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Cylinder with Conical Cap

Scattering Filters at the Cylinder-Cone Junction

Wave impedance at frequency $\omega$ rad/sec in a converging cone:

$$Z_\xi(j\omega) = \frac{\rho c}{S(\xi)} \frac{j\omega}{j\omega - c/\xi}$$

(converging cone impedance)
where
\[ \xi = \text{distance to the apex of the cone} \]
\[ S(\xi) = \text{cross-sectional area of cone} \]
\[ \rho_c = \text{wave impedance in open air} \]

In the limit as \( \xi \to \infty \),
\[ Z_\infty(j\omega) = \frac{\rho_c}{S} \] (cylindrical tube impedance)

**Reflectance** of the conical cap, *seen from cylinder*:
\[ R(s) = -\frac{c/\xi}{c/\xi - 2s} \]

**Transmittance** to the right:
\[ T(s) = 1 + R(s) = -\frac{2s}{c/\xi - 2s} \]

- \( R(s) \) and \( T(s) \) are first-order transfer functions, each having a single real pole at \( s = c/(2\xi) \Rightarrow \text{unstable} \)
- \( R(s) \) and \( T(s) \) identical from left and right given no wavefront area discontinuity.

**Reflectance of the Conical Cap**

- Let \( t_\xi \triangleq \xi/c \) denote the time to propagate across the length of the cone in one direction
- Reflectance of complete (lossless) cone is \(-1\) for pressure waves (reflects like an open-ended cylinder)
- Round-trip transfer function from cone entrance to tip and back is
\[ R_{t\xi}(s) \triangleq -e^{-2st_\xi} = e^{-st_\xi}(-1)e^{-st_\xi} \]

(\text{reflectance seen inside the cone})

**Reflectance of Conical Cap Seen from Cylinder**

From the figure, we can derive the conical cap reflectance
A transfer function \( R_J(s) = \frac{N(s)}{D(s)} \) is stable if there are no poles in the right-half \( s \) plane. That is, for each zero \( s_i \) of \( D(s) \), we must have \( \text{re} \{ s_i \} \leq 0 \). If this can be shown, along with \( |R_J(j\omega)| \leq 1 \), then the reflectance \( R_J \) is shown to be passive.

We must also study the system zeros (roots of \( N(s) \)) in order to determine if there are any pole-zero cancellations (common factors in \( D(s) \) and \( N(s) \)).

Since \( \text{re} \{ st\xi \} \geq 0 \) if and only if \( \text{re} \{ s \} \geq 0 \), for \( t\xi > 0 \), we may set \( t\xi = 1 \) without loss of generality. Thus, we need only study the roots of

\[
N(s) = 1 - e^{-2s} - 2se^{-2s} \\
D(s) = 2s - 1 + e^{-2s}
\]

If this system is stable, we have stability also for all \( t\xi > 0 \).

Since \( e^{-2s} \) is not a rational function of \( s \), the reflectance \( R_J(s) \) may have infinitely many poles and zeros.

**Stability Proof**

First consider the roots of the denominator

\[
D(s) = 2s - 1 + e^{-2s}.
\]

At any pole (solution \( s \) of \( D(s) = 0 \)), we must have

\[
s = \frac{1 - e^{-2s}}{2}
\]

To obtain separate equations for the real and imaginary parts, take the real and imaginary parts of

\[
D(\sigma + j\omega) = 0
\]

to get

\[
\text{re} \{ D(s) \} = (2\sigma - 1) + e^{-2\sigma} \cos(2\omega) = 0 \\
\text{im} \{ D(s) \} = 2\omega - e^{-2\sigma} \sin(2\omega) = 0
\]
Both of these equations must hold at any pole of the reflectance. For stability, we further require \( \sigma \leq 0 \).

Defining \( \tau = 2\sigma \) and \( \nu = 2\omega \), we obtain the simpler conditions

\[
e^{\tau}(1 - \tau) = \cos(\nu)
\]

\[
e^{\tau} = \frac{\sin(\nu)}{\nu}
\]

For any poles of \( R_j(s) \) on the \( j\omega \) axis, we have \( \tau = 0 \), and the second equation reduces to \( \text{sinc}(\nu) = 1 \). It is well known that the sinc function is less than 1 in magnitude at all \( \nu \) except \( \nu = 0 \). Therefore, this relation can hold only at \( \omega = \nu = 0 \), and so

\textbf{Any right-half-plane poles occur at } \omega = 0.\]

\textbf{Stability Proof, continued}

The same argument can be extended to the entire right-half plane as follows. Going back to

\[
\frac{\sin(\nu)}{\nu} = e^{\tau},
\]

since \( |\sin(\nu)/\nu| \leq 1 \) for all real \( \nu \), and since \( |e^{\tau}| > 1 \) for \( \tau > 0 \), this equation clearly has no solutions in the right-half plane. Therefore,

\textbf{Any right-half-plane poles occur at } \omega = 0.\]

\textbf{A Pole at DC}

Since both of the conditions

\[
e^{\tau}(1 - \tau) = \cos(\nu)
\]

\[
e^{\tau} = \frac{\sin(\nu)}{\nu}
\]

are clearly satisfied for \( \tau = \nu = 0 \), we see that there is in fact a pole in the reflectance at dc \( (s = 0) \), provided it is not canceled by a zero at dc in the numerator \( N(s) \).

\textbf{The Left-Half Plane}

In the left-half plane, there are many potential poles:

- The first of the two equations

  \[
e^{\tau}(1 - \tau) = \cos(\nu)
\]

  has infinitely many solutions for each \( \tau < 0 \), since the elementary inequality \( 1 - \tau \leq e^{-\tau} \) implies

  \[
e^{\tau}(1 - \tau) < e^{\tau}e^{-\tau} = 1
\]

- The second equation,

  \[
e^{\tau} = \frac{\sin(\nu)}{\nu}
\]

  has an increasing number of solutions as \( \tau \) grows more and more negative.
• As $\tau \to -\infty$, the number of solutions becomes infinite and are given by the zeros of $\sin(\nu)$

• At $\tau \to -\infty$, the solutions of the other equation converge to the zeros of $\cos(\nu)$

• Thus, the solutions of $e^{\tau(1-\tau)} = \cos(\nu)$

$$e^{\tau} = \frac{\sin(\nu)}{\nu}$$

may not necessarily occur together for $\tau < 0$, as they must.

Poles at $s=0$

We know from the foregoing that the denominator of the cone reflectance has at least one root at $s = 0$. We now investigate the “dc behavior” more thoroughly.

• A hasty analysis based on the reflection and transmission filters (see figure) might conclude that the reflectance of the conical cap converges to $-1$ at dc, since $R(0) = -1$ and $T(0) = 0$. However, this is incorrect.

• Instead, it is necessary to take the limit as $\omega \to 0$ of the complete conical cap reflectance $R_J(s)$:

$$R_J(s) = \frac{1 - e^{-2s} - 2se^{-2s}}{2s \left(1 + e^{-2s}\right)}$$

We already discovered a root at $s = 0$ in the denominator in the context of the preceding stability proof. However, note that the numerator also goes to zero at $s = 0$. This indicates a pole-zero cancellation at dc.

• To find the reflectance at dc, we may use L’Hospital’s rule to obtain

$$R_J(0) = \lim_{s \to 0} \frac{N'(s)}{D'(s)} = \lim_{s \to 0} \frac{4se^{-2s}}{2 - 2e^{-2s}}$$

and once again the limit is an indeterminate $0/0$ form.

• We apply L’Hospital’s rule again to obtain

$$R_J(0) = \lim_{s \to 0} \frac{N''(s)}{D''(s)} = \lim_{s \to 0} \frac{(4 - 8s)e^{-2s}}{4e^{-2s}} = +1$$

Thus, two poles and zeros cancel at dc, and the dc reflectance is $+1$, not $-1$ as an analysis based only on the scattering filters would indicate.

• From a physical point of view, it makes more sense that the cone should “look like” a simple rigid
termination of the cylinder at dc, since its length becomes vanishingly small compared with the wavelength in the limit.

- Another method of showing this result is to form a Taylor series expansion of the numerator and denominator:

\[
N(s) = 2s^2 - \frac{8s^3}{3} + 2s^4 + \ldots
\]

\[
D(s) = 2s^2 - \frac{4s^3}{3} + \frac{2s^4}{3} + \ldots
\]

Both series begin with the term \(2s^2\) which means both the numerator and denominator have two roots at \(s = 0\). Hence, again the conclusion is two pole-zero cancellations at dc.

- The series for the conical cap reflectance is

\[
R_J(s) = 1 - \frac{2s}{3} + \frac{2s^2}{9} - \frac{4s^3}{135} - \frac{2s^4}{405} + \ldots
\]

which approaches +1 as \(s \to 0\).