n) + f_d(n-1)\right]$$

We see that this is the same as the backward Euler scheme plus a new term $(T/2m)f_d(n-1)$.

Hybrid Euler-Bilinear Mapping

We can easily interpolate between Backward Euler and Bilinear Transform:

$$\boxed{s \to \frac{1+\alpha}{T}\frac{1-z^{-1}}{1+\alpha\, z^{-1}}}$$

- $\alpha = 0$ gives Backward Euler (high-frequency modes artificially damped)
- $\alpha = 1$ gives Bilinear Transform (high-frequency modes artificially squeezed in frequency)
- Intermediate α allows optimization of another consideration, such as decay time
- Low-frequency response approximately invariant, dc maps to dc in every case

Example: Leaky Integrator

$$H_a(s) \;=\; \frac{1}{s+\epsilon} \;\longrightarrow\; H_d(z) \;=\; \frac{1}{\frac{1+\alpha}{T}\frac{1-z^{-1}}{1+\alpha\, z^{-1}}+\epsilon}$$

$$= g\frac{1+\alpha z^{-1}}{1-pz^{-1}}, \quad \boxed{p = \frac{1-\alpha\frac{\epsilon T}{1+\alpha}}{1+\frac{\epsilon T}{1+\alpha}}}, \quad g = \frac{T}{1+\alpha+\epsilon T}$$

Accuracy of Trapezoidal Rule

For the *Trapezoid Rule* (bilinear transform),

$$f_n \;=\; m\left.\frac{dv}{dt}\right|_{nT} + \mathcal{O}(T^2)$$

so it is *second-order* accurate in T

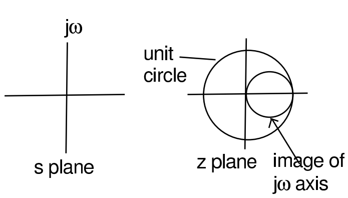
We will come back to this below

Backward Difference Conformal Map

We saw that the *backwards difference* substitution can be seen as a *conformal map* taking the s plane to the z plane:

$$\boxed{s \to \frac{1-z^{-1}}{T}}$$

Look at the image of the $j\omega$ axis under this mapping:



The continuous-time frequency axis, $s = j\omega$, is *not* mapped to the discrete-time frequency axis (unit circle):

- dc ($s = 0$) mapped to dc ($z = 1$)
- infinite frequency mapped to ($z = 0$)

This means *artificial damping* will be introduced for high-frequency system resonances

Laplace Analysis of Trapezoidal Rule

The z transform of the trapezoid rule yields

$$F(z) = \frac{2m}{T}\frac{1-z^{-1}}{1+z^{-1}}V(z)$$

Since $F(s) = ms\,V(s)$, the s to z mapping has become

$$\boxed{s \to \frac{2}{T}\frac{1-z^{-1}}{1+z^{-1}}}$$

which is of course the standard bilinear transform:

- $s = j\omega$ axis maps to the $|z| = 1$ unit circle where it belongs
- dc maps to dc
- Infinite frequency maps to half the sampling rate
- Frequency axis is *warped*, especially at high frequencies
- Stability preserved precisely

Trapezoidal Rule Frequency Mapping

Let's look at the s to z mapping,

$$s = \frac{2}{T}\frac{1-z^{-1}}{1+z^{-1}}$$

on the unit circle, where $s = j\omega_a$ and $z = e^{j\omega_d T}$:

$$j\omega_a = \frac{2}{T}\frac{1-e^{-j\omega_d T}}{1+e^{-j\omega_d T}} = j\frac{2}{T}\tan(\omega_d T/2)$$

or

$$\boxed{\frac{\omega_a T}{2} = \tan\left(\frac{\omega_d T}{2}\right)}$$

- Near dc ($\omega_d = 0$), we have

  $$\omega_a = \frac{2}{T}\tan(\omega_d T/2) = \omega_d + \mathcal{O}(T^3)$$

  where, since $\tan(\theta)$ is odd, there are no even-order terms in its series expansion

In general, the trapezoid rule is a second-order accurate approximation to a derivative, in the limit of small T (i.e., near dc). Here, it is third-order accurate along the unit circle at dc.

Summary of Backward Euler vs. Trapezoidal Rule

For

$$\begin{aligned} f(t) &= m\,a(t) = m\,\dot{v}(t) = \dot{p}(t) \\ &= \lim_{T\to 0}\frac{p(t)-p(t-T)}{T} \approx \frac{p(t)-p(t-T)}{T} \end{aligned}$$

- Backward Euler (BE)

  $$f_n = \frac{1}{T}(p_n - p_{n-1})$$

  is $\mathcal{O}(T)$ (first-order accurate in T)
- Bilinear Transform, or *Trapezoid Rule* (TR)

  $$f_n = \frac{2}{T}(p_n - p_{n-1}) - f_{n-1},$$

  is $\mathcal{O}(T^2)$ (second-order accurate in T)
- A *continuum* of transforms

  $$s = \frac{1+\alpha}{T}\frac{1-z^{-1}}{1+\alpha z^{-1}}$$

  exists between BE and TR and can be optimized for the application at hand (see Kurt Werner thesis and Germain and Werner DAFx-15 paper for details—Germain thesis coming soon)

Why Don't We Always Use the Bilinear Transform?

- Backward Euler (BE) is still sometimes needed:
  - Damps out unwanted high-frequency oscillations (warped)
  - Avoids oscillations at half the sampling rate from a real exponential
    * TR warps high-frequency poles toward half the sampling rate:

      $$s = g' \cdot (1-z^{-1})/(1+z^{-1})$$

      toward $z = -1 \leftrightarrow (-1)^n$
    * BE warps high-frequency poles toward $z = 0$ so it *never* introduces alternating-sign oscillations:

      $$s = g \cdot (1-z^{-1})$$

    * Alternating-sign oscillations due to BLT can be problematic in nonlinear circuits such as those containing diodes (see Kurt Werner thesis for a real-world example)
- Recall also that Forward Euler (FE) can break a *delay-free loop*, and pairs well with BE in series

Physical Model Formulations

Reminder of the various kinds of physical model representations we are considering:

- Ordinary Differential Equations (ODE)
- Partial Differential Equations (PDE)
- Difference Equations (DE)
- Finite Difference Schemes (FDS)
- (Physical) State Space Models
- Transfer Functions (between physical signals)
- Modal Representations (Parallel Second-Order Filters)
- Equivalent Circuits
- Impedance Networks
- Wave Digital Filters (WDF)
- Digital Waveguide (DW) Networks

We are mainly concerned with *real-time computational physical models*

State-Space Models

The *state space* formulation replaces an Nth-order ODE by a *vector* first-order ODE.

Review of discrete-time case:

$$\begin{aligned} \underline{x}(n+1) &= \mathbf{A}\,\underline{x}(n) + \mathbf{B}\,\underline{u}(n) \\ \underline{y}(n) &= \mathbf{C}\,\underline{x}(n) + \mathbf{D}\,\underline{u}(n) \end{aligned}$$

where

- $\underline{x}(n) \in \mathbb{R}^N =$ *state vector* at time n
- $\underline{u}(n) = p \times 1$ vector of inputs
- $\underline{y}(n) = q \times 1$ output vector
- $\mathbf{A} = N \times N$ *state transition matrix*
- $\mathbf{B} = N \times p$ *input coefficient matrix*
- $\mathbf{C} = q \times N$ *output coefficient matrix*
- $\mathbf{D} = q \times p$ *direct path coefficient matrix*

The state-space representation is especially powerful for

- *multi-input, multi-output* (MIMO) linear systems
- *time-varying* linear systems
(every matrix can have a time subscript n)

Continuous-Time State Space Models:

In continuous time, we obtain a first-order vector ODE in which a vector of *state time-derivatives* is driven by linear combinations of state variables:

$$\begin{aligned} \dot{\underline{x}}(t) &= \mathbf{A}\,\underline{x}(t) + \mathbf{B}\,\underline{u}(t) \\ \underline{y}(t) &= \mathbf{C}\,\underline{x}(t) + \mathbf{D}\,\underline{u}(t) \end{aligned}$$

State-Space Advantages:

- State-space models are used extensively in advanced modeling applications
- Extensive support in Matlab, with many numerically excellent associated tools and techniques (such as the singular value decomposition, to name one)
- Analytically powerful for theory work
- Example: Solution of $\dot{\underline{x}}(t) = \mathbf{A}\,\underline{x}(t)$ is $\underline{x}(t) = e^{At}\,\underline{x}(0)$, where the *matrix exponential* is defined as
$$e^{\mathbf{A}t} \triangleq I + \mathbf{A}t + \frac{1}{2}\mathbf{A}^2t^2 + \frac{1}{3!}\mathbf{A}^3t^3 + \cdots$$
- We won't do much with state-space modeling in this class, but you should know it exists and that it should be considered for larger, more complex systems than we will be dealing with

Digitizing State Space Models (Simplistically)

Starting with a continuous-time state-space model

$$\begin{aligned} \dot{\underline{x}}(t) &= \mathbf{A}\,\underline{x}(t) + \mathbf{B}\,\underline{u}(t) \\ \underline{y}(t) &= \mathbf{C}\,\underline{x}(t) + \mathbf{D}\,\underline{u}(t) \\ \longleftrightarrow \quad s\underline{X}(s) - \underline{x}(0) &= \mathbf{A}\,\underline{X}(s) + \mathbf{B}\,\underline{U}(s) \\ \underline{Y}(s) &= \mathbf{C}\,\underline{X}(s) + \mathbf{D}\,\underline{U}(s) \end{aligned}$$

we can, e.g., apply Backward Euler, Trapezoidal Rule (Bilinear Transform), or anything in between:

$$\boxed{s = g\frac{1-z^{-1}}{1+\alpha z^{-1}}, \quad \alpha \in [0,1]}$$

to get, letting $g = (1+\alpha)/T$ and defining $\underline{x}_n = \underline{x}(nT)$,

$$\begin{aligned} \frac{\underline{x}_n - \underline{x}_{n-1}}{T} &= \mathbf{A}\left[\frac{\underline{x}_n + \alpha\underline{x}_{n-1}}{1+\alpha}\right] + \mathbf{B}\left[\frac{\underline{u}_n + \alpha\underline{u}_{n-1}}{1+\alpha}\right] \\ \underline{y}_n &= \mathbf{C}\,\underline{x}_n + \mathbf{D}\,\underline{u}_n \end{aligned}$$

for zero initial conditions $\underline{x}(0) = \underline{0} \;\Rightarrow$

$$\begin{aligned} \underline{x}_{n+1} &= \left(I - \mathbf{A}\frac{T}{1+\alpha}\right)^{-1}\left(I + \mathbf{A}\frac{\alpha T}{1+\alpha}\right)\underline{x}_n \\ &+ \left(I - \mathbf{A}\frac{T}{1+\alpha}\right)^{-1}\mathbf{B}T\left(\frac{z+\alpha}{1+\alpha}\right)u_n \end{aligned}$$

where $z\,u_n \triangleq u_{n+1}$

More sophisticated methods will digitize in a manner that conserves energy and/or momentum

Recommended Related Courses at Stanford

- Math 226
- AA 214 A/B/C
- ME 300 A/B/C
- ME 335 A/B/C