Nullspace Method of Spectral Analysis

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0 Introduction

We are interested in the spectral analysis of discrete-time signals to a very high degree of accuracy.

The intended tone is tutorial.

The reader should at least be comfortable with linear algebra and vector spaces as taught in Strang [1], and discrete-time signals and systems as in Oppenheim & Willsky [2].

1 The Vast NullSpace

It is a curious fact in linear algebra [1,ch.2.2] that there is no unique solution x to the platonic problem,

$$Ax = b \qquad ; A \in \mathrm{R}^{m \times n}, \qquad m < n \tag{1}$$

given a *full rank*[1] fat matrix A and a vector b. Because of the assumption of full rank, the vector b has consequently no choice but to belong to the range of A. Hence, any chosen solution x must be exact. Because $m < n$, there exists an infinite continuity of solutions $\{x\}$ for which the problem is solved exactly.

1.1 Calculating $\{x\}$

The solution set is determined as follows: Suppose we know the *nullspace* of A. The nullspace $\mathcal{N}(A)$ is defined as the set of all vectors x such that $Ax = 0$; i.e.,

$$\mathcal{N}(A) = \{ x \mid Ax = 0 \} \in \mathrm{R}^n \tag{2}$$

Because $\mathcal{N}(A)$ is a subspace of the real vector space R^n, $\mathcal{N}(A)$ possesses a basis just like any other vector space. [7,ch.2.1] Letting any particular *orthonormal basis*[2] for the nullspace constitute the columns of a matrix N, we can easily express the nullspace of A in terms of the range of N;

$$\mathcal{N}(A) = \mathcal{R}(\mathrm{N}) \tag{3}$$

If we denote any *arbitrary solution* to Eq.(1) as x_o, then we can express any one of the infinity of solutions $\{x\}$ as

$$x \triangleq x_o + \mathrm{N}\,\xi \qquad ; \mathrm{N} \in \mathrm{R}^{n \times (n-m)} \tag{4}$$

Because $A\,\mathrm{N} = 0$, then it follows for any vector ξ that $Ax = b$ for those x described by Eq.(4).

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[1] The *rank* r counts the number of independent columns in A. In general, $r \le \min(m,n)$. When A is *full rank*, $r = \min(m,n)$. r is the *dimension* of the range of the column space; $r = \dim \mathcal{R}(A)$.

[2] A basis is not unique, in general. Whatever basis is selected, we only demand that it be orthonormal.

1.2 The Shortest Solution

The particular solution to Eq.(1) which is traditionally recommended as optimal, is that one having shortest length. [4,ch.5.6] Thus, a sense of uniqueness is instilled into the problem by ascribing a notion of goodness to the attribute of Euclidean length. [3,ch.5.5.1]

When the matrix A is fat, the inverse of A does not exist by definition, for if it did, then the resultant solution $x = A^{-1} b$ would be unique. Hence the concept of the *pseudoinverse*, invented to satisfy a need for uniqueness. With a slight change of notation, $x_r = A^{+} b$ is deemed to produce the shortest solution [1,App.A]; which is itself unique among all solutions. If we set x_o in Eq.(4) to this particular solution x_r, then it stands to reason that any other solution x should possess a nullspace component;

$$x \triangleq x_r + x_n \qquad ; x_n \triangleq \mathrm{N}\ \xi \qquad (4a)$$

This conjecture is proven by the following principle:

Nullspace Projection Principle
Given $\mathcal{N}(A) = \mathcal{R}(\mathrm{N})$ from Eq.(3) and any solution x that satisfies $A\,x = b$, the vector $x_r = x - \mathrm{N}\,\mathrm{N}^+ x$ where[3] $\mathrm{N}^+ = \mathrm{N}^\mathrm{T}$, $\mathrm{N}^+\mathrm{N} = \mathrm{I}$, and $A\,\mathrm{N} = 0$, is the shortest solution that satisfies $A\,x = b$ and lies completely in the rowspace of A.

Proof: Because the rowspace is the orthogonal complement[4] of the nullspace, any particular x from R^n may be uniquely decomposed into a sum of orthogonal components,

$$x = x_r + x_n = A^\mathrm{T}\rho + \mathrm{N}\,\eta \qquad (5)$$

where A is assumed full rank and fat, $\eta = \mathrm{N}^+ x$, and $\rho = (A\,A^\mathrm{T})^{-1}A\,x$. ρ is found by equating $A^\mathrm{T}\rho$ to $x - \mathrm{N}\,\mathrm{N}^+ x$. The rowspace component x_r must be orthogonal to every vector in the nullspace;

$$(\mathrm{N}\,\xi)^\mathrm{T} x_r = \xi^\mathrm{T}\mathrm{N}^\mathrm{T} A^\mathrm{T}\rho = \xi^\mathrm{T}\mathrm{N}^\mathrm{T}\left(x - \mathrm{N}\,\mathrm{N}^\mathrm{T} x\right) = 0 \qquad ;\forall\,\xi \qquad (6)$$

which shows that $x - \mathrm{N}\,\mathrm{N}^+ x \in \mathcal{R}(A^\mathrm{T})$.

$$x_r = A^\mathrm{T}\rho = A^\mathrm{T}(A\,A^\mathrm{T})^{-1}A\,x \triangleq A^+ A\,x = A^+ b \qquad (7)$$

is recognized as the orthogonal projection of $\{x \mid A\,x = b\}$ onto the rowspace of A. $x_n = \mathrm{N}\,\mathrm{N}^+ x$ is the orthogonal projection of $\{x \mid A\,x = b\}$ onto the nullspace of A. Hence by Eq.(5), $x - \mathrm{N}\,\mathrm{N}^+ x$ yields x_r; the same x_r as in Eq.(7). [7,ch.3.3] Likewise, $x - A^+ A\,x$ yields x_n. [4,ch.6.9.2] Since the component projections of x are orthogonal, it follows that

$$\|x\|^2 = \|x_r\|^2 + \|x_n\|^2 \qquad (8)$$

Because $A\,x_n = 0$, the shortest solution is $x = x_r$. ◊

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[3]The $^+$ exponent denotes the pseudoinverse.

[4]*Orthogonal complement:* $\mathcal{R}(A^\mathrm{T}) \perp \mathcal{N}(A)$ and $\mathcal{R}(A^\mathrm{T}) \oplus \mathcal{N}(A) = \mathrm{R}^n$

In other words, given any solution x to $Ax = b$, the solution $x_r = x - \mathrm{N}\,\mathrm{N}^+ x$ removes the nullspace component from x. It follows from this and Eq.(7) that the particular solution obtained via the pseudoinverse $x_r = A^+ b$ lies completely in the rowspace of A. [1,App.A]

underdetermined solution derivation

There are problems for which the shortest solution x is inappropriate or undesirable, but there are not many alternative criteria offered in the literature for which length does not play a role. Consider, for example, the Pareto optimization problem. [%%%ref]

%%%
||Ax-b|| + w||x||
cite Pareto solution, note Ax!=b
note lim w-> 0 = pseudoinverse

One cannot help but wonder whether there are other desirable attributes besides length, to divine alternative meaningful solutions from Eq.(1).

2 A Simple Novel Criterion for the Solution of $\boldsymbol{A\,x = b}$

Let's begin with a rudimentary example that we will later understand as an antecedent of the more difficult spectral analysis problem.

Example 1.

Consider the problem $A\,x = b$ and the numerically simple matrices

$$A = \begin{pmatrix} 1 & 1 & 8 & 1 & 1 \\ 3 & 2 & 8 & \frac{1}{2} & \frac{1}{3} \\ 9 & 4 & 8 & \frac{1}{4} & \frac{1}{9} \end{pmatrix}, \qquad b = \begin{pmatrix} 1 \\ \frac{1}{2} \\ \frac{1}{4} \end{pmatrix} \tag{9}$$

The number of rows selected for A here is arbitrary, but the number of columns chosen for this example is five so as to yield a two-dimensional nullspace.[5] The *obvious solution* given this particular A and b is

$$x = \mathbf{1}_4 \qquad ; \; \mathbf{1}_4 \triangleq \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} \tag{10}$$

The obvious solution is indeed the desired solution in our spectral analysis which asks the question: "Which of these *exponentials*[6], constituting the columns of A, most closely resembles the *signal* b?"

An arbitrary solution determined by MATLAB [5] is,[7]

$$x_o = \begin{pmatrix} \frac{-2}{128} \\ 0 \\ \frac{5}{128} \\ 0 \\ \frac{90}{128} \end{pmatrix} \tag{11}$$

Note that the 1 and 2-norm of this arbitrary x_o are each less than 1. The same is true of the shortest solution Eq.(7). These observations are noteworthy because the obvious solution for this example Eq.(10) can not be obtained by

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[5] $\dim \mathcal{N}(A) = n-m$ when A is full rank [1]; cf. Eq.(4).

[6] An exponential is a signal of the general form $a\,z^n$; $a, z \in \mathbb{C}$ (complex), $n \in \mathbb{Z}$ (integer). $\mathbb{R}$ denotes the real numbers.

[7] This solution is produced by the MATLAB statement `xo=A\b` .

cite Karhunen = principal component analysis
where x = U U′ x U from SVD

%%%%%%%%traditional methods of norm minimization. It is fascinating that none of the traditional methods of solution can point to the fourth column of A as the answer, when that answer is so easily obtainable by inspection. We have no choice but to invent some algorithm to find the obvious solution.

Taking the first step, we form N from any orthonormal basis for the nullspace of A; say,

$$\mathsf{N} = \begin{pmatrix} -0.267248 & 0 \\ 0.864635 & -0.0413757 \\ -0.142473 & 0.0387897 \\ 0.188782 & -0.827514 \\ 0.353619 & 0.558572 \end{pmatrix} \triangleq \begin{pmatrix} \mathsf{n}_1^T \\ \mathsf{n}_2^T \\ \mathsf{n}_3^T \\ \mathsf{n}_4^T \\ \mathsf{n}_5^T \end{pmatrix} \tag{12}$$

Then we write the equation for all possible solutions Eq.(4), but we set it to zero;

$$x = x_o + \mathsf{N}\,\xi = 0 \tag{13}$$

(We will explain the reason for doing this in a later section. For now we just go through the steps of the algorithm for solution.)

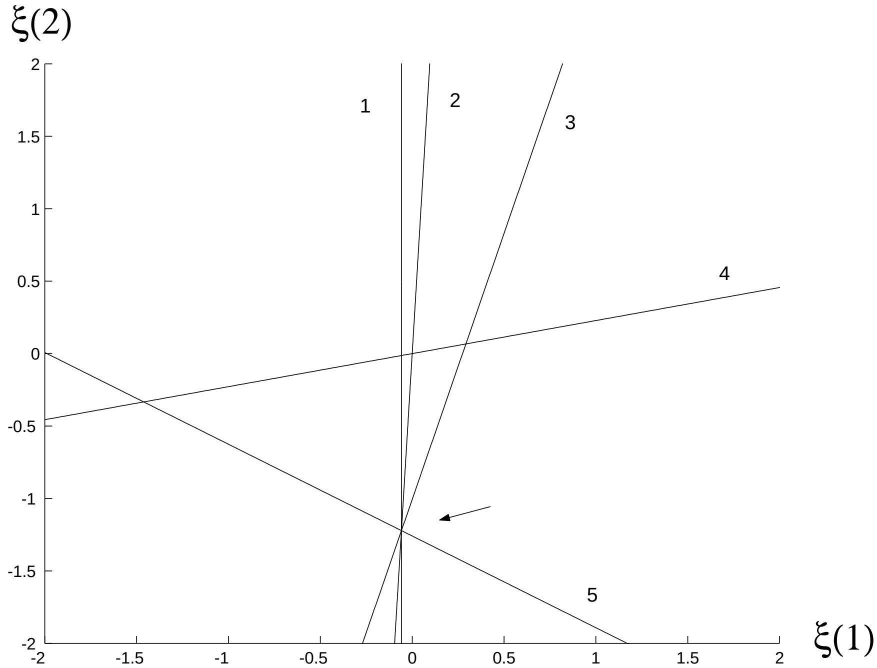

Figure 1. Hyperplanes $\mathsf{N}\,\xi = -x_o$. The corresponding rows of N and x_o are labelled. The arrow points toward the obvious solution.

2.1 Derivation of the Nullspectrum

3 From the Platonic to the Quantitative

The prototypic problem Eq.(1) [1,ch.3.6] finds wide application in engineering and the sciences; indeed, volumes have been authored in its regard. [8,ch.3.6] The study of spectral analysis forces a move away from the Platonic because, as we saw, there are no simple closed form solutions for simple spectral problems; spectral analysis demands algorithms.

%%% cite Murray example 1, successive match to columns of A algo.

3.1 Brief History of Spectral Analysis

The serious development of spectral analysis is indeed brief beginning in the twentieth century. Fourier invented his famous formulae in ???1917???, while more modern methods are typified in the nondeterministic approach of Kay [9].

Appendix I: The Half-Space and Hyperplane

Rudiments of affine geometry. [6]

Notes

maximum number of nonzero elements of x sufficient for Eq.(1) to be satisfied is m.

Solution in Example 1 not changed by scaling.

Boyd does not believe in Fourier analysis. In his view, there are only signals.

References

[1] Gilbert Strang, *Linear Algebra and its Applications*, third edition, Harcourt Brace Jovanovich, 1988

[2] Oppenheim, Willsky

[3] Gene H. Golub, Charles F. Van Loan, *Matrix Computations*, third edition, Johns Hopkins University Press, 1996

[4] Philip E. Gill, Walter Murray, Margaret H. Wright, *Numerical Linear Algebra and Optimization*, Volume 1, Addison Wesley, 1991

[5] MATLAB, version 5.3, 1999

[6] Boyd, Vandenberghe

[7] Erwin Kreyszig, *Introductory Functional Analysis with Applications*, Wiley, 1989

[8] S. Lawrence Marple, Jr., *Digital Spectral Analysis with Applications*, Prentice–Hall, 1987

[9] Kay, *Modern Spectral Estimation*, Prentice