

MINIMUM PEAK IMPULSE FIR FILTER DESIGN

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Abstract. Saturation pulses $rf(t)$ are essential to many imaging applications. Criteria for desirable saturation profile are flat passband magnitude and sharp transition with minimum peak $rf(t)$ amplitude. Design parameters for RF pulses include passband and stopband ripple tolerances and time-bandwidth product. The well-known Shinnar-Le Roux RF pulse design technique is a transform that relates magnetization profile to two polynomials A_N and B_N . In the past, B_N has been obtained by traditional digital filter design techniques. A conventional approach, to minimum-peak $rf(t)$, is to design a maximum-phase polynomial B_N , factor B_N to obtain its roots, then combinatorially search by root inversion over all possible phase patterns. But this conventional method is limited to time-bandwidth ≈ 18 before number of combinations becomes prohibitive. For time-bandwidth well in excess of that, we propose a convex optimization technique that determines a B_N yielding the global minimum peak $rf(t)$ amplitude.

A novel convex optimization method for finding a minimum peak finite impulse response digital filter is presented. This digital filter frequency domain magnitude obeys specified tolerances while its phase is unconstrained. This problem can be formulated as an optimization, with the exception of one constraint, that is convex. To overcome nonconvexity, we employ an iterative convex method in which the concept of direction-matrix is introduced. This results in solving a noncombinatorial sequence of convex problems.

Key words. digital filter design, convex optimization, iterative convex method, rank constraint

AMS subject classifications. 46N10, 90C25

1. Introduction. Finite Impulse Response (FIR) filter design, subject to upper and lower bounds on frequency response magnitude, is generally a nonconvex problem. Wu *et al.* [1] show how such a problem can be formulated in a convex manner by employing spectral factorization. Their design procedure lies in the domain of power spectral density $R(\omega)$ of the filter's impulse response $h(t)$. They formulated a lowpass filter design example as a convex optimization problem to minimize stopband ripples subject to constraints on passband frequency, stopband frequency, and passband ripples. Other common filter design problems, such as minimizing maximum approximation error (between target and desired magnitude response functions) and magnitude equalizer design, can be formulated as convex problems as well.

2. Methods. This purpose of this paper is to show how to design a lowpass filter that minimizes the impulse response peak amplitude $\|h(t)\|_\infty$ with a given a frequency magnitude response as constraint, *i.e.*,

$$(2.1) \quad \begin{array}{ll} \text{minimize} & \|h(t)\|_\infty \\ \text{subject to} & 1 - \delta_1 \leq |H(\omega)| \leq 1 + \delta_1, \quad \omega \in [0, \omega_p] \\ & |H(\omega)| \leq \delta_2, \quad \omega \in [\omega_s, \pi] \end{array}$$

where ω_p , ω_s , δ_1 , and δ_2 are the passband frequency, stopband frequency, maximum passband ripple, and maximum stopband ripple respectively. And $H(\omega)$ is the frequency filter response, $H(\omega) = \mathfrak{F}(h(t))$, where \mathfrak{F} represents Fourier Transform operation.

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But this problem statement is nonconvex (*i.e.* solution not necessarily globally optimal) [2]. So instead, define an autocorrelation matrix of h as

$$(2.2) \quad G \triangleq hh^H \in \mathbb{C}^{n \times n}$$

where G is positive semidefinite with rank 1. G is of rank 1 because every column of G is made up of vector h shifted by different amount and scaled differently (also by elements of h , Fig.2.1). Summing along each of $2N-1$ subdiagonals produces entries of the autocorrelation function r of h , where

$$(2.3) \quad r(i) = \sum_{j=0}^{N-1-i} h(j)h(j+i)^*, \quad \text{where } * \text{ denotes complex conjugate.}$$

$$\text{and } r \triangleq r_{\text{re}} + i r_{\text{im}} \in \mathbb{C}^n.$$

To write $r(n)$ in terms of G , define $I_0 \triangleq I$ and define I_n as a zero matrix having vector $\mathbf{1}$ along the n^{th} superdiagonal when n is positive or $\mathbf{1}$ along the n^{th} subdiagonal when n is negative.

For example, $r(0)$ is the sum of all elements along the main diagonal of G , *i.e.*

$$(2.4) \quad \begin{aligned} r(0) &= h(0)h(0)^* + h(1)h(1)^* + \dots + h(N-1)h(N-1)^* \\ &= \text{trace} \left(\begin{bmatrix} 1 & 0 & \dots & 0 \\ 0 & 1 & & 0 \\ & & \ddots & \\ 0 & \dots & 0 & 1 \end{bmatrix} G \right) \\ &= \text{trace}(IG) \end{aligned}$$

and $r(1)$ is the sum of all elements along the first super diagonal of G .

$$(2.5) \quad \begin{aligned} r(1) &= h(0)h(1)^* + h(1)h(2)^* + \dots + h(N-2)h(N-1)^* \\ &= \text{trace} \left(\begin{bmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & & 0 \\ & & & \ddots & \\ & & & & 1 \\ 0 & \dots & 0 & 0 & \end{bmatrix}^T G \right) \\ &= \text{trace}(I_1^T G) \end{aligned}$$

An important property of G is the fact that its main diagonal holds squared absolute entries of h (red circles in Fig. 2.1. Minimizing $\|h\|_\infty$ is therefore equivalent to minimizing $\|\text{diag}(G)\|_\infty$.

Define $R(\omega)$ as the Fourier transform of $r(n)$ and recall $R(\omega)$ is the Fourier transform of h defined in Eq. 2.1, $R(\omega)$ is the power spectral density function of h , *i.e.*

$$\begin{aligned} R(\omega) &= \mathfrak{F}(r(n)) \\ H(\omega) &= \mathfrak{F}(h) \\ R(\omega) &= |H(\omega)|^2 \end{aligned}$$

By spectral factorization [8], an equivalent problem to Eq. 2.1 is expressed in Eq. 2.6.

$$\begin{aligned}
 & \text{minimize} \quad \|\text{diag}(G)\|_\infty \\
 & \text{subject to} \quad R(\omega) = r_{\text{re}}(0) + 2 \sum_{n=1}^{N-1} (r_{\text{re}} \cos(\omega n) + r_{\text{im}} \sin(\omega n)) \\
 (2.6) \quad & (1 - \delta_1)^2 \leq R(\omega) \leq (1 + \delta_1)^2, & \omega \in [-\omega_p, \omega_p] \\
 & R(\omega) \leq \delta_2^2, & \pm\omega \in [\omega_s, \pi] \\
 & R(\omega) \geq 0, & \omega \in [-\pi, \pi] \\
 & r(n) = \text{trace}(I_n^T G), & n = 0 \dots N-1 \\
 & G \succeq 0 \\
 & \text{rank}(G) = 1
 \end{aligned}$$

Each line of the constraints in Eq. 2.6 is explained below:

1. Definition of $R(\omega)$. It is equal to taking the Fast Fourier Transform of $r(n)$. The expression used in Eq. 2.6 is a direct translation to the implementation in Matlab using CVX [3, 4].
2. Constraint in passband. Since $R(\omega) = |H(\omega)|^2$, the tolerance is squared.
3. Constraint in stopband.
4. Nonnegativity constraint due to $R(\omega) = |H(\omega)|^2$.
5. Relating $r(n)$ to G by summing each sub-diagonal of G .
6. and 7. These last two constraints (G is positive semi-definite and $\text{rank}(G) = 1$) enforce G to be an autocorrelation matrix.

The optimal G and r exist in subspace where all constraints intersect. Excepting the rank constraint, this problem is convex. To overcome the rank constraint limitation, an *iterative convex method* is used and is described in the following session.

$$\begin{bmatrix}
 |h(0)|^2 & h(0)h(1)^* & h(0)h(2)^* & \cdots & h(0)h(N-3)^* & h(0)h(N-2)^* & h(0)h(N-1)^* \\
 h(1)h(0)^* & |h(1)|^2 & h(1)h(2)^* & \cdots & h(1)h(N-3)^* & h(1)h(N-2)^* & h(1)h(N-1)^* \\
 h(2)h(0)^* & h(2)h(1)^* & |h(2)|^2 & \cdots & h(2)h(N-3)^* & h(2)h(N-2)^* & h(2)h(N-1)^* \\
 \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\
 h(N-3)h(0)^* & h(N-3)h(1)^* & h(N-3)h(2)^* & \cdots & |h(N-3)|^2 & h(N-3)h(N-2)^* & h(N-3)h(N-1)^* \\
 h(N-2)h(0)^* & h(N-2)h(1)^* & h(N-2)h(2)^* & \cdots & h(N-2)h(N-3)^* & |h(N-2)|^2 & h(N-2)h(N-1)^* \\
 h(N-1)h(0)^* & h(N-1)h(1)^* & h(N-1)h(2)^* & \cdots & h(N-1)h(N-3)^* & h(N-1)h(N-2)^* & |h(N-1)|^2
 \end{bmatrix}$$

FIG. 2.1. Autocorrelation matrix G . Red circles highlight the elements on the main diagonal which sum to $r(0)$.

2.1. iterative convex method. Using the iterative convex method (see Appendix), we can rewrite Eq. 2.6 as a sequence of convex optimization problems by introducing a new *direction vector* W :

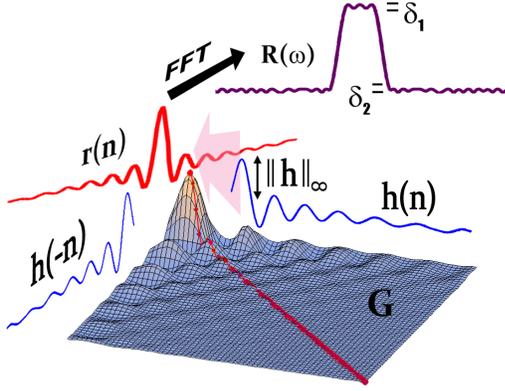


FIG. 2.2. Graphical depiction of the relationship between key variables.

While $\text{rank}(G) > 1$

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$$\begin{aligned}
 & \text{minimize}_{G \in \mathbb{S}^N, r \in \mathbb{R}^N} \quad \langle G, W \rangle \\
 & \text{subject to} \quad R(\omega) = r_{\text{re}}(0) + 2 \sum_{n=1}^{N-1} (r_{\text{re}}(n) \cos(\omega n) + r_{\text{im}}(n) \sin(\omega n)) \\
 & \quad (1 - \delta_1)^2 \leq R(\omega) \leq (1 + \delta_1)^2 \quad \omega \in [-\omega_p, \omega_p] \\
 & \quad R(\omega) \leq \delta_2^2 \quad \pm \omega \in [\omega_s, \pi] \\
 & \quad R(\omega) \geq 0 \quad \omega \in [-\pi, \pi] \\
 & \quad r(n) = \text{trace}(I_n^T G) \quad n = 0 \dots N - 1 \\
 & \quad G \succeq 0 \\
 & \quad \text{diag}(G) \leq h_{\text{max}}^2
 \end{aligned}
 \tag{2.7}$$

Update W

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where h_{max} is the desired impulse response peak magnitude. This constraint on $\text{diag}(G)$ replaces the objective from problem 2.6. This sequence of convex optimization problems is executed sequentially beginning with an initialization of W (usually the identity matrix). At the end of each iteration, the direction vector W is updated. If $\text{rank}(G) > 1$, then the next iteration will be computed using an updated W . This process is repeated until $\text{rank}(G) = 1$.

One way to update the direction vector W is described as follows. Eigenvectors of W are chosen to be identical to G . Suppose G has an eigendecomposition that is the diagonalization

$$G = Q \Lambda Q^T
 \tag{2.8}$$

where Q is an orthogonal matrix containing eigenvectors of G , and Λ is a diagonal matrix holding eigenvalues corresponding to those eigenvectors. The direction vector W is updated by setting its eigenvectors as Q and by setting its eigenvalues, whose locations corresponding to the smallest $N - 1$ eigenvalues of G , to be 1 and 0 elsewhere. The idea is, on the next

iteration, the smallest eigenvalues of G get minimized. We choose

$$(2.9) \quad W = Q\Phi Q^T, \quad \Phi = \begin{bmatrix} 0 & & & \mathbf{0} \\ & 1 & & \\ & & 1 & \\ & & & \ddots \\ \mathbf{0} & & & & 1 \end{bmatrix}$$

This way,

$$(2.10) \quad \langle G, W \rangle = \text{tr}(GW) = \text{tr}(Q\Lambda\Phi Q^T) = \sum_{i=2}^N \Lambda_{ii}$$

the vector inner product is a sum of the $N - 1$ smallest eigenvalues of G . Rank of G must equal 1 when those $N - 1$ eigenvalues vanish because

$$(2.11) \quad \text{rank } G = \text{rank } \Lambda$$

Once a rank-one matrix G is found, polynomial h can be obtained simply by extracting the first column of G followed by normalization (see Fig. 2.1): $h = G(:, 1) / \sqrt{G(1, 1)}$.

3. Results.

3.1. Low filter order. We compared our optimization technique with the conventional zero-flipping technique when the filter order is low and trackable. Figure 3.1 shows two impulse response filters with minimum peak found by zero-flipping (blue) and by the proposed optimal technique (red). The impulse response maximum peak obtained by these two techniques are 0.1189. The filter order $N = 40$, passband frequency $\omega_p = 0.2\pi$, stopband frequency $\omega_s = 0.3\pi$, passband and stopband ripple tolerance $\delta_p = \delta_s = 0.01$. The number of zeros required to flip in order to determine the minimum peak is 10. Figure 3.2 illustrates the frequency domain magnitude profiles using our optimal technique and that from zero-flipping.

3.2. High filter order. When the filter order is high and the number of zeros required for flipping exceeded reasonable computation time (we gave up running our zero-flipping program when there was a power outage after one week of computation), we display the optimal minimum peak filter calculated by our technique and the minimum phase filter as reference in figure 3.3. The design criteria are: filter order $N = 84$, passband frequency $\omega_p = 0.26\pi$, stopband frequency $\omega_s = 0.3\pi$, passband and stopband ripple tolerance $\delta_p = \delta_s = 0.01$ (fig. 3.4). The number of zeros required to flip is 21 (fig. 3.5).

4. Discussion. We presented a method to design a globally minimum peak time-domain filter without the need for exhaustive root-inversion search. Minimum peak time-domain filter can be obtained by a series of Convex Optimization for which the objective is to find a h having the lowest possible peak amplitude while simultaneously satisfying the given filter design parameters. Instead of working directly with time domain impulse response function or its Fourier transform in the frequency domain, our technique employs its autocorrelation and imposes a rank constraint on the autocorrelation matrix G . The main diagonal of G is $[h(0)h(0)^*, h(1)h(1)^*, \dots]^T$. Minimizing $\|h\|_\infty$ is therefore equivalent to minimizing $\|\text{diag}(G)\|_\infty$. Once the program has achieved global minimum, h can be extracted from

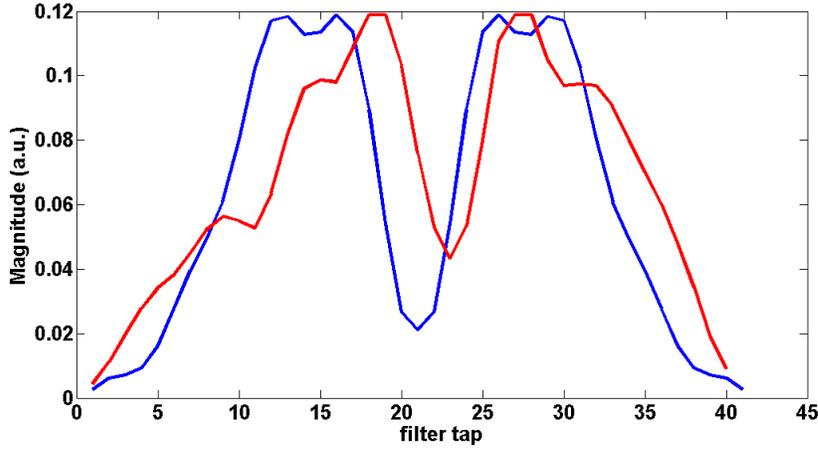


FIG. 3.1. Impulse response determined by (blue) zero-flipping method and (red) proposed optimal method.

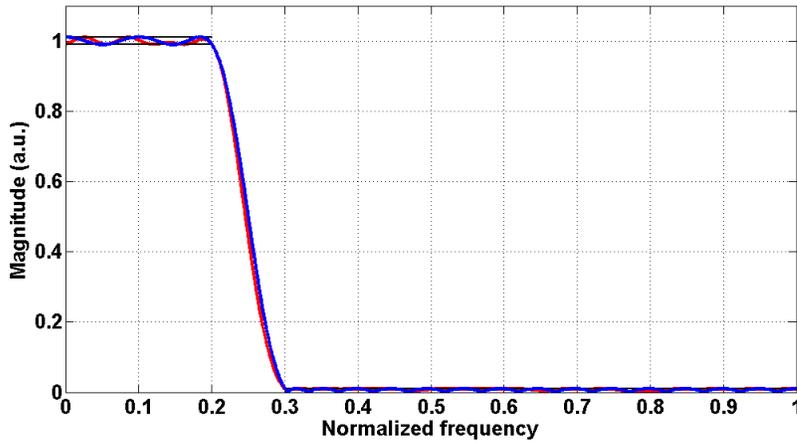


FIG. 3.2. Frequency domain filter profiles determined by (blue) zero-flipping method and (red) proposed optimal method.

the first column of G . Columns of autocorrelation matrix G are simply copies of h having different scale factors; consequently, $\text{rank}(G)=1$. But a rank constraint is not convex. We overcome this by using an iterative convex method. The number of iterations varies and depend on the complexity of the problem.

Applications for this specific example, *i.e.* minimizing FIR filter impulse response peak for specific magnitude frequency response, is only applicable when the frequency domain phase is unconstrained. One such situation is for designing radio frequency (RF) saturation pulse in magnetic resonance imaging (MRI). The purpose of saturation pulse is to remove unwanted signals from certain anatomy or tissue (for instance, suppressing signal from outside the volume of interest to prevent aliasing). In MRI, RF pulses design is related to FIR digital filter design [5]. Phase of saturation pulse in the frequency domain is not important but a sharp transition width and flat passband are often desired. The peak magnitude of the

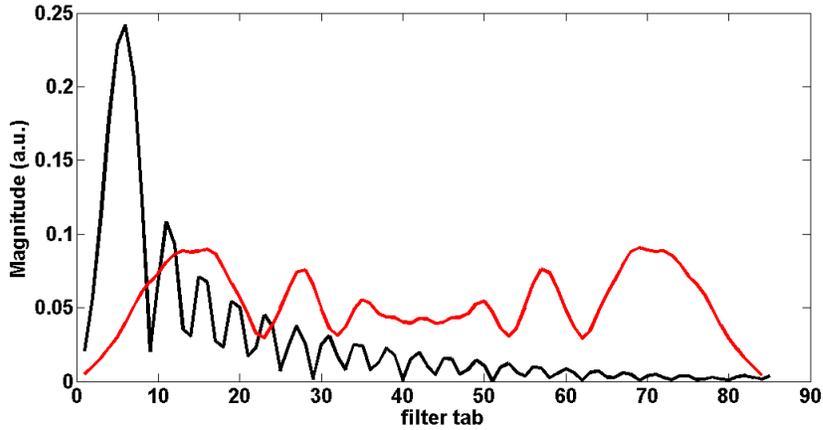


FIG. 3.3. Impulse response of a (black) minimum phase filter and (red) corresponding filter determined by our proposed optimal method.

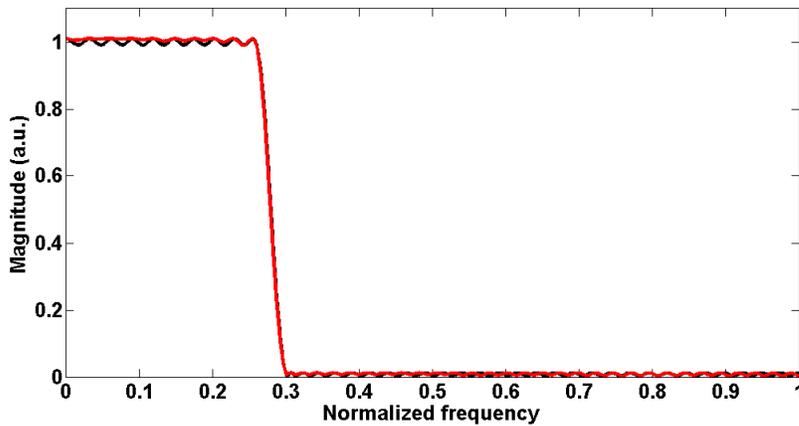


FIG. 3.4. Frequency domain filter profiles of the (black) minimum phase filter and (red) proposed optimal method.

saturation pulse in time domain is proportional to the specific absorption rate (SAR) of RF energy being deposited onto patients [6, 7, 8]. There are straight regulations of maximum SAR one can receive during MRI, which result in tradeoff between saturation pulse peak magnitude and its frequency domain specifications. The technique we presented in this paper is applicable for FIR filter design. More work is needed to extend this technique for direct RF saturation pulse design.

5. APPENDIX. The problem solve in this paper is rank-constrained. To visualize the process of our iterative convex method pictorially, we will look at a closely related cardinality-constrained problem instead. Consider the following cardinality-constrained problem:

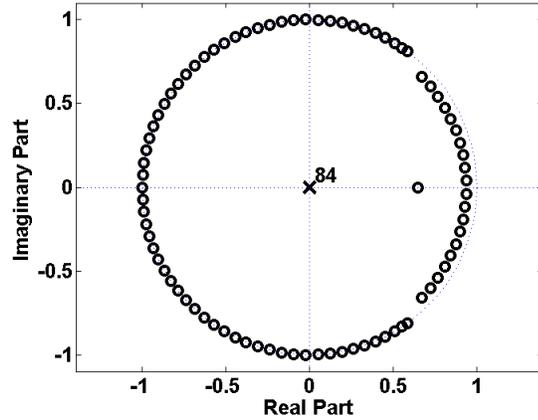


FIG. 3.5. Pole-zero diagram of the minimum phase filter.

$$(5.1) \quad \begin{array}{ll} \text{find} & x \in \mathbb{R}^n \\ \text{subject to} & x \in \mathcal{E} \\ & x \succeq 0 \\ & \|x\|_0 \leq k \end{array}$$

The objective in problem 5.1 is to find a real vector x subject to the constraints that x belongs to the ellipsoid \mathcal{E} , x belongs to the positive orthant, and the desired cardinality of x is k , i.e. the number of nonzero elements in x is at most k . A cardinality constraint is nonconvex. Thus, the entire problem becomes nonconvex. To overcome the nonconvexity, an *iterative convex method* is used to rewrite problem 5.1 as a sequence of convex problems where the cardinality constraint is replaced by a vector inner-product:

$$(5.2) \quad \begin{array}{ll} \underset{x \in \mathbb{R}^n}{\text{minimize}} & \langle x, y \rangle \\ \text{subject to} & x \in \mathcal{E} \\ & x \succeq 0 \end{array}$$

Convex problem 5.2 introduces a new parameter y which is called the *direction vector* and is the key to the iterative convex method technique. If an optimal direction vector y^* is known at the beginning, then substituting that y^* into problem 5.2 would give an optimal solution x^* that is also optimal to problem 5.1. In that case, no iteration is required. We know that such a y^* exists because we assume that a cardinality- k solution is feasible to problem 5.1: Optimal y^* has 1 in each entry corresponding to 0 in cardinality- k x^* . If we knew where those zeros in x^* were supposed to be, then y^* would be known at the beginning.

In practice, optimal direction vector y^* is not known in advance. So we choose an initial y , solve problem 5.2, update y , and then solve problem 5.2 using the updated y . This sequence of solving problem 5.2 and updating y is repeated until an optimal cardinality- k solution is found. Direction vector y is updated in much the same way as W in Eq. 2.9. Instead of eigenvalues, we look directly at entries of vector x : Optimal x^* from problem 5.2 is first sorted. Then direction vector y is set to 1 in each entry corresponding to the smallest $N - k$ entries in x^* and 0 elsewhere.

$$(5.3) \quad \langle x^*, y \rangle = \sum_{i=k+1}^N \text{smallest entries } x_i^* \text{ in } x^*$$

Figure 5.1 illustrates this iteration process.

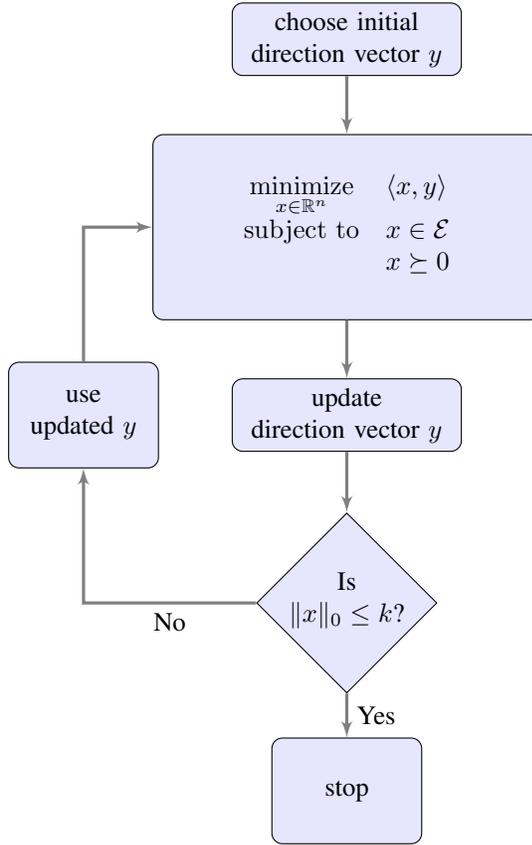


FIG. 5.1. Flow chart 1.

Let us visualize this problem in \mathbb{R}^2 , define $x \triangleq [x_1, x_2]^T$. Assume $k = 1$, then if either x_1 or x_2 is zero, an optimal solution is found. If the initial choice of direction vector y (e.g. $y = [1, 1]^T$) does not yield an optimal solution, x_1 and x_2 are nonzero. Without loss of generality, assume the elements in the resulting vector x is $x_2 > x_1 > 0$. We update the direction vector y by setting the element of y that corresponds to the location of x_1 be 1. i.e. the updated y is $y = [1, 0]^T$. Using $y = [1, 0]^T$, problem 5.2 is solved again and this cycle is repeated until an optimal solution is found. Figures 5.2-5.6 illustrate this concept.

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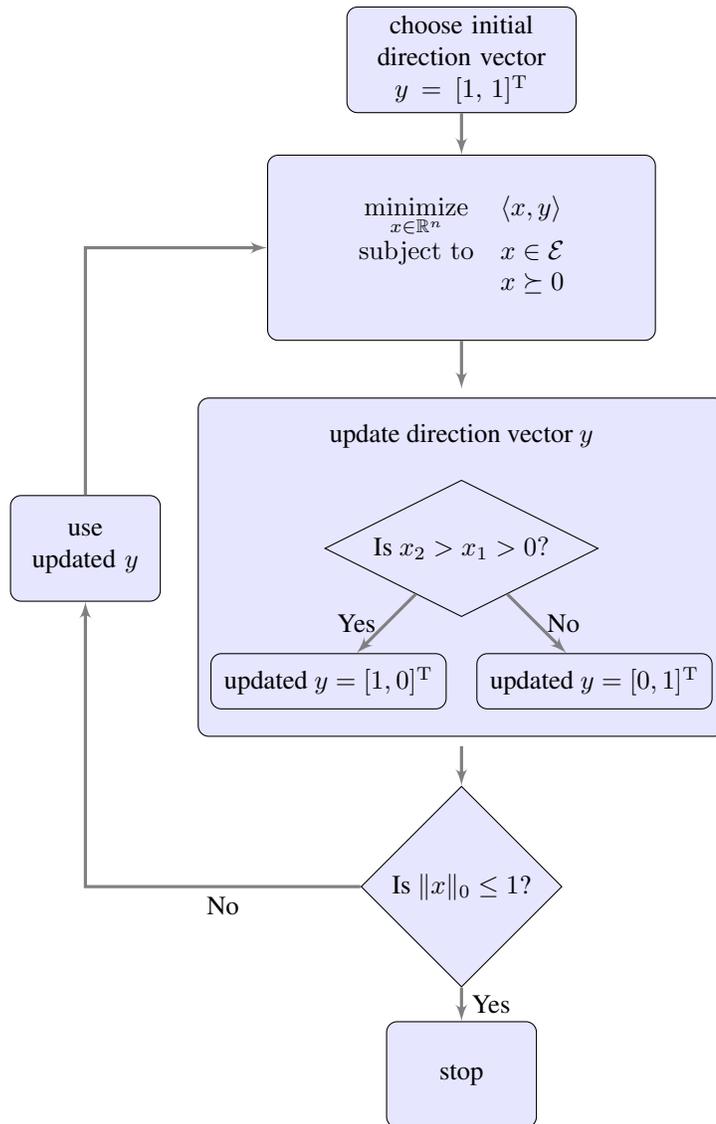


FIG. 5.2. Flow chart 2.

- [3] CVX RESEARCH, INC., *CVX: Matlab Software for Disciplined Convex Programming, version 2.0 beta*, Sep 2012.
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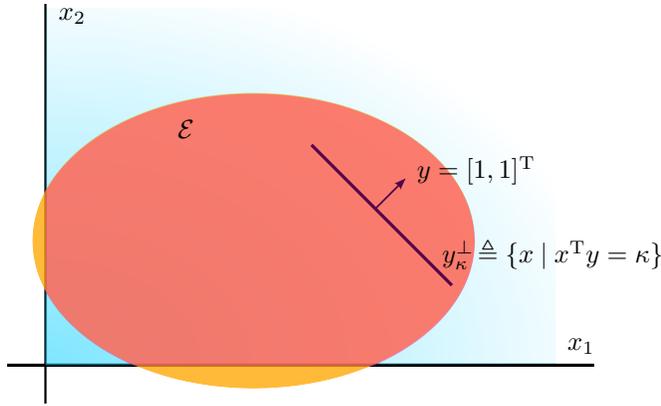


FIG. 5.3. Choose initial direction vector y . In this \mathbb{R}^2 example, the feasible-solution set is the intersection (shown in red) between the ellipsoid \mathcal{E} (orange) and the positive orthant $x \succeq 0$ (blue). The initial direction vector y is chosen as $[1, 1]^T$. Also drawn is the vector normal y_{κ}^{\perp} .

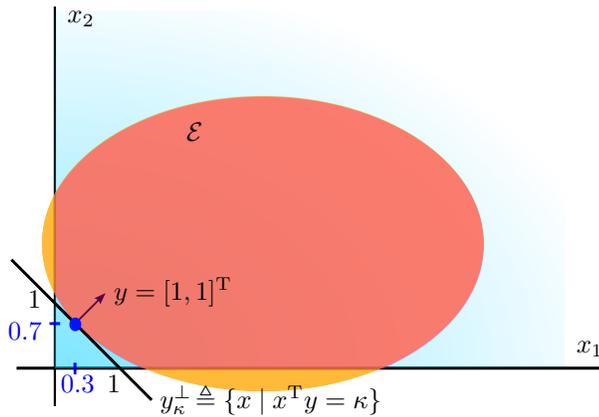


FIG. 5.4. First iteration. The next step is to solve $\min_{x \in \mathbb{R}^n} \langle x, y \rangle$ for x within the feasible-solution set. Graphically, pushing y_{κ}^{\perp} in the direction of y will lead to an increase of $x^T y = \kappa$. Since the objective is to minimize $x^T y$, we push the y_{κ}^{\perp} in the direction opposite to y until it reaches the boundary of the feasible set. The intersection between y_{κ}^{\perp} and the feasible set boundary is the optimal solution x^* to problem 5.2 for $y = [1, 1]^T$. Here, $x^* = [0.3, 0.7]^T$.

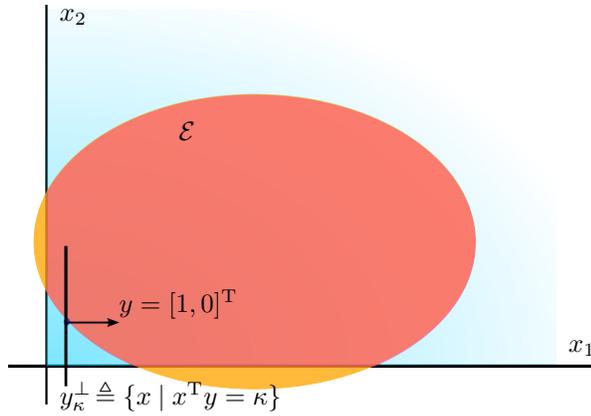


FIG. 5.5. Update y . We sort the entries of $x^* = [0.3, 0.7]^T$. Here, $x_2 > x_1$. Thus, we update y by placing 1 in its second entry and 0 elsewhere; i.e. updated $y = [1, 0]^T$.

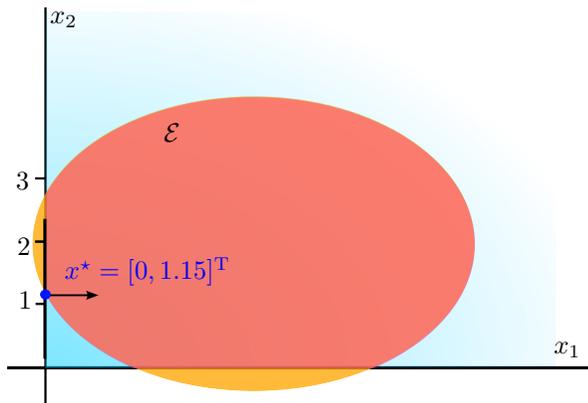


FIG. 5.6. Second iteration. We solve the problem $\text{minimize}_{x \in \mathbb{R}^n} \langle x, y \rangle$ again using $y = [1, 0]^T$. We push y_{κ}^{\perp} in the direction opposite to y until it reaches the boundary of the feasible set. Here, we have $x^* = [0, 1.15]^T$ whose cardinality is 1. Thus, $x^* = [0, 1.15]^T$ is also an optimal solution to the original problem 5.1. In fact, the set $x_1 = 0, x_2 \in [1.15, 2.7]$ is a solution to problems 5.1 and 5.2.