Prelude

> *The constant demands of my department and university and the ever increasing work needed to obtain funding have stolen much of my precious thinking time, and I sometimes yearn for the halcyon days of Bell Labs.*
>
> −Steven Chu, Nobel laureate [90]

Convex Analysis is an emerging calculus of inequalities while Convex Optimization is its application. Analysis is inherently the domain of a mathematician while Optimization belongs to the engineer. A convex optimization problem is conventionally regarded as minimization of a convex objective function subject to an artificial convex domain imposed upon it by the problem constraints. The constraints comprise equalities and inequalities of convex functions whose simultaneous solution set generally constitutes the imposed convex domain: called *feasible set.*

It is easy to minimize a convex function over any convex subset of its domain because any local minimum must be a global minimum. But it is difficult to find the maximum of a convex function over some convex domain because there can be many local maxima; although this has practical application (Eternity II §4.7.0.0.15, §C.5), it is not a convex problem. Tremendous benefit accrues when a mathematical problem can be transformed to an equivalent convex optimization, primarily because any locally optimal solution is then guaranteed globally optimal.[0.1] An *optimal* solution is a best solution to the problem posed; a certificate can be obtained guaranteeing that no better solution exists.

To provide a concrete example of what it meant by *optimal,* recall the ordinary *least squares* problem espoused by Gauss and Legendre over 200 years ago: (§E.0.1.0.1)

$$\underset{x}{\text{minimize}}\ \ \|Ax-b\|_2^2$$

Suppose we were to pose this problem a bit differently by *constraining* variable vector x simultaneously with the minimization. In particular, let's suppose that each entry of x were bounded above by the same maximum allowable value:

$$\begin{array}{cl}\underset{x}{\text{minimize}} & \|Ax-b\|_2^2\\ \text{subject to} & x\preceq x_{\max}\end{array}$$

Would a constrained solution, so obtained, be equivalent to an ordinary least squares solution whose entries (exceeding the prescribed bound) are simply clipped to the maximum value? The two solutions are, generally, different when clipping occurs. We argue that a constrained solution is better than a clipped solution; indeed, it is optimal.

[0.1] Solving a nonlinear system for example, by instead solving an equivalent convex optimization problem, is therefore highly preferable and what motivates *geometric programming*; a form of convex optimization invented in 1960s [65] [88] that has driven great advances in the electronic circuit design industry. [36, §4.7] [283] [453] [456] [114] [211] [220] [221] [222] [223] [224] [300] [301] [351]

Both of the foregoing ordinary and bounded least squares problems are convex. Recognizing a problem as convex is an acquired skill; that being, to know when an objective function is convex and when constraints specify a convex feasible set. The challenge, which is indeed an art, is how to express difficult problems in a convex way: perhaps, problems previously believed nonconvex. Practitioners in the art of Convex Optimization engage themselves with discovery of which hard problems can be transformed into convex equivalents; because, once convex form of a problem is found, then a globally optimal solution is close at hand - the hard work is finished: Finding convex expression of a problem is itself, in a very real sense, its solution.

Yet, that skill acquired by understanding the geometry and application of Convex Optimization will remain more an art for some time to come; the reason being, there is generally no unique transformation of a given problem to its convex equivalent. This means, two researchers pondering the same problem are likely to formulate a convex equivalent differently; hence, one solution is likely different from the other although any convex combination of those two solutions remains optimal. Any presumption of only one right or correct solution becomes nebulous. Study of equivalence & sameness, uniqueness, and duality therefore pervade study of Optimization.

It can be difficult for the engineer to apply convex theory without an understanding of Analysis. These pages comprise my journal over a ten year period bridging gaps between engineer and mathematician; they constitute a translation, unification, and cohering of about four hundred papers, books, and reports from several different fields of mathematics and engineering. Although beacons of historical accomplishment are cited throughout, much of what is written here will not be found elsewhere. Care to detail, clarity, accuracy, consistency, and typography accompanies removal of ambiguity and verbosity out of respect for the reader. But the book is nonlinear in its presentation. Consequently there is much indexing, cross referencing, linkage to online sources, and background material provided in the text, footnotes, and appendices so as to be more self-contained and to provide understanding of fundamental concepts.

Looking toward the future, there remains much to be done in the area of machine computation if mathematical Optimization is to become fully embraced by the signal processing community. Wordlength of contemporary computers and numerical burdens upon them prohibit real time solution and accuracy sufficient to embed optimization problems within a recursive mathematical setting. When optimization problems constitute only intermediate solution to much larger problems, acquiring only a "few digits" accuracy can throw off subsequent dependent calculations. *Barrier* methods of solution are the principal obstacle to accuracy while *simplex* methods are the principal setback to speed. Novel, not hybrid, methods of solution are needed.

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