

Tutorial on Wave Digital Filters

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Outline

- 1 Introduction
 - Motivation
 - Classical Network Theory
- 2 Wave Digital Formulation
 - Wave Digital One-Ports Derivation
 - Wave Digital Adaptors
 - Nonlinearity
- 3 Summary
 - Examples
 - Conclusions
- 4 Appendix
 - Scattering Junction Derivations
 - Mechanical Impedance Analogues

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Wave digital filters model circuits used for filtering.

Overview

- Fettweis (1986), Wave Digital Filters: Theory and Practice.
- Wave Digital Filters (WDF) mimic structure of classical filter networks.
 - Low sensitivity to component variation.
- Use wave variable representation to break delay free loop.
- WDF adaptors have low sensitivity to coefficient quantization.
 - Direct form with second order section biquads are also robust
 - Transfer function abstracts relationship between component and filter state
 - WDF provides direct one-to-one mapping from physical component to filter state variable

Wave digital filters model circuits used for filtering

Applications

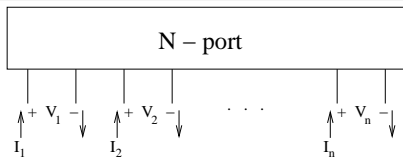
- Modeling physical systems with equivalent circuits.
 - Piano hammer mass spring interaction
 - Generally an ODE solver
 - Element-wise discretization and connection strategy
 - Real time model of loudspeaker driver with nonlinearity
 - Multidimensional WDF solves PDEs
- Ideal for interfacing with digital waveguides (DWG).

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Classical Network Theory

N-port linear system



- Describe a circuit in terms of voltages (across) and current (thru) variables
- General N-port network described by V and I of each port
- Impedance or admittance matrix relates V and I

$$\bullet \begin{pmatrix} V_1 \\ V_2 \\ \vdots \\ V_N \end{pmatrix} = \underbrace{\begin{pmatrix} Z_{11} & Z_{12} & \dots & Z_{1N} \\ Z_{21} & \ddots & & Z_{2N} \\ \vdots & & & \vdots \\ Z_{N1} & \dots & & Z_{NN} \end{pmatrix}}_Z \begin{pmatrix} I_1 \\ I_2 \\ \vdots \\ I_N \end{pmatrix}$$

Classical Network Theory

Element-wise discretization for digital computation

- For example, use Bilinear transform

$$s = \frac{2}{T} \frac{1-z^{-1}}{1+z^{-1}}$$

- Capacitor: $Z(s) = \frac{1}{sC}$

$$Z(z^{-1}) = \frac{T}{2C} \frac{1+z^{-1}}{1-z^{-1}} = \frac{V(z^{-1})}{I(z^{-1})}$$

$$v[n] = \frac{T}{2C}(i[n] + i[n-1]) + v[n-1]$$

- $v[n]$ depends instantaneously on $i[n]$ with $R_0 = \frac{T}{2C}$
- This causes problems when trying to make a signal processing algorithm
- Can also solve for solution using a matrix inverse (what SPICE does).

Classical Network Theory

Wave variable substitution and scattering

$$A = V + RI$$

$$B = V - RI$$

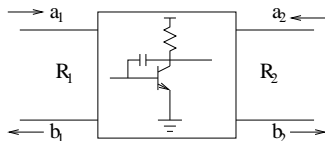
$$V = \frac{A+B}{2}$$

$$I = \frac{A-B}{2R}$$

- Variable substitution from V and I to incident and reflected waves, A and B
- An N -port gives an $N \times N$ scattering matrix
- Allows use of scattering concept of waves

Classical Network Theory

Two port is commonly used in microwave electronics to characterize amplifiers



- Input port (1) and output port (2)
- Represent as scattering matrix and wave variables

$$\begin{pmatrix} V_1 \\ V_2 \end{pmatrix} = \begin{pmatrix} Z_{11} & Z_{12} \\ Z_{21} & Z_{22} \end{pmatrix} \begin{pmatrix} I_1 \\ I_2 \end{pmatrix} \qquad \begin{pmatrix} b_1 \\ b_2 \end{pmatrix} = \begin{pmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \end{pmatrix}$$

- Scattering matrix **S** determines reflected wave b_n as a linear combination of N incident waves a_1, \dots, a_n
- Guts of the circuit abstracted away into **S** or **Z** matrix

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Wave Digital Elements

Basic one port elements

- Work with voltage wave variables b and a . Substitute into Kirchhoff circuit equations and solve for b as a function of a .
- Wave reflectance between two impedances is well known

$$\rho = \frac{b}{a} = \frac{R_2 - R_1}{R_2 + R_1}$$

- Define a port impedance R_p
- Input wave comes from port and reflects off the element's impedances.
 - Resistor $Z_R = R$, $\rho_R(s) = \frac{1 - R_p/R}{1 + R_p/R}$
 - Capacitor $Z_C = \frac{1}{sC}$, $\rho_C(s) = \frac{1 - R_pCs}{1 + R_pCs}$
 - Inductor $Z_L = sL$, $\rho_L(s) = \frac{s - R_p/L}{s + R_p/L}$

Wave Digital Elements

Discretize the capacitor by bilinear transform

- Plug in bilinear transform

$$\frac{b_n}{a_n} = \frac{1 - R_p C \frac{2}{T} \frac{1-z^{-1}}{1+z^{-1}}}{1 + R_p C \frac{2}{T} \frac{1-z^{-1}}{1+z^{-1}}}$$

$$(1 + R_p C \frac{2}{T})b[n] + (1 - R_p C \frac{2}{T})b[n-1] = \\ (1 - R_p C \frac{2}{T})a[n] + (1 + R_p C \frac{2}{T})a[n-1]$$

- Choose R_p to eliminate dependence of $b[n]$ on $a[n]$, e.g., $R_p = \frac{T}{2C}$, resulting in:

$$b[n] = a[n-1]$$

- Note that chosen R_p exactly the instantaneous resistance of the capacitor when discretized by the bilinear transform

Wave Digital Elements

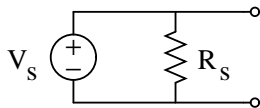
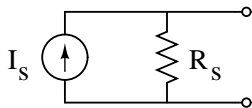
T-ports and I-ports

- *T-port*. Port resistance can be chosen to perfectly match element resistance to eliminate instantaneous reflection and avoid delay-free loop.
- *I-port*. If port is not matched, $b[n]$ depends on $a[n]$ instantaneously. R_p can be chosen as any positive value.
 - Short circuit
 $b[n] = -a[n]$
 - Open circuit
 $b[n] = a[n]$
 - Voltage source of voltage V
 $b[n] = -a[n] + 2V$
 - Current source of current I
 $b[n] = a[n] - 2R_p I$

Wave Digital Elements

One port summary for voltage waves

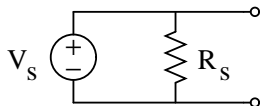
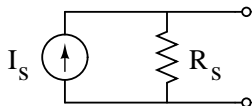
Element	Port Resistance	Reflected wave
Resistor	$R_p = R$	$b[n] = 0$
Capacitor	$R_p = \frac{T}{2C}$	$b[n] = a[n - 1]$
Inductor	$R_p = \frac{2L}{T}$	$b[n] = -a[n - 1]$
Short circuit	$R_p = 0$	$b[n] = -a[n]$
Open circuit	$R_p = \infty$	$b[n] = a[n]$
Voltage source V_s	$R_p = 0$	$b[n] = -a[n] + 2V_s$
Current source I_s	$R_p = \infty$	$b[n] = a[n] + 2R_p I_s$
Terminated V_s	$R_p = R_s$	$b[n] = V_s$
Terminated I_s	$R_p = R_s$	$b[n] = R_p I_s$



Wave Digital Elements

One port summary for current waves. Signs are flipped for some reflectances.

Element	Port Resistance	Reflected wave
Resistor	$R_p = R$	$b[n] = 0$
Capacitor	$R_p = \frac{T}{2C}$	$b[n] = -a[n - 1]$
Inductor	$R_p = \frac{2L}{T}$	$b[n] = a[n - 1]$
Short circuit	R_p	$b[n] = a[n]$
Open circuit	R_p	$b[n] = -a[n]$
Voltage source V_s	R_p	$b[n] = a[n] + 2V_s$
Current source I_s	R_p	$b[n] = -a[n] + 2R_p I_s$
Terminated V_s	$R_p = R_s$	$b[n] = V_s$
Terminated I_s	$R_p = R_s$	$b[n] = R_p I_s$



Wave Digital Elements

Two ports

- Series
- Parallel
- Transformer
- Unit element

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Adaptors

Adaptors perform the signal processing calculations

- Treat connection of N circuit elements as an N -port
- Derive scattering junction from Kirchhoff's circuit laws and port impedances determined by the attached element
- Scattering matrix is $N \times N$
- Parallel and series connections can be simplified to linear complexity
- Signal flow diagrams to reduce number of multiply or add units
 - Dependent port - one coefficient can be implied,
 - Reflection free port - match impedance to eliminate reflection

Two-Port Parallel Adaptor

$$b_1 = a_2 + \gamma(a_2 - a_1)$$

$$b_2 = a_1 + \gamma(a_2 - a_1)$$

$$\gamma = (R_1 - R_2)/(R_1 + R_2)$$

N-Port Parallel Adaptor

- G_ν are the port conductances

$$G_n = G_1 + G_2 + \cdots + G_{n-1}, G_\nu = 1/R_\nu$$

- Find scattering parameters

$$\gamma_\nu = \frac{G_\nu}{G_n}, \nu = 1 \text{ to } n - 1$$

- Note γ sum to 2

$$\gamma_1 + \gamma_2 + \cdots + \gamma_n = 2$$

- Use intermediate variable to find reflected waves

$$a_o = \gamma_1 a_1 + \gamma_2 a_2 + \cdots + \gamma_n a_n$$

$$b_\nu = a_o - a_\nu$$

Parallel Adaptor

with port n reflection free (RFP)

- G_ν are the port conductances
- R_n is set equal to equivalent resistance looking at all the other ports (their R s in parallel) to make port n RFP

$$G_n = G_1 + G_2 + \cdots + G_{n-1}, G_\nu = 1/R_\nu$$

$$\gamma_n = 1$$

$$\gamma_1 + \gamma_2 + \cdots + \gamma_{n-1} = 1$$

$$\gamma_\nu = \frac{G_\nu}{G_n}, \nu = 1 \text{ to } n-1$$

$$b_n = \gamma_1 a_1 + \gamma_2 a_2 + \cdots + \gamma_{n-1} a_{n-1}, \quad \text{RFP}$$

$$b_\nu = b_n + a_n - a_\nu$$

Series Adaptor

with port n dependent

- R_ν are the port resistances.
- Find scattering parameters

$$\gamma_\nu = \frac{2R_\nu}{R_1 + R_2 + \cdots + R_n}$$

- Note scattering parameters sum to 2

$$\gamma_1 + \gamma_2 + \cdots + \gamma_n = 2$$

- Use intermediate variable to find reflected waves

$$a_o = a_1 + a_2 + \cdots + a_n$$

$$b_\nu = a_\nu - \gamma_\nu a_o$$

Series Adaptor

with port n reflection free (RFP)

- R_ν are the port resistances
- R_n is set equal to equivalent resistance looking at all the other ports (their R s in series) to make port n RFP

$$R_n = R_1 + R_2 + \cdots + R_{n-1}$$

$$\gamma_n = 1$$

$$\gamma_1 + \gamma_2 + \cdots + \gamma_{n-1} = 1$$

$$\gamma_\nu = \frac{R_\nu}{R_n}, \nu = 1 \text{ to } n-1$$

$$b_n = -(a_1 + a_2 + \cdots + a_{n-1}), \quad \text{RFP}$$

$$b_\nu = a_\nu - \gamma_\nu(a_n - b_n)$$

Adaptors

Dependent ports

- Adaptors have property of low coefficient sensitivity, e.g., coefficients can be rounded or quantized.
- Dependent ports take advantage of property that γ s sum to two.
- Use this fact along with quantization to ensure that adaptor is (pseudo-)passive.

Connection Strategy

Parameter updates

- Parameter updates propagate from leaf through its parents to the root
- Each adaptor's RFP must be recalculated when a child's port resistance changes
- Parameter update is more complicated than solving the Kirchhoff's equations directly, where the parameters are just values in the resistance matrix.

Connection Strategy

Avoid delay-free loops with adaptors connected as a tree

- Sarti et. al., Binary Connection Tree - implement WDF with three-port adaptors
- Karjalainen, BlockCompiler - describe WDF in text, produces efficient C code
- Scheduling to compute scattering
 - Directed tree with RFP of each node connected to the parent
 - Label each node (a, b, c, ...)
 - Label downward going signals d by node and port number
 - Label upward going signals u by node
 - Start from leaves, calculate all u going up the tree
 - Then start from root, calculate all d going down the tree

Connection Strategy

Automatic generation of WDF tree structure

- SPQR tree algorithms find biconnected and triconnected graphs (Fränken, Ochs, and Ochs, 2005. Generation of Wave Digital Structures for Networks Containing Multiport Elements.)
- Q nodes are one ports
- S and P nodes are Series and Parallel adaptors
- R nodes are triconnected elements
 - Implemented with similarity transform of $N \times N$ scattering matrix into two-port adaptors (Meerkötter and Fränken, Digital Realization of Connection Networks by Voltage-Wave Two-Port Adaptors”
 - Includes bridge connections and higher order connections
- Implemented in WDInt package for Matlab (<http://www-nth.uni-paderborn.de/wdint/index.html>)

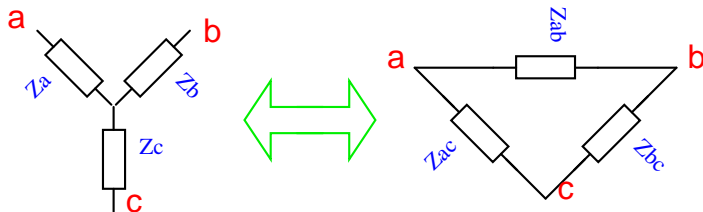
WDF Interconnections

More about the R nodes

- For example, bridge connection
- higher order triconnections also common
- N-port scattering junction
 - can be reduced to implementation by two port adaptors using similarity transform - keeps robust properties for quantization
 - reduces operations for filtering vs scatter matrix
- In general parameter update is complicated

R-Nodes

Other strategies: $Y - \Delta$ $\Delta - Y$ transformations

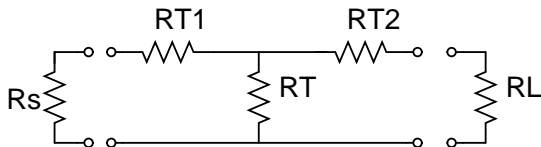


- Circuit analysis technique to replace triconnected impedances with equivalents that can be connected in series or parallel.
- Must discretize general impedances, no longer correspondence between prototype circuit element and WDF element

Tone Stack

Formulate blocks compatible with scattering

- Observe that tone stack is a specific two-port
- Direct implementation of a 2-port scattering matrix
- Or convert into an equivalent circuit with impedances and use adaptors
- Tabulate the scattering parameters or impedances as they vary with parameter changes



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Nonlinearity

Literature

- Meerkötter and Scholz (1989), Digital Simulation of Nonlinear Circuits by Wave Digital Filter Principles.
- Sarti and De Poli (1999), Toward Nonlinear Wave Digital Filters.
- Karjalainen and Pakarinen (2006), Wave Digital Simulation of Vacuum-Tube Amplifier
- Petrusch and Rabenstein (2004), Wave Digital Filters with Multiple Nonlinearities
- Either conceive as nonlinear resistor or dependent source
- Introduces an I-port, may lead to delay-free loops
- DFL must be solved as a system of equations in wave variables

Nonlinear Conductance

Meerkötter and Scholz (1989).

- Current is a nonlinear function of voltage, $i = i(v)$
- In wave variables

$$a = f(v) = v + R_p i(v)$$

$$b = g(v) = v - R_p i(v)$$

- Substituting wave variables into Kirchhoff variable definition of nonlinear resistance and solving for $b(a)$

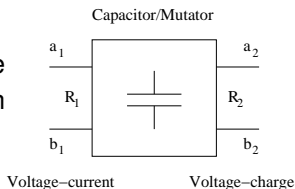
$$b = b(a) = g(f^{-1}(a))$$

- f^{-1} must exist
- Port resistance R_p can be chosen arbitrarily within constraints that $f(v)$ be invertible
- Instantaneous dependence exists regardless of R_p

Extension to nonlinear reactances

Nonlinear Capacitor. Sarti and De Poli (1999).

Use a “mutator” to integrate the Kirchoff variable so that the nonlinear reactance can be defined in terms of wave variables.



- Define wave variable such that port resistance can be an impedance.

$$A(s) = V(s) + R(s)I(s)$$

$$B(s) = V(s) - R(s)I(s)$$

- Recall that standard wave definitions for a one port such as a capacitor are

$$A_1 = V_1 + R_1 I_1$$

$$B_1 = V_1 - R_1 I_1$$

Extension to nonlinear reactances

Nonlinear Capacitor. Sarti and De Poli (1999).

For the capacitor with the usual single resistive port, define a second port across the capacitor with its port impedance $R(s) = \frac{1}{sC}$

$$A_2(s) = V_2(s) + \frac{1}{sC} I_2(s)$$

$$B_2(s) = V_2(s) - \frac{1}{sC} I_2(s)$$

This looks like the usual wave variable definitions if $I(s)$ is replaced by its integral $Q(s)$, charge, and port impedance $R_2 = 1/C$.

$$A_2(s) = V_2(s) + \frac{1}{C} Q(s)$$

$$B_2(s) = V_2(s) - \frac{1}{C} Q(s)$$

A_2 and B_2 can be substituted into the definition of a generic nonlinear capacitance $Q = f(V) = C(V) \cdot V$ to find the nonlinear “reflection” as it is done for the nonlinear resistor. R_2 can be chosen rather arbitrarily as before.

Extension to nonlinear reactances

Voltage-current to voltage-charge wave conversion (Felderhoff 1996).

Compute scattering relations between the resistive and the integrated port using two relations: consistency of voltage for the two ports $V_1 = V_2$, and $I_1 + I_2 = 0$ across junction .

$$V = \frac{A_1 + B_1}{2} = \frac{A_2 + B_2}{2} \quad (1)$$

$$I_1 = \frac{A_1 - B_1}{2R_1} = -\frac{A_2 - B_2}{2R(s)} = -\frac{A_2 - B_2}{2\frac{1}{sC}} \quad (2)$$

bilinear transform $s \rightarrow z$

$$\frac{A_1 - B_1}{R_1} = -\frac{A_2 - B_2}{\frac{1}{C} \frac{T}{2} \frac{1+z^{-1}}{1-z^{-1}}}$$

$$(1 + z^{-1})(A_1 - B_1) = -\frac{2}{T} CR_1 (1 - z^{-1})(A_2 - B_2)$$

$$(A_1 + z^{-1}A_1 - B_1 - z^{-1}B_1) = -\frac{2}{T} CR_1 (A_2 - z^{-1}A_2 - B_2 + z^{-1}B_2)$$

$$a_1[n] + a_1[n-1] - b_1[n] - b_1[n-1] = -\frac{2}{T} CR_1 (a_2[n] - a_2[n-1] - b_2[n] + b_2[n-1])$$

substitute $b_2 = a_1 + b_1 - a_2$ using (1)

$$a_1[n] + a_1[n-1] - b_1[n] - b_1[n-1] = -\frac{2}{T} CR_1 (a_2[n] - a_2[n-1])$$

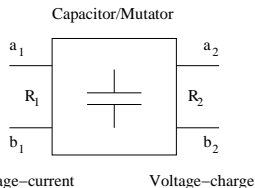
$$- (a_1[n] + b_1[n] - a_2[n]) + a_1[n-1] + b_1[n-1] - a_2[n-1]$$

Extension to nonlinear reactances

Voltage-current to voltage-charge wave conversion.

Set $R_1 = T/(2C)$ to eliminate reflection at port 1.

$$a_1[n-1] - b_1[n] = -a_2[n] + a_2[n-1]$$



Resulting scattering junction (or mutator according to Sarti and De Poli) converts between voltage-current and voltage-charge waves:

$$b_2 = a_1 + (a_1[n-1] - a_2[n-1])$$

$$b_1 = a_2 + (a_1[n-1] - a_2[n-1])$$

Port resistance for new mutated waves corresponding to voltage and charge is $R_2 = \frac{1}{C}$.

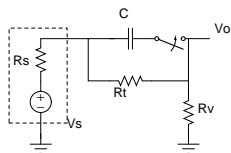
Port resistance for usual waves corresponding to voltage and current is $R_1 = \frac{T}{2C} = \frac{T}{2} R_2$.

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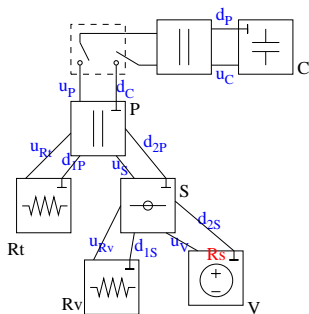
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Parametrized Linear Circuit Example

Volume pot with bright switch



Output is voltage over R_V , $V_o = \frac{(d_{1S} + u_{Rt})}{2}$
 $R_V = R_{pot}(vol)$, $R_t = R_{pot}(1 - vol)$. Changes in vol require recomputation of γ 's starting from bottom of tree. Use RFP to allow open circuit when C is disconnected.



$$u_{R_V} = u_{R_t} = 0, \quad u_V = V$$

$$u_S = -(u_{R_V} + u_V)$$

$$u_P = \gamma_{1P} u_{R_t} + \gamma_{2P} u_S$$

$$u_C = d_P [n - 1]$$

$$d_P = u_P + \gamma_P (u_C - u_P)$$

$$d_C = u_C + \gamma_P (u_C - u_P), \quad \text{or} \quad d_C = u_P$$

$$d_{1P} = u_P + d_C - u_{R_t}, \quad \text{don't care}$$

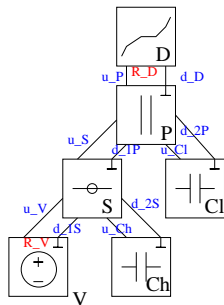
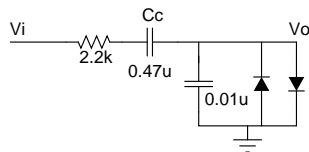
$$d_{2P} = u_P + d_C - u_S$$

$$d_{1S} = u_{R_V} - \gamma_{1S} (d_{2P} - u_S), \quad \text{output value}$$

$$d_{2S} = u_V - \gamma_{2S} (d_{2P} - u_S), \quad \text{don't care}$$

Nonlinear Circuit Example

Diode clipper



- Diode clipper circuit found in guitar distortion pedals
- Treat diodes together as single nonlinear one-port

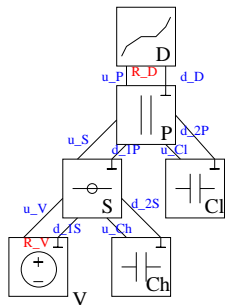
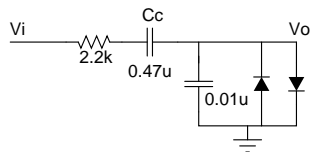
$$I(V) = 2I_s \sinh(V/V_d)$$

$$\text{Solve for } b(a), \frac{a-b}{2R_p} = 2I_s \sinh\left(\frac{a+b}{2V_d}\right)$$

- Isolate nonlinearity at root of tree
- Incorporate resistor into voltage source

Nonlinear Circuit Example

Diode clipper: Computational algorithm



$$u_V = V, \quad u_{Ch} = d_{2S}[n-1]$$

$$u_S = -(u_V + u_{Ch}), \quad u_{Cl} = d_{2P}[n-1]$$

$$u_P = \gamma_{1P}u_S + \gamma_{2P}u_{Cl}$$

$$d_D = f(u_P), \quad \text{nonlinear function}$$

$$d_{1P} = u_P + d_D - u_S$$

$$d_{2P} = u_P + d_D - u_{Cl}$$

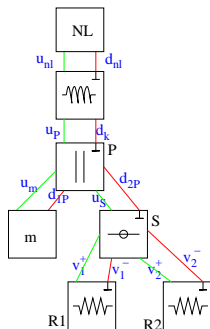
$$d_{1S} = u_V - \gamma_{1S}(d_{1P} - u_S), \quad \text{don't care}$$

$$d_{2S} = u_{Ch} - \gamma_{2S}(d_{1P} - u_S)$$

Nonlinear Musical Acoustics Example

WDF hammer and nonlinear felt

- Draw mass/spring/waveguide system in terms of equivalent circuits
- Waveguides look like resistors to the lumped hammer. Waves enter lumped junction directly.
- WDF result in tree like structures with adaptors/scattering junctions at the nodes, and elements at the leaves.
- The root of the tree allowed to have instantaneous reflections
- Nonlinearity gives instantaneous reflection, WDF handles only 1 nonlinearity naturally.
- Compression ($d = \frac{u_{nl} - d_{nl}}{2R_{nl}}$ from next slide) must ≥ 0 , otherwise hammer is not in contact.



To left and right waveguides

Nonlinear Musical Acoustics Example

Computations

v_1^+, v_2^- from waveguides

$$u_S = -(v_1^+ + v_2^-)$$

$$u_m = d_{1P}[n-1]$$

$$u_P = \gamma_{1P}u_m + \gamma_{2P}u_S$$

$$u_{nl} = u_P + (u_P[n-1] - d_{nl}[n-1])$$

$$d_{nl} : \text{solve } \left\{ \frac{u_{nl} + d_{nl}}{2} = k \left(\frac{u_{nl} - d_{nl}}{2R_{nl}} \right)^\gamma \right\}$$

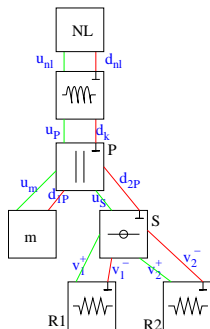
$$d_k = d_{nl} + (u_P[n-1] - d_{nl}[n-1])$$

$$d_{1P} = u_P + d_k - u_m$$

$$d_{2P} = u_P + d_k - u_S$$

$$v_1^- = v_1^+ - \gamma_{1S}(d_{2P} - u_S)$$

$$v_2^+ = v_2^- - \gamma_{2S}(d_{2P} - u_S)$$



To left and right waveguides

Outline

- 1 Introduction
 - Motivation
 - Classical Network Theory
- 2 Wave Digital Formulation
 - Wave Digital One-Ports Derivation
 - Wave Digital Adaptors
 - Nonlinearity
- 3 Summary
 - Examples
 - **Conclusions**
- 4 Appendix
 - Scattering Junction Derivations
 - Mechanical Impedance Analogues

Observations

- Scattering formulation works well with DWG - DWG looks like resistor in WDF
- For a standalone simulation of nonlinear circuits, may not be the best choice
- More difficult for parameter update than direct solving
- Root node is special - easy to implement nonlinearity or parameter changes
- May be able to design circuits without bridge connections that have equivalent transfer function

Summary

- Wave digital formulation uses matched (reflection-free) ports to eliminate reflections and avoid delay-free loops
- Elements are connected in tree structure with reflection-free ports connected to the parent node.
- Useful for building up a model component-wise.

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Two-port adaptor reflectance and transmittance

N-port parallel adaptor

N-port series adaptor

Capacitor

Inductor

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Mechanical Impedance Analogues from Music 420

The following mechanical examples are taken from:
“Lumped Elements, One-Ports, and Passive Impedances”, by
Julius O. Smith III, (From Lecture Overheads, Music 420).

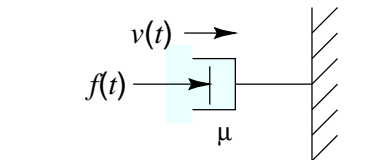
<http://ccrma.stanford.edu/jos/OnePorts/>

<http://ccrma.stanford.edu/~jos/OnePorts/>

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Dashpot

Ideal dashpot characterized by a constant impedance μ



- Dynamic friction law

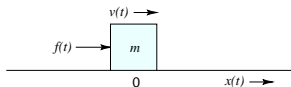
$f(t) \approx \mu v(t)$ “Ohm’s Law” (Force = Friction_coefficient \times Velocity)

- Impedance

$$R_{\mu}(s) \triangleq \mu \geq 0$$

- Dashpot = *gain* for force input and velocity output
- Electrical analogue: Resistor $R = \mu$
- More generally, losses due to friction are
 - *frequency dependent*
 - *hysteretic*

Mass



Ideal mass of m kilograms sliding on a frictionless surface

- Newton's 2nd Law

$$f(t) = ma(t) \triangleq m\dot{v}(t) \triangleq m\ddot{x}(t) \text{ (Force = Mass} \times \text{Acceleration)}$$

- *Differentiation Theorem*

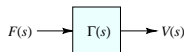
$$F(s) = m[sV(s) - v(0)] = msV(s)$$

for Laplace Transform when $v(0) = 0$.

- *Impedance*

$$R_m(s) \triangleq \frac{F(s)}{V(s)} = ms$$

Mass



“Black Box” Description

- *Admittance*

$$\Gamma_m(s) \triangleq \frac{1}{R_m(s)} = \frac{1}{ms}$$

- *Impulse Response*
(unit-momentum input)

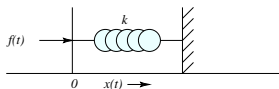
$$\gamma_m(t) \triangleq \mathcal{L}^{-1} \{ \Gamma_m(s) \} = \frac{1}{m} u(t)$$

- *Frequency Response*

$$\Gamma_m(j\omega) = \frac{1}{mj\omega}$$

- Mass admittance = *Integrator*
(for force input, velocity output)
- Electrical analogue: *Inductor*
 $L = m$.

Spring (Hooke's Law)



Ideal spring

- Hooke's law

$$f(t) = kx(t) \triangleq k \int_0^t v(\tau) d\tau \text{ (Force = Stiffness} \times \text{Displacement)}$$

- Impedance

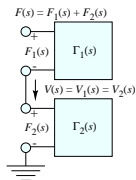
$$R_k(s) \triangleq \frac{F(s)}{V(s)} = \frac{k}{s}$$

- Frequency Response

$$\Gamma_k(j\omega) = \frac{j\omega}{k}$$

- Spring = *differentiator* (force input, velocity output)
- Velocity $v(t)$ = “compression velocity”
- Electrical analogue: *Capacitor* $C = 1/k$ (1/stiffness = “compliance”)

Series Connection of One-Ports



- Series Impedances *Sum*:

$$R(s) = R_1(s) + R_2(s)$$

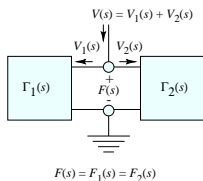
- Admittance:

$$\Gamma(s) = \frac{1}{\frac{1}{\Gamma_1(s)} + \frac{1}{\Gamma_2(s)}} = \frac{\Gamma_1 \Gamma_2}{\Gamma_1 + \Gamma_2}$$

- Physical Reasoning:

- Common Velocity* \Rightarrow *Series* connection
- Summing Forces* \Rightarrow *Series* connection

Parallel Combination of One-Ports



- Parallel Admittances *Sum*

$$\Gamma(s) = \Gamma_1(s) + \Gamma_2(s)$$

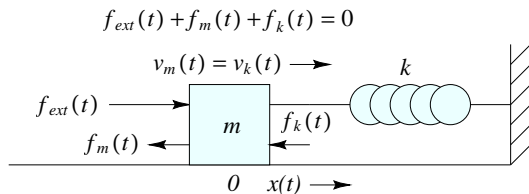
- Impedance:

$$R(s) = \frac{1}{\frac{1}{R_1(s)} + \frac{1}{R_2(s)}} = \frac{R_1 R_2}{R_1 + R_2}$$

or, for EEs, $R = R_1 \parallel R_2$

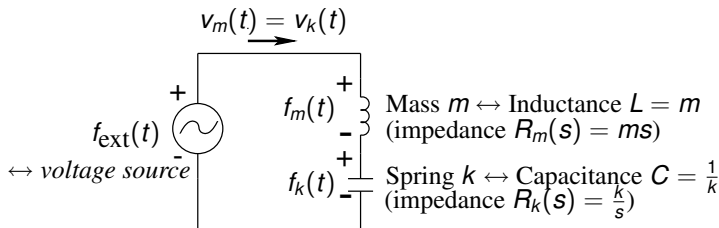
- Physical Reasoning:
 - *Common Force* \Rightarrow *Parallel* connection
 - *Summing Velocities* \Rightarrow *Parallel* connection

Mass-Spring-Wall (Series)

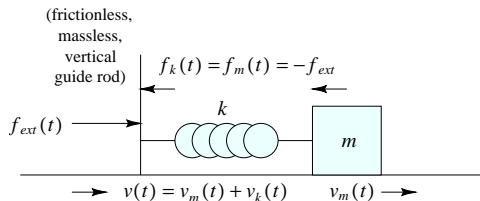


Physical Diagram:

Electrical Equivalent Circuit:



Spring-Mass (Parallel)



Physical Diagram:

Electrical Equivalent Circuit:

