Appendix F

Notation, Definitions, Glossary

\( b \) scalar or column vector (italic \( abcd..xyz \))

\( b_i \) \text{\textit{i}th entry of vector } b = [b_i , i=1..n] \text{ or } \textit{i}th \text{ vector from a list } \{b_j , j=1..n\}

or \textit{i}th iterate of vector \( b \)

\( b_{i:j} \) or \( b(i:j) \): truncated vector comprising i\text{th} through j\text{th} entry of vector \( b \) (294)

\( b_{k(i:j)} \) or \( b_{i:j,k} \): truncated vector comprising \( i\text{th} \) through \( j\text{th} \) entry of vector \( b_k \)

\( b^T \) vector transpose or row vector

\( b^H \) complex conjugate transpose \( b^*^T \)

\( A \) matrix (italic \( ABC..Z \))

\( A^T \) Matrix transpose \([A_{ij}] \rightarrow [A_{ji}]\) is a linear operator. Regarding \( A \) as a linear operator, \( A^T \) is its adjoint.

\( A^{-T} \) matrix transpose of inverse; and vice versa, \( (A^{-1})^T = (A^T)^{-1} \) (confer p.489 no.47)

\( A^{T1} \) first of various transpositions of a cubix or quartix \( A \) (p.555, p.559)

\( A^{-1} \) inverse of matrix \( A \)

\( A^\dagger \) Moore-Penrose pseudoinverse of matrix \( A \) (§E)

\( A_{ij} \) or \( A(i,j) \): \textit{ij}th entry of matrix \( A = \begin{bmatrix} 1 & 2 & 3 \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \end{bmatrix} \) or rank-1 matrix \( a_ia_j^T \) (§4.10)

\( A(i,j) \): \( A \) is a function of \( i \) and \( j \)

\( A_i \) \text{\textit{i}th matrix from a set} \text{ or } \textit{i}th principal submatrix (1329) or \textit{i}th iterate of \( A \)

\( A(i,:) \) or \( A_i : \) \textit{i}th row of matrix \( A \)

\( A(:,j) \) or \( A_j : \) \textit{j}th column of matrix \( A \) [185, §1.1.8]

\( A(i:j,k:\ell) \) or \( A_{i:j,k:\ell} : \) submatrix; \textit{i}th through \textit{j}th row and \textit{k}th through \textit{\ell}th column of \( A \)

\( \sqrt{\cdot} \) positive square root

\( \sqrt[\cdot]{\cdot} \) entrywise positive square root of vector \( x \)

APPENDIX F. NOTATION, DEFINITIONS, GLOSSARY

\( \sqrt{\cdot} \) positive \( \ell \)\textsuperscript{th} root

\( A^{1/2} \) and \( \sqrt{A} \) \( A^{1/2} \) is any matrix such that \( A^{1/2}A^{1/2} = A \).
For \( A \in S^n_+ \), \( \sqrt{A} \in S^n_+ \) is unique and \( \sqrt{A}A = A \). [59, §1.2] (§A.5.1.4)

\( \sqrt{D} = [\sqrt{d_{ij}}] \) absolute distance matrix (1517) or
Hadamard positive square root: \( D = \sqrt{D} \circ \sqrt{D} \)

\textbf{thin} \ a skinny matrix; meaning, more rows than columns:

\[
\begin{bmatrix}
& \\
& 
\end{bmatrix}
\]

\textit{When there are more equations than unknowns, we say that the system \( Ax = b \) is overdetermined.} [185, §5.3]

\textbf{wide} \ a fat matrix; meaning, more columns than rows:

\textit{underdetermined}

\textbf{hollow} \ matrix having \( 0 \) main diagonal

\( A \) some set (calligraphic \( ABCDEFGHIJKLMNOPQRSTUVWXYZ \))

\( \mathcal{A} \) set of vectors or matrices (blackboard \( ABCDEFGHIJKLMNOPQRSTUVWXYZ \))

\( \mathcal{F} \) discrete Fourier transform (932)
(Euler Fraktur \( ABCDEFGHIJKLMNOPQRSTUVWXYZ \))

\( \mathcal{F}(C \supset A) \) smallest face (176) that contains element \( A \) of set \( C \)

\( \mathcal{G}(\mathcal{K}) \) generators (§2.13.4.2.1) of set \( \mathcal{K} \);
any collection of points and directions whose hull constructs \( \mathcal{K} \)

\( \mathcal{L}^\nu \) level set (571)

\( \mathcal{L}_\nu \) sublevel set (575)

\( \mathcal{L}^\nu \) superlevel set (669)

\( \mathcal{L} \) Lagrangian (516)

\( \mathcal{E} \) member of elliptope \( \mathcal{E}_t \) (1258) parametrized by scalar \( t \)

\( \mathcal{E} \) ellipope (1237)

\( E \) elementary matrix

\( E_{ij} \) member of standard orthonormal basis for symmetric (62) or symmetric hollow (78) matrices

\textit{id est} \ from the Latin meaning \textit{that is}

\textit{e.g} \ \textit{exempli gratia}, from the Latin meaning \textit{for sake of example}

\textit{sic} \ from the Latin meaning \textit{so or thus or in this manner}; something meant as written

\textit{videlicet} \ from the Latin meaning \textit{it is permitted to see}

\textit{ibidem} \ from the Latin meaning \textit{in the same place}

\textit{no.} \ \textit{number}, from the Latin \textit{numero}

\textit{vs.} \ \textit{versus}, from the Latin
a.i. affinely independent (§2.4.2.3)
c.i. conically independent (§2.10)
l.i. linearly independent
w.r.t with respect to
a.k.a also known as
  re real part
  im imaginary part
i or j \sqrt{-1}
\subseteq \supseteq subset, superset
\subset \supset proper subset, proper superset
\cap \cup intersection, union
\in membership, element belongs to, or element is a member of
\exists membership, contains as in \mathcal{C} \ni y (\mathcal{C} contains element y)
\exists such that
\exists there that
:\therefore therefore
\forall for all, or over all
& (ampersand) and
& (ampersand italic) and
\propto proportional to
\infty infinity
\equiv equivalent to
\triangleq defined equal to, equal by definition
\approx approximately equal to
\simeq isomorphic to or with
\cong congruent to or with

  Hadamard quotient as in, for \( x, y \in \mathbb{R}^n \), \( \frac{x}{y} \triangleq [x_i/y_i, \ i=1\ldots n] \in \mathbb{R}^n \)
  Hadamard product of matrices: \( x \circ y \triangleq [x_i y_i, \ i=1\ldots n] \in \mathbb{R}^n \) (§D.1.2.2, §A.1.1)
  Kronecker product of matrices (§D.1.2.1, §A.1.1)

\oplus vector sum of sets \( \mathcal{X} = \mathcal{Y} \oplus \mathcal{Z} \) where every element \( x \in \mathcal{X} \) has unique expression \( x = y + z \) where \( y \in \mathcal{Y} \) and \( z \in \mathcal{Z} \); [349, p.19] then summands are algebraic complements. \( \mathcal{X} = \mathcal{Y} \oplus \mathcal{Z} \Rightarrow \mathcal{X} = \mathcal{Y} + \mathcal{Z} \). Now assume \( \mathcal{Y} \) and \( \mathcal{Z} \) are nontrivial subspaces. \( \mathcal{X} = \mathcal{Y} + \mathcal{Z} \Rightarrow \mathcal{X} = \mathcal{Y} \oplus \mathcal{Z} \Leftrightarrow \mathcal{Y} \cap \mathcal{Z} = 0 \) [350, §1.2] [123, §5.8]. Each element from a vector sum (+) of subspaces has unique expression (⊕) when a basis from each subspace is linearly independent of bases from all the other subspaces.
likewise, unique vector difference of sets

orthogonal vector sum of sets $X = \mathcal{Y} \oplus \mathcal{Z}$ where every element $x \in X$ has unique orthogonal expression $x = y + z$ where $y \in \mathcal{Y}$, $z \in \mathcal{Z}$, and $y \perp z$. \[372, \text{p.51}\]

If $\mathcal{Z} \subseteq \mathcal{Y}$ then $X = \mathcal{Y} \oplus \mathcal{Z} \iff X = \mathcal{Y} \oplus \mathcal{Z}$. \[123, \S 5.8\]

If $\mathcal{Z} = \mathcal{Y}$ then summands are orthogonal complements.

plus or minus or plus and minus

as in $\mathcal{A}$ means logical not $\mathcal{A}$, or relative complement of set $\mathcal{A}$; id est, $\mathcal{A} = \{x \notin \mathcal{A}\}$; e.g, $B \setminus \mathcal{A} \triangleq \{x \in B \mid x \notin \mathcal{A}\} \equiv B \cap \mathcal{A}$

and $\iff$ sufficient and necessary, implies and is implied by; e.g,

$A$ is sufficient: $A \Rightarrow B$, $A \Rightarrow B \iff \mathcal{A} \not\subseteq \mathcal{B}$, $A \not\subseteq \mathcal{B}$

if $A$ then $B$, $A \not\subseteq \mathcal{B}$

A only if $B$, $A \not\subseteq \mathcal{B}$

if and only if (iff) or corresponds with or necessary and sufficient or logical equivalence

is plural are, as in $A$ is $B$ means $A \Rightarrow B$; conventional usage of English language imposed by logicians

and $\iff$ insufficient and unnecessary, does not imply and is not implied by; e.g,

$A$ is insufficient: $A \not\Rightarrow B$, $A \not\Rightarrow B \iff \mathcal{A} \not\subseteq \mathcal{B}$, $A \not\subseteq \mathcal{B}$

if $B$ then $A$, $A \not\subseteq \mathcal{B}$

$B$ only if $A$.

is replaced with; substitution, assignment

goes to, or approaches, or maps to

t $\to 0^+$ $t$ goes to $0$ from above; meaning, from the positive \[230, \text{p.2}\]

\[\cdot \ldots \cdot 1\] as in $1 \ldots 1$ means ones in a row or

$[s_1 \ldots s_N]$ means continuation; a matrix whose columns are $s_i$ for $i = 1 \ldots N$

or as in $n(n-1)(n-2)\ldots 1$ means continuation of a product

\[\ldots i = 1 \ldots N\] as in $i = 1 \ldots N$ meaning, $i$ is a sequence of successive integers beginning with 1 and ending with $N$; id est, $1 \ldots N = 1:N$

$[\ldots i : j = i \ldots j]$ (if $j < i$ then sequence is descending)

$f$ real function or multidimensional function a.k.a operator

$f : \mathcal{M} \to \mathcal{R}$ meaning $f$ is a mapping from ambient space $\mathcal{M}$ to ambient $\mathcal{R}$, not necessarily denoting either domain or range

$\mid$ as in $f(x) \mid x \in \mathcal{C}$ means with the condition(s) or such that or evaluated for, or as in $\{f(x) \mid x \in \mathcal{C}\}$ means evaluated for each and every $x$ belonging to set $\mathcal{C}$

$g \mid_{x_p}$ expression $g$ evaluated at $x_p$

$\| \parallel$ parallel

$A, B$ as in, for example, $A, B \in \mathbb{S}^N$ means $A \in \mathbb{S}^N$ and $B \in \mathbb{S}^N$
(a, b) open interval between a and b in \( \mathbb{R} \) or variable pair perhaps of disparate dimension

\([a, b]\) closed interval or line segment between a and b in \( \mathbb{R} \)

\(\binom{n}{k}\) binomial coefficient on \( \mathbb{Z} \)

\(\binom{n}{k} \triangleq \begin{cases} 
1, & k = 0 \\
\frac{-k(-n+k-1)!}{k!(-n-1)!}, & k > 0 \\
-1^{n-k}\frac{(-n)!}{(n-k)!(-n-k)!}, & k \leq n \\
0, & n < k < 0 \\
0, & k < 0 \text{ or } k > n \\
\frac{n!}{k!(n-k)!}, & \text{otherwise} 
\end{cases} \)

\(\delta(A)\) (a.k.a diag(A) , §A.1) vector made from main diagonal of A if A is a matrix; otherwise, diagonal matrix made from vector A

\(\delta^2(A) \equiv \delta(\delta(A))\). For vector or diagonal matrix \(\Lambda\), \(\delta^2(\Lambda) = \Lambda\)

\(\delta(A)^2 = \delta(\delta(A))\) where A is a vector
\( \lambda_i(X) \) \( i^{th} \) entry of vector \( \lambda \) is function of \( X \)

\( \lambda(X) \) \( i^{th} \) entry of vector-valued function of \( X \)

\( \lambda(A) \) vector of eigenvalues of matrix \( A \) (§A.5) (a nonlinear vector-valued function)

\( \lambda(A) \) spectral cone \( K_\lambda \) for matrix set \( A \) (§5.11.2.3)

\( \sigma(A) \) vector of singular values of matrix \( A \) (§A.6) (a nonlinear vector-valued function always arranging in nonincreasing order), or support function in direction \( A \)

\( \Sigma \) diagonal matrix of singular values, not necessarily square

\( \sum \) sum. Empty sum equals, conventionally, 0 or 0

\( \pi(\gamma) \) nonlinear permutation operator (or presorting function) arranges vector \( \gamma \) into nonincreasing order (§7.1.3). \( \pi_i \) is a permutation matrix; e.g., (961).

\( \Xi \) permutation matrix

\( \Pi \) doublet or permutation operator or matrix or set of all permutation matrices

\( \Pi \) product. Empty product equals, conventionally, 1 or \( I \)

\( \psi(Z) \) signum-like step function that returns a scalar for matrix argument (767), it returns a vector for vector argument (1752)

\( D \) symmetric hollow matrix of distance-square or Euclidean distance matrix

\( D \) Euclidean distance matrix operator

\( D^T(X) \) adjoint operator

\( D(X)^T \) transpose of \( D(X) \)

\( D^{-1}(X) \) inverse operator

\( D(X)^{-1} \) inverse of \( D(X) \)

\( D^* \) optimal value of variable \( D \)

\( D^\ast \) dual to variable \( D \)

\( V \) geometric centering operator, \( V(D) = -VDV \frac{1}{2} \) (1157)

\( V_N \) \( V_N(D) = -V_N^T D V_N \) (1171)

\( V \) \( N \times N \) symmetric elementary, auxiliary, projector, geometric centering matrix, \( R(V) = N(1^T) \), \( N(V) = R(1) \), \( V^2 = V \) (§B.4.1)

\( V_N \) \( N \times N - 1 \) Schoenberg auxiliary matrix \( R(V_N) = N(1^T) \), \( N(V_N^T) = R(1) \) (§B.4.2)

\( V_X \) \( V_X V_X^T \equiv V^TX^TXV \) (1349)

\( X \) point list (§79) having cardinality \( N \) arranged columnar in \( \mathbb{R}^{n \times N} \), or set of generators, or extreme directions, or matrix variable

\( G \) Gram matrix \( X^TX \) (1058)

\( r \) affine dimension (1203)
\( \alpha_c \) geometric center (1134)
\( \rho \) rank of matrix or bound on affine dimension
\( k \) number of conically independent generators
\( k \) raw-data domain of Magnetic Resonance Imaging (MRI) machine, as in \( k \)-space
\( n \) Euclidean (ambient spatial) dimension of list \( X \in \mathbb{R}^{n \times N} \), or integer
\( \eta \) noise factor or noise signal or normal vector or \( \min\{m, n\} \)
\( N \) cardinality of list \( X \in \mathbb{R}^{n \times N} \), or integer
\( \text{epi} \) function epigraph
\( \text{dom} \) function domain
\( \mathcal{R}f \) function range
\( \mathcal{R}(A^T) \) the subspace: rowspace of \( A \) (145) or span basis \( \mathcal{R}(A^T) \); \( \mathcal{R}(A^T) \perp \mathcal{N}(A) \)
\( \mathcal{R}(A) \) the subspace: range of \( A \) (146) or span basis \( \mathcal{R}(A) \); \( \mathcal{R}(A) \perp \mathcal{N}(A^T) \)
\( \text{span} \) as in \( \text{span} A = \mathcal{R}(A) = \{Ax \mid x \in \mathbb{R}^n\} \) when \( A \) is a matrix
\( \mathcal{B}(A) \) overcomplete columnar basis for range of \( A \)
or minimal set constituting generators for vertex-description of \( \mathcal{R}(A) \)
or linearly independent set of vectors spanning \( \mathcal{R}(A) \)
\( \mathcal{N}(A) \) the subspace: nullspace of \( A \) (147) a.k.a kernel of \( A \); \( \mathcal{N}(A) \perp \mathcal{R}(A^T) \)
\( \mathbb{R}^n \) Euclidean \( n \)-dimensional real vector space (nonnegative integer \( n \))
a subspace, conventionally, but not a proper subspace. [259, §2.1]
\( \mathbb{R}^0 = \{0\} \). \( \mathbb{R} = \mathbb{R}^1 \) or vector space of unspecified dimension. [450]
\( \mathbb{R}^{m \times n} \) Euclidean vector space of \( m \) by \( n \) dimensional real matrices
\( \times \) Cartesian product. \( \mathbb{R}^{m \times n} \leq \mathbb{R}^{m \times n} \). \( \mathcal{K}_1 \times \mathcal{K}_2 = \begin{bmatrix} \mathcal{K}_1 \\ \mathcal{K}_2 \end{bmatrix} \)
\( \mathbb{Z} \) the real integers
\( \mathbb{N} \) the nonnegative natural numbers; \( \text{id est} \), \( \mathbb{Z}_+ \)
\( \mathbb{B}^n \), \( \mathbb{B}^{n \times n} \) \( \{0,1\}^n \) and \( \{0,1\}^{n \times n} \) binary vectors of respective dimension \( n \) and \( n \times n \)
\( \mathbb{B}_{\pm}^n \), \( \mathbb{B}_{\pm}^{n \times n} \) \( \{-1,1\}^n \) and \( \{-1,1\}^{n \times n} \) bipolar binary vectors of dimension \( n \) and \( n \times n \)
\( \mathbb{C}^n \), \( \mathbb{C}^{n \times n} \) Euclidean complex vector space of respective dimension \( n \) and \( n \times n \)
\( \mathbb{R}_{+}^n \), \( \mathbb{R}_{+}^{n \times n} \) nonnegative orthant in Euclidean vector space of respective dimension \( n \) and \( n \times n \)
\( \mathbb{R}_{-}^n \), \( \mathbb{R}_{-}^{n \times n} \) nonpositive orthant in Euclidean vector space of respective dimension \( n \) and \( n \times n \)
\( \mathbb{S}^n \) subspace of real symmetric \( n \times n \) matrices; the symmetric matrix subspace.
\( \mathbb{S}^0 = \{0\} \). \( \mathbb{S} = \mathbb{S}^1 \) or symmetric subspace of unspecified dimension.
\( \mathbb{S}^{n \perp} \) orthogonal complement of \( \mathbb{S}^n \) in \( \mathbb{R}^{n \times n} \), the antisymmetric matrices (54)
\( S^+ \) convex cone comprising all (real) symmetric positive semidefinite \( n \times n \) matrices, the *positive semidefinite cone*

\( \text{intr} S^+ \) interior of convex cone comprising all (real) symmetric positive semidefinite \( n \times n \) matrices; *id est*, positive definite matrices

\( S^n_+(\rho) = \{ X \in S^n_+ \mid \text{rank} X \geq \rho \} \) (269) convex set of all positive semidefinite \( n \times n \) symmetric matrices whose rank equals or exceeds \( \rho \)

\( EDM^N \) cone of \( N \times N \) Euclidean distance matrices in the symmetric hollow subspace

\( \sqrt{EDM} \) nonconvex cone of \( N \times N \) Euclidean absolute distance matrices in the symmetric hollow subspace (§6.3)

\( S^n_0 \) subspace comprising all symmetric \( n \times n \) matrices having all zeros in first row and column (2220) (§5.4.2.1)

\( S^n_h \) subspace comprising all symmetric hollow \( n \times n \) matrices (0 main diagonal), the *symmetric hollow subspace* (69)

\( S^n_{h\perp} \) orthogonal complement of \( S^n_h \) in \( S^n \), the set of all diagonal matrices (70)

\( S^n_c \) subspace comprising all geometrically centered symmetric \( n \times n \) matrices; *geometric center subspace* \( S^n_c = \{ Y \in S^n \mid Y1 = 0 \} \) (2216)

\( S^n_{c\perp} \) orthogonal complement of \( S^n_c \) in \( S^n \) (2218); *translation-invariant subspace*

\( R^{m \times n}_c \) subspace comprising all geometrically centered \( m \times n \) matrices (2215)

\( R^{n \times n}_h \) subspace of symmetric [sic] matrices having 0 main diagonal; *a.k.a.* real hollow subspace (66)

\( R^{n \times n}_{h\perp} \) subspace of antisymmetric antihollow matrices (67)

\( X^\perp \) basis \( N(X^T) \) (§2.13.10, §E.3.4)

\( x^\perp \) \( N(x^T) \); \( \{ y \in \mathbb{R}^n \mid x^T y = 0 \} \) (§2.13.11.1.1)

\( \perp \) as in \( A \perp B \) meaning \( A \) is orthogonal to \( B \) (and *vice versa*), where \( A \) and \( B \) are sets, vectors, or matrices. When \( A \) and \( B \) are vectors (35) (36) (or matrices (40) under Frobenius’ norm), \( A \perp B \iff \langle A, B \rangle = 0 \iff \| A + B \|_2 = \| A \|_2 + \| B \|_2 \)

\( \mathcal{R}(P)^\perp = \mathcal{N}(P^T) \); orthogonal complement of \( \mathcal{R}(P) \) (fundamental subspace relations (141))

\( \mathcal{N}(P)^\perp = \mathcal{R}(P^T) \)

\( \mathcal{R}^\perp = \{ y \in \mathbb{R}^n \mid \langle x, y \rangle = 0 \ \forall \ x \in \mathcal{R} \} \) (381).

*Orthogonal complement of \( \mathcal{R} \) in \( \mathbb{R}^n \) when \( \mathcal{R} \) is a subspace*

\( \mathcal{K}^\perp \) normal cone (458)

\( \mathcal{A}^\perp \) normal cone to affine subset \( \mathcal{A} \) (§3.1.1.2.2)

\( \mathcal{K} \) cone

\( \mathcal{K}^\circ \) dual cone \( -\mathcal{K}^\circ \)

\( \mathcal{K}^\circ \) polar cone \( -\mathcal{K}^* \)

\( D^\circ \) polar variable \( D \)
$360^\circ$ angular degree; e.g., $360^\circ \Leftrightarrow 2\pi$ radians

$K_{\mathcal{M}^+}$ monotone nonnegative cone

$K_{\mathcal{M}}$ monotone cone

$K_\lambda$ spectral cone

$K_{\lambda\delta}^*$ cone of majorization

$\mathcal{H}$ halfspace

$\mathcal{H}_-$ halfspace described using an outward-normal $^\perp$ to the hyperplane partially bounding it

$\mathcal{H}_+$ halfspace described using an inward-normal $^\perp$ to the hyperplane partially bounding it

$\partial \mathcal{H}$ hyperplane; id est, partial boundary of halfspace

$\partial \mathcal{H}^+$ supporting hyperplane

$\partial \mathcal{H}^-$ a supporting hyperplane having outward-normal with respect to set it supports

$\partial \mathcal{H}^+$ a supporting hyperplane having inward-normal with respect to set it supports

$\partial$ partial derivative or partial differential or matrix of distance-square squared (1557) or boundary of set $K$ as in $\partial K$ (18) (25)

$d$ derivative or differential

$\sqrt{d_{ij}}$ (absolute) distance scalar

$d_{ij}$ distance-square scalar, EDM entry

$d$ vector of distance-square

$d_{ij}$ lower bound on distance-square $d_{ij}$

$d_{ij}^2$ upper bound on distance-square $d_{ij}$

$\overline{AB}$ closed line segment between points A and B

$AB$ matrix multiplication of A and B

$\overline{C}$ closure of set $C$

$g'$ first derivative of possibly multidimensional function with respect to real argument

$g''$ second derivative with respect to real argument

$\partial Y$ first directional derivative of possibly multidimensional function $g$ in direction $Y \in \mathbb{R}^{K \times L}$ (maintains dimensions of $g$)

$\partial Y^2$ second directional derivative of $g$ in direction $Y$

$\nabla$ gradient from calculus, $\nabla f$ is shorthand for $\nabla_x f(x)$. $\nabla f(y)$ means $\nabla_y f(y)$ or gradient $\nabla_x f(y)$ of $f(x)$ with respect to $x$ evaluated at $y$

$\nabla^2$ second-order gradient
\( \Delta \) difference or discrete differential or distance scalar (Figure 28) or first-order difference matrix (945) or infinitesimal difference operator (§D.1.4)

\( \triangle_{ijk} \) triangle made by vertices \( i, j, \) and \( k \)

\( \hat{\nu} \) coefficient vector for two spectral factors, Figure 112 level 2

\( \bar{\nu} \) coefficient vector corresponding to four spectral factors, Figure 112 level 3

\( \overline{\nu} \) vector containing numerator or denominator coefficients of eight spectral factors; level 4 in a bifurcation tree like Figure 112

CPU central processing unit

dB decibel

3D three-dimensional or three dimensions

DC direct current (0 Hz)

DCT discrete cosine transform

DFT discrete Fourier transform \( \mathcal{F} \) (932)

Hz hertz (cycles per second), kHz means kilohertz, MHz megahertz, GHz gigahertz

\( F_s = 1/T \), sample rate in Hz

EDM Euclidean distance matrix

PSD positive semidefinite

SDP semidefinite program

SOCP second-order cone program

LP linear program

QUBO quadratic unconstrained binary optimization

PCA principal component analysis

SVD singular value decomposition

SNR signal to noise ratio

USA United States of America

*in* function \( f \) in \( x \) means \( x \) as argument to \( f \)
or \( x \) in \( \mathcal{C} \) means element \( x \) is a member of set \( \mathcal{C} \)

*on* function \( f(x) \) on \( \mathcal{A} \) means \( \mathcal{A} \) is \( \text{dom} f \)
or relation \( \preceq \) on \( \mathcal{A} \) means \( \mathcal{A} \) is set whose elements are subject to \( \preceq \)
or projection of \( x \) on \( \mathcal{A} \) means \( \mathcal{A} \) is body on which projection is made
or operating on vector identifies argument type to \( f \) as “vector”

*over* function \( f(x) \) over \( \mathcal{C} \) means \( f \) evaluated at each and every element of set \( \mathcal{C} \)

one-to-one injective map or unique correspondence between sets

onto function \( f(x) \) maps onto \( \mathcal{M} \) means \( f \) over its domain is a surjection w.r.t \( \mathcal{M} \)
injection \( f \) that is one-to-one

surjection \( f \) that is onto

bijection \( f \) that is one-to-one and onto

orthant generalization of two-dimensional quadrant \( \perp \) to higher dimension

orthogonality generalization of two-dimensional perpendicularity \( \perp \) to higher dimension

decomposition orthornomal (2117, p.582), biorthogonal (2092, p.577)

expansion orthogonal (2127, p.584), biorthogonal (413, p.149)

vector column vector in \( \mathbb{R}^n \); identifiable by Cartesian coordinates of point at its head

entry scalar element or real variable constituting a vector or matrix

cubix member of \( \mathbb{R}^{M \times N \times L} \)

quartix member of \( \mathbb{R}^{M \times N \times L \times K} \)

feasible as in feasible solution, means satisfies the (“subject to”) constraints of an optimization problem, may or may not be optimal

feasible set most simply, the set of all variable values satisfying all constraints of an optimization problem

active set an inequality constraint is termed active when it is met with equality; the set of all active constraints

solution set most simply, the set of all optimal solutions to an optimization problem; a subset of the feasible set and not necessarily a single point

set collection of elements in which order and multiplicity are ignored. A set member is called element [444]

list ordered set retaining multiplicity

optimal as in optimal solution, means a solution to an optimization problem. An optimal solution is not necessarily unique, but there is no better solution. optimal \( \Rightarrow \) feasible

optimum optimal value, usually the objective. Can be unique

same as in same problem, means optimal solution set for one problem is identical to optimal solution set of another (without transformation)

equivalent as in equivalent problem, means optimal solution to one problem can be derived from optimal solution to another via suitable transformation

convex as in convex problem, essentially means a convex objective function optimized over a convex set (§4)

objective the three objectives of Optimization are minimize (not min), maximize (not max), and find
Semidefinite program is any convex minimization, maximization, or feasibility problem constraining a variable to a subset of a positive semidefinite cone. Prototypical semidefinite program conventionally means: a semidefinite program having linear objective, affine equality constraints, but no inequality constraints except for cone membership. (§4.1.1)

Linear program is any feasibility problem, or minimization or maximization of a linear objective, constraining the variable to some polyhedron. (§2.13.1.1.2) Prototypical linear program conventionally means: a linear program having linear objective, affine equality constraints, but no inequality constraints except for membership to a nonnegative orthant. (§4.1)

natural order with reference to stacking columns in a vectorization means a vector made from superposing column 1 on top of column 2 then superposing the result on column 3 and so on; as in a vector made from entries of the main diagonal $\delta(A)$ means taken from left to right and top to bottom

partial order relation $\preceq$ is a partial order, on a set, if it possesses reflexivity, antisymmetry, and transitivity (§2.7.2.2)

operator mapping to a vector space (a multidimensional function)

projector short for projection operator; not necessarily minimum-distance nor representable by matrix

sparsity ratio of number of nonzero entries to matrix-dimension product

tight with reference to a bound means a bound that can be met, with reference to an inequality means equality is achievable

trivial with reference to $0$ matrix, function, solution, or $\{0\}$ subspace

$\emptyset$ empty set, an implicit member of every set

$0$ real zero

$0$ origin or vector or matrix of zeros

$O$ sort-index matrix

$O$ order of magnitude or polynomial order or computational intensity: $O(N)$ is first-order, $O(N^2)$ is second-, and so on

$1$ real one

$1$ vector of ones. $1 = \delta(1)$, $\delta(1) = I$

$1_m$ $1 \in \mathbb{R}^m$

$e_i$ vector whose $i^{th}$ entry is 1 (otherwise 0); $i^{th}$ member of the standard basis for $\mathbb{R}^m$ (63)

$I$ Roman numeral one or capital i

$I$ Identity operator or matrix $I = \delta(I)$, $\delta(I) = 1$

$I_m$ $I \in \mathbb{S}^m$

$I$ index set, a discrete set of indices
max \quad \text{maximum [230, §0.1.1] or largest element of a totally ordered set}

maximal \quad \text{characterizes a maximum that is, somehow, not necessarily unique; id est, maximum } \Rightarrow \text{ unique maximum}

\text{maximize } x \quad \text{find maximum of objective function w.r.t independent variables } x. \quad \text{Subscript } x \leftarrow x \in C \text{ may hold implicit constraints if context clear; e.g., semidefiniteness}

\text{arg } \quad \text{argument of operator or function, or variable of optimization}

\text{sup } X \quad \text{supremum of totally ordered set } X, \text{ least upper bound, may or may not belong to set [230, §0.1.1]; e.g., range } X \text{ of real function}

\text{arg sup } f(x) \quad \text{argument } x \text{ at supremum of function } f; \text{ not necessarily unique or a member of function domain}

\text{subject to} \quad \text{specifies constraints of an optimization problem; generally, inequalities and affine equalities. Subject to implies: anything not an independent variable is constant, an assignment, or substitution}

\text{min} \quad \text{minimum [230, §0.1.1] or smallest element of a totally ordered set}

minimal \quad \text{describes a minimum that is, in some sense, not necessarily unique; id est, minimum } \Rightarrow \text{ unique minimum}

\text{minimize } x \quad \text{find objective function minimum w.r.t independent variables } x. \quad \text{Subscript } x \leftarrow x \in C \text{ may hold implicit constraints if context clear; e.g., semidefiniteness}

\text{find } x \quad \text{find any feasible solution, specified by the (“subject to”) constraints, w.r.t independent variables } x. \quad \text{Subscript } x \leftarrow x \in C \text{ may hold implicit constraints if context clear; e.g., semidefiniteness. “find” denotes a feasibility problem; it is the third objective of Optimization}

\text{inf } X \quad \text{infimum of totally ordered set } X, \text{ greatest lower bound, may or may not belong to set [230, §0.1.1]; e.g., range } X \text{ of real function}

\text{arg inf } f(x) \quad \text{argument } x \text{ at infimum of function } f; \text{ not necessarily unique or a member of function domain}

\text{iff} \quad \text{if and only if, necessary and sufficient; also the meaning indiscriminately attached to appearance of the word “if” in the statement of a mathematical definition, [148, p.106] [294, p.4] an esoteric practice worthy of abolition because of ambiguity thus conferred}

\text{rel} \quad \text{relative}

\text{intr} \quad \text{interior}

\text{lim} \quad \text{limit}

\text{sgn} \quad \text{signum function or sign; for } x \in \mathbb{R}^n, \text{ sgn}(x) = \begin{cases} x_i/|x_i|, & x_i \neq 0 \\ 0, & x_i = 0 \end{cases}

\text{round} \quad \text{round to nearest integer}

\text{mod} \quad \text{modulus function}
APPENDIX F. NOTATION, DEFINITIONS, GLOSSARY

tr matrix trace
rank as in rank \( A \), rank of matrix \( A \); \( \dim \mathcal{R}(A) \) (143)
dim dimension, \( \dim \mathbb{R}^n = n \), \( \dim \mathbb{R}^{m \times n} = m \times n \)
\( \dim(x \in \mathbb{R}^n) = n \), \( \dim \mathcal{R}(x \in \mathbb{R}^n) = 1 \)
\( \dim(A \in \mathbb{R}^{m \times n}) = m \times n \), \( \dim \mathcal{R}(A \in \mathbb{R}^{m \times n}) = \text{rank} \ A \)
aff affine hull
dim aff affine dimension \( r \)
card cardinality, number of nonzero entries \( \text{card} x \triangleq \| x \|_0 \)
or \( N \) is cardinality of list \( X \in \mathbb{R}^{n \times N} \) (p. 261)
conv convex hull (§2.3.2)
cone conic (§2.3.3)
cenv convex envelope (§7.2.2.1)
content of high-dimensional bounded polyhedron, volume in \( \mathbb{R}^3 \), area in \( \mathbb{R}^2 \), and so on
cof matrix of cofactors corresponding to matrix argument
dist absolute distance between point or set arguments; e.g., \( \text{dist}(x, \mathcal{B}) \)
vec columnar vectorization of \( m \times n \) matrix, Euclidean dimension \( mn \) (39)
svec columnar vectorization of symmetric \( n \times n \) matrix, Euclidean dimension \( n(n+1)/2 \) (59)
dvec columnar vectorization of symmetric hollow \( n \times n \) matrix, Euclidean dimension \( n(n-1)/2 \) (76)
\( \angle(x, y) \) complex sinusoid phase or angle between vectors \( x \) and \( y \), or dihedral angle between affine subsets
\( \succeq \) generalized inequality, membership to pointed closed convex cone; e.g., \( A \succeq 0 \) means:
  - vector or matrix \( A \) must be expressible in a biorthogonal expansion having nonnegative coordinates with respect to extreme directions of some implicit pointed closed convex cone \( \mathcal{K} \) (§2.13.2.0.1, §2.13.8.1.1),
  - or comparison to the origin with respect to some implicit pointed closed convex cone (2.7.2.2),
  - or (when \( \mathcal{K} = \mathbb{S}^n_+ \)) matrix \( A \) belongs to the positive semidefinite cone of symmetric matrices (nonnegative eigenvalues, §2.9.0.1),
  - or (when \( \mathcal{K} = \mathbb{R}^n_+ \)) vector \( A \) belongs to the nonnegative orthant (each vector entry is nonnegative, §2.3.1.1)
\( \succeq \) as in \( x \succeq z \) means \( x - z \in \mathcal{K} \) (189)
\( \succeq \mathcal{K} \)
\( \succ \) strict generalized inequality, membership to cone interior; \( A \succ 0 \) means:
• vector or matrix $A$ must be expressible in a biorthogonal expansion having positive coordinates with respect to extreme directions of some implicit pointed closed convex cone $K$ (§2.13.2.0.1, §2.13.8.1.1),

• or comparison to the origin with respect to the interior of some implicit pointed closed convex cone (2.7.2.2),

• or (when $K = \mathbb{S}_+^n$) matrix $A$ belongs to the interior of the positive semidefinite cone of symmetric matrices (positive eigenvalues, §2.9.0.1),

• or (when $K = \mathbb{R}_+^n$) vector $A$ belongs to the interior of the nonnegative orthant (each vector entry is positive, §2.3.1.1)

⊁ not positive definite

≥ scalar inequality, total order, greater than or equal to; comparison of scalars, or entrywise comparison of vectors or matrices with respect to $\mathbb{R}_+$

nonnegative for $a \in \mathbb{R}^n$, $a \geq 0$; id est, nonnegative entries when w.r.t nonnegative orthant; coefficients of vector on boundary of or interior to pointed closed convex cone $K$

> greater than

positive for $a \in \mathbb{R}^n$, $a > 0$; id est, positive (nonzero) entries when w.r.t nonnegative orthant; coefficients to no vector on boundary of pointed closed convex cone $K$

⌊ ⌋ floor function, $\lfloor x \rfloor$ is greatest integer not exceeding $x$

| | download button

| | entrywise absolute value of scalars, vectors, and matrices

log natural (or Napierian) logarithm

det matrix determinant

\[
\|x\| = \sqrt{\sum_{j=1}^{n} |x_j|^2} \quad \text{Euclidean norm or vector 2-norm} \quad \|x\|_2
\]

\[
\|x\|_2^2 = x^T x = \langle x , x \rangle \quad \text{Euclidean norm square}
\]

\[
\|x\|_{\ell} = \sqrt[\ell]{\sum_{j=1}^{n} |x_j|^\ell} \quad \text{vector } \ell\text{-norm, convex for } \ell \geq 1, \text{ nonconvex (not a norm) for } 0 < \ell < 1
\]

\[
\|x\|_0 = \text{card } x \quad \text{“0-norm” or cardinality of vector } x
\]

\[
\|x\|_1 = 1^T |x| \quad 1\text{-norm, dual infinity-norm}
\]

\[
\|x\|_{\infty} = \max \{|x_j| \ \forall \ j\} \quad \text{infinity-norm}
\]

\[
\|x\|_{k} = \sum_{i=1}^{k} \pi(|x|)_i \quad k\text{-largest norm}
\]

\[
\|X\|_2 = \sup_{\|a\|=1} \|Xa\|_2 = \sigma_1 = \sqrt{\lambda(X^TX)_1} \quad \text{matrix 2-norm or spectral norm}
\]

\[
\|X\|_2 = \sigma_1 = \sqrt{\lambda(X^TX)_1} \quad \text{matrix 2-norm or spectral norm, largest singular value} \quad [185, \text{p.56}]. \text{ For } x \text{ a vector: } \|x\|_2 = \|x\|_{\infty}.
\]

\[
\|Xa\|_2 \leq \|X\|_2 \|a\|_2
\]
\[ \|X\|_2^n = 1^T \sigma(X) \quad \text{nuclear norm, dual spectral norm} \quad (\S C.2) \]
\[ \|X\| = \sqrt{\sum_{i,j} X_{ij}^2} \quad \text{Frobenius’ matrix norm } \|X\|_F \quad (\S 2.2.1) \]