Appendix F

Proof of EDM composition

F.1 EDM-entry exponential

(§4.10)

$$D \in \mathbb{EDM}^n \;\Leftrightarrow\; [1 - e^{-\lambda d_{ij}}] \in \mathbb{EDM}^n \quad \forall \lambda > 0 \tag{656}$$

Lemma 2.1. from *A Tour d'Horizon ... on Completion Problems.* [154] The following assertions are equivalent: for $D = [d_{ij}\,,\; i\,,j = 1 \ldots n] \in \mathbb{S}_h^n$ and $\mathcal{E}^n$ the elliptope in $\mathbb{S}^n$ (§4.9.1.0.1),

(i) $D \in \mathbb{EDM}^n$

(ii) $e^{-\lambda D} \triangleq [e^{-\lambda d_{ij}}] \in \mathcal{E}^n$ for all $\lambda > 0$

(iii) $\mathbf{1}\mathbf{1}^T - e^{-\lambda D} \triangleq [1 - e^{-\lambda d_{ij}}] \in \mathbb{EDM}^n$ for all $\lambda > 0$ $\diamond$

Citation: Jon Dattorro, *Convex Optimization & Euclidean Distance Geometry,* Meboo Publishing USA, 2005.

Proof. [211] (*confer* [149])

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Date: Fri, 06 Jun 2003 10:42:47 +0200
From: Monique Laurent <M.Laurent@cwi.nl>
To: Jon Dattorro <dattorro@Stanford.EDU>
Subject: Re: Tour

Hallo Jon,

I looked again at the paper of Schoenberg and what I can see is
the following:

1) the equivalence of Lemma 2.1 (i) (ii) (my paper) is stated in
Schoenberg's Theorem 1 (page 527).

2) (ii) implies (iii) can be seen from the statement in the beginning
of section 3, saying that a distance space embeds in $L_2$ iff
some associated matrix is PSD. Let me reformulate it:

  Let $d=(d_{ij})_{i,j=0,1,..,n}$ be a distance space on $n+1$ points
  (i.e., symmetric matrix of order $n+1$ with zero diagonal)
  and let $p=(p_{ij})_{i,j=1,..,n}$ be the symmetric matrix of order
  $n$ related by relations:

      (A) $p_{ij} = {d_{0i}+d_{0j}-d_{ij}\2}$ for $i,j=1,..n$

      or equivalently

      (B) $d_{0i} = p_{ii},  d_{ij} = p_{ii}+p_{jj}-2p_{ij}$
                                          for $i,j=1,..,n$

   Then, $d$ embeds in $L_2$ iff $p$ is positive semidefinite matrix
   iff $d$ is of negative type
   (see second half of page 525 and top of page 526)
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For the implication from (ii) to (iii), set:
    $p = exp(-\lambda d)$ and define $d'$ from $p$ using (B) above.
    Then, $d'$ is a distance space on $n+1$ points that embeds
    in $L_2$. Thus its subspace of $n$ points also embeds in $L_2$
    and is precisely $1-exp(-\lambda d)$.

Note that (iii) implies (ii) cannot be read immediately from this
argument since (iii) involves the subdistance of $d'$ on $n$ points
(and not the full $d'$ on $n+1$ points).

3) Show (iii) implies (i) by using the series expansion of the function
$1 - exp(-lambda d)$; the constant term cancels; $\lambda$ factors out;
remains a summation of $d$ plus a multiple of $\lambda$; letting
$\lambda$ go to 0 gives the result.

As far as I can see this is not explicitly written in Schoenberg.
But Schoenberg also uses such an argument of expansion of the
exponential function plus letting $\lambda$ go to 0
(see first proof in page 526).

I hope this helps. If not just ask again.
Best regards, Monique

> Hi Monique
>
> I'm reading your "A Tour d'Horizon..." from the AMS book "Topics in
> Semidefinite and Interior-Point Methods".
>
> On page 56, Lemma 2.1(iii),  1-exp(-\lambda D) is EDM  <=>  D is EDM.
> You cite Schoenberg 1938; a paper I have acquired.  I am missing the
> connection of your Lemma(iii) to that paper; most likely because of my
> lack of understanding of Schoenberg's results.  I am wondering if you
> might provide a hint how you arrived at that result in terms of
> Schoenberg's results.
>
> Jon Dattorro
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