Outline

- Coupled Planes of Vibration in One String
- Two Coupled Strings
- Digital Waveguide Formulation
- Bridge Velocity Transmission Filter
- Eigenanalysis of Coupled Strings

Uncoupled Transverse Planes

- Digital waveguide model of a rigidly terminated string vibrating in three-dimensional space
- Two uncoupled planes of vibration, simulating
  - Orthogonal planes of vibration on one string
  - Single plane of vibration on two different strings
- Because the bridge is normally more yielding in one direction than another, orthogonal planes of vibration are typically slightly out of tune
- A “cheap hack” for coupled string simulation is to simply sum two separate string simulations as in the above figure (but slightly detuned)

Linearly Coupled Planes of Vibration

- Horizontal and vertical waves coupled at the bridge
- Coupling caused by yielding termination at the bridge
- Linear, time-invariant coupling = two-by-two matrix transfer function:

\[
\begin{bmatrix}
  F_v^-(z) \\
  F_h^-(z)
\end{bmatrix} = 
\begin{bmatrix}
  H_{vv}(z) & H_{vh}(z) \\
  H_{hv}(z) & H_{hh}(z)
\end{bmatrix}
\begin{bmatrix}
  F_v^+(z) \\
  F_h^+(z)
\end{bmatrix}
\]

Coupled Piano Strings

- One to three strings per key
- Two-stage decay due to coupling of two strings:
  - Initial attack time constant
  - “Aftersound” time constant
- Mistuning is used to shape the amplitude envelope
- Each string has a horizontal and vertical transverse vibration plane
- Longitudinal waves are audible and are sometimes tuned

See (if interested)

http://www.speech.kth.se/music/5_lectures/weinreic/weinreic.html
Two Strings Coupled at a Load

Two strings terminated at a common bridge impedance.

Digital Waveguide Formulation (Commuted)

General linear coupling of two equal-impedance strings using a common bridge filter.

Eigenanalysis of Coupled Strings

Eigenanalysis of the coupling matrix yields formulas for damping and mode tuning caused by coupling. It also explains the “attack” and “aftersound” components of piano string tones.

General LTI coupling matrix for two strings ($s$ plane):

\[
\begin{bmatrix}
V_1^-(s) \\
V_2^-(s)
\end{bmatrix} =
\begin{bmatrix}
H_{11}(s) & H_{12}(s) \\
H_{21}(s) & H_{22}(s)
\end{bmatrix}
\begin{bmatrix}
V_1^+(s) \\
V_2^+(s)
\end{bmatrix}
\]

(1)

where

\[
C(s) =
\begin{bmatrix}
1 - H_b(s) & -H_b(s) \\
-H_b(s) & 1 - H_b(s)
\end{bmatrix}
\]

where

\[
H_b(s) = \frac{2}{2 + R_b}
\]

and

\[
\hat{R}_b \triangleq \frac{R_b}{R}
\]

is the bridge impedance divided by string impedance.

Treating $C(s)$ as a constant complex matrix for each fixed $s$, the eigenvectors are easily checked to be

\[
\xi_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \quad \xi_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}
\]

with eigenvalues

\[
\lambda_1(s) = 1 - 2H_b(s), \quad \lambda_2 = 1.
\]

Note that only one eigenvalue depends on $s = j\omega$, and neither eigenvector is a function of $s$.

- “In-phase vibrations” see a longer effective string length
  - Length increase for in-phase vibrations given by phase delay of
    \[
    1 - 2H_b = \frac{\hat{R}_b(s) - 2}{\hat{R}_b(s) + 2} = \frac{\hat{R}_b(s) - 2R}{\hat{R}_b(s) + 2R}
    \]
    (reflectance seen from two in-phase strings of impedance $R$).
  - In-phase vibrations move the bridge vertically a lot, causing more rapid decay of the in-phase mode.
- “Anti-phase vibrations” see no length correction at all because the bridge is rigid with respect to anti-phase vibration of the two strings connected to that point.
- This analysis predicts that the “initial fast decay” in a piano note should be a measurably flatter than the “aftersound” which should remain “in tune”.

Bridge Velocity Transmission Filter

\[
V_b(z) = H_b(z)[V_1^+(z) + V_2^+(z)]
\]

\[
H_b(z) \triangleq \frac{2}{2 + R_b(z)}
\]

where

\[
R_b(z) = \text{Bridge Impedance} \\
R = \text{String Wave Impedance}
\]

- Bridge filter input = sum of incoming velocity waves
- Bridge filter output = physical bridge velocity
- Coupled strings are simulated properly this way
- In reduced-cost implementations, the bridge filter $H_b(z)$ can be the only loss filter
- For passive bridges, $R_b(z)$ is always a positive real function (stable, min-phase, $|\text{phase}| \leq \pi/2$)
- The trivial filter $H_b(z) = z^{-1}$ is not passive because it corresponds to $R_b(z)/R = 2(z - 1)$ (non-causal).