

Convex Optimization

Euclidean Distance Geometry^{2ε}

1	Overview	19
2	Convex geometry	31
2.1	Convex set	31
2.2	Vectorized-matrix inner product	42
2.3	Hulls	50
2.4	Halfspace, Hyperplane	58
2.5	Subspace representations	69
2.6	Extreme, Exposed	74
2.7	Cones	77
2.8	Cone boundary	85
2.9	Positive semidefinite (PSD) cone	90
2.10	Conic independence (c.i.)	111
2.11	When extreme means exposed	115
2.12	Convex polyhedra	116
2.13	Dual cone & generalized inequality	122
3	Geometry of convex functions	169
3.1	Convex real and vector-valued function	169
3.2	Practical norm functions, absolute value	173
3.3	Powers, roots, and inverted functions	181
3.4	Affine function	183
3.5	Epigraph, Sublevel set	186
3.6	Gradient	193
3.7	First-order convexity condition, real function	198
3.8	First-order convexity condition, vector-valued	202
3.9	Second-order convexity condition, vector-valued	202
3.10	Convex matrix-valued function	203
3.11	First-order convexity condition, matrix-valued	205
3.12	Epigraph of matrix-valued function, sublevel sets	206
3.13	Second-order convexity condition, matrix-valued	207
3.14	Quasiconvex	209
3.15	Salient properties	220
4	Semidefinite programming	223
4.1	Conic problem	224
4.2	Framework	230
4.3	Rank reduction	240

4.4	Cardinality reduction	246
4.5	Rank constraint by Convex Iteration	249
4.6	Constraining cardinality	273
4.7	Cardinality and rank constraint examples	288
4.8	Quantum optimization	326
4.9	Constraining rank of indefinite matrices	331
4.10	Convex Iteration rank-1	335
5	Euclidean Distance Matrix	343
5.1	EDM	343
5.2	First metric properties	344
5.3	\exists fifth Euclidean metric property	345
5.4	EDM definition	348
5.5	Invariance	372
5.6	Injectivity of \mathbf{D} & unique reconstruction	375
5.7	Embedding in affine hull	380
5.8	Euclidean metric <i>versus</i> matrix criteria	384
5.9	Bridge: Convex polyhedra to EDMs	389
5.10	EDM-entry composition	394
5.11	EDM indefiniteness	397
5.12	List reconstruction	401
5.13	Reconstruction examples	405
5.14	Fifth property of Euclidean metric	409
6	Cone of distance matrices	417
6.1	Defining EDM cone	418
6.2	Polyhedral bounds	420
6.3	$\sqrt{\text{EDM}}$ cone is not convex	421
6.4	EDM definition in $\mathbf{11}^T$	421
6.5	Correspondence to PSD cone \mathbb{S}_+^{N-1}	428
6.6	Vectorization & projection interpretation	432
6.7	A geometry of completion	438
6.8	Dual EDM cone	443
6.9	Theorem of the alternative	455
6.10	Postscript	455
7	Proximity problems	457
7.1	First prevalent problem:	462
7.2	Second prevalent problem:	470
7.3	Third prevalent problem:	478
7.4	Conclusion	485
A	Linear algebra	487
A.1	Main-diagonal δ operator, λ , tr , vec , \circ , \otimes	487
A.2	Semidefiniteness: domain of test	490
A.3	Proper statements of positive semidefiniteness	493
A.4	Schur complement	501
A.5	Eigenvalue decomposition	505
A.6	Singular value decomposition, SVD	508
A.7	Zeros	512

B	Simple matrices	517
B.1	Rank-1 matrix (dyad)	517
B.2	Doublet	521
B.3	Elementary matrix	522
B.4	Auxiliary V -matrices	524
B.5	Orthomatrices	527
B.6	Arrow matrix	532
C	Some analytical optimal results	535
C.1	Properties of infima	535
C.2	Trace, singular and eigen values	536
C.3	Orthogonal Procrustes problem	541
C.4	Two-sided orthogonal Procrustes	543
C.5	Quadratics	546
D	Matrix calculus	549
D.1	Gradient, Directional derivative, Taylor series	549
D.2	Tables of gradients and derivatives	564
E	Projection	573
E.1	Idempotent matrices	577
E.2	$I - P$, Projection on algebraic complement	580
E.3	Symmetric idempotent matrices	581
E.4	Algebra of projection on affine subsets	585
E.5	Projection examples	586
E.6	Vectorization interpretation	593
E.7	Projection on matrix subspaces	598
E.8	Range, Row space interpretation	600
E.9	Projection on convex set	601
E.10	Alternating projection	612
F	Notation, Definitions, Glossary	629
	Bibliography	645
	Index	667

List of Tables

2 Convex geometry	
Table 2.9.2.3.1 , rank <i>versus</i> dimension of \mathbb{S}_+^3 faces	97
Table 2.10.0.0.1 , maximum number of c.i. directions	111
Cone Table 1	151
Cone Table S	152
Cone Table A	153
Cone Table 1*	157
4 Semidefinite programming	
Faces of \mathbb{S}_+^3 corresponding to faces of \mathcal{S}_+^3	228
Quantum impulse	328
Quantum step	330
Quantum AND function	330
5 Euclidean Distance Matrix	
<i>Précis</i> 5.7.2 : affine dimension in terms of rank	383
B Simple matrices	
Auxiliary <i>V</i> -matrix Table B.4.4	526
D Matrix calculus	
Table D.2.1 , algebraic gradients and derivatives	565
Table D.2.2 , trace Kronecker gradients	566
Table D.2.3 , trace gradients and derivatives	567
Table D.2.4 , logarithmic determinant gradients, derivatives	569
Table D.2.5 , determinant gradients and derivatives	570
Table D.2.6 , logarithmic derivatives	570
Table D.2.7 , exponential gradients and derivatives	571

List of Figures

1	Overview	19
1	Sigma delta quantizer	20
2	Room geometry estimation by first acoustic reflections	20
3	<i>Orion nebula</i>	21
4	Application of trilateration is localization	22
5	Molecular conformation	23
6	Facial recognition	24
7	<i>Swiss roll</i>	25
8	USA map reconstruction	26
9	Honeycomb, Hexabenzocoronene molecule	27
10	Robotic vehicles	28
11	Reconstruction of David	29
12	David by distance geometry	29
2	Convex geometry	31
13	Slab	33
14	Open, closed, convex sets	35
15	Intersection of line with boundary	36
16	Tangentials	38
17	Inverse image	41
18	Inverse image under linear map	41
19	<i>Tesseract</i>	44
20	Linear injective mapping of Euclidean body	45
21	Linear noninjective mapping of Euclidean body	46
22	Convex hull of a random list of points	50
23	Hulls	52
24	Two Fantopes	54
25	Nuclear Norm Ball	55
26	Convex hull of rank-1 matrices	56
27	A simplicial cone	59
28	Hyperplane illustrated $\partial\mathcal{H}$ is a partially bounding line	60
29	Hyperplanes in \mathbb{R}^2	62
30	Affine independence	64
31	$\{z \in \mathcal{C} \mid a^T z = \kappa_i\}$	65
32	Hyperplane supporting closed set	66
33	Minimizing hyperplane over affine subset in nonnegative orthant	72
34	Maximizing hyperplane over convex set	73
35	Closed convex set illustrating exposed and extreme points	78

36	Two-dimensional nonconvex cone	78
37	Nonconvex cone made from lines	79
38	Nonconvex cone is convex cone boundary	79
39	Union of convex cones is nonconvex cone	79
40	Truncated nonconvex cone \mathcal{X}	80
41	Cone exterior is convex cone	80
42	Not a cone	81
43	Minimum element, Minimal element	83
44	\mathcal{K} is a pointed polyhedral cone not full-dimensional	86
45	Exposed and extreme directions	89
46	Positive semidefinite cone	92
47	Convex Schur-form set	93
48	Projection of truncated PSD cone	95
49	Circular cone showing axis of revolution	103
50	Circular section	104
51	Polyhedral inscription	106
52	Conically (in)dependent vectors	112
53	Pointed six-faceted polyhedral cone and its dual	113
54	Minimal set of generators for halfspace about origin	115
55	Venn diagram for cones and polyhedra	117
56	Range form polyhedron	118
57	Simplex	120
58	Two views of a simplicial cone and its dual	121
59	Two equivalent constructions of dual cone	123
60	Dual polyhedral cone construction by right angle	124
61	Orthogonal cones	126
62	<i>Blades</i> \mathcal{K} and \mathcal{K}^*	127
63	\mathcal{K} is a halfspace about the origin	128
64	Iconic primal and dual objective functions	129
65	Membership w.r.t \mathcal{K} and orthant	137
66	Shrouded polyhedral cone	142
67	Simplicial cone \mathcal{K} in \mathbb{R}^2 and its dual	146
68	Monotone nonnegative cone $\mathcal{K}_{\mathcal{M}+}$ and its dual	154
69	Monotone cone $\mathcal{K}_{\mathcal{M}}$ and its dual	155
70	Two views of monotone cone $\mathcal{K}_{\mathcal{M}}$ and its dual	156
71	First-order optimality condition	159
3 Geometry of convex functions		169
72	Convex functions having unique minimizer	170
73	Minimum/Minimal element, dual cone characterization	172
74	Norm balls	173
75	1-norm ball \mathcal{B}_1 from compressed sensing/compressive sampling	176
76	Cardinality minimization, phase transition, signed <i>versus</i> unsigned variable	177
77	1-norm variants	177
78	Affine function	184
79	Supremum of affine functions	185
80	Epigraph	186
81	Log function constraint	192
82	Quadratic bowl and 1-norm gradients in \mathbb{R}^2 evaluated on grid	193
83	Quadratic function convexity in terms of its gradient	199
84	Contour plot of convex real function at selected levels	200

85	Tangent hyperplane to nonconvex surface	201
86	Taxicab distance on nonuniform rectangular grid	205
87	Iconic quasiconvex function	209
88	Quasiconcave monotonic function xu	211
89	Operational Amplifier implementation of third-order filter having a zero	212
90	Mason flowgraph for operational amplifier arbitrary magnitude filter circuit	213
91	Bisection method linearity	216
92	Arbitrary magnitude analog filter design	218
93	Sum of convex functions	221
4	Semidefinite programming	223
94	Venn diagram of convex program types	226
95	Visualizing positive semidefinite cone in high dimension	227
96	Primal/Dual transformations	234
97	Projection <i>versus</i> convex iteration	251
98	Trace heuristic	252
99	Sensor-network localization	255
100	2-lattice of sensors and anchors for localization example	257
101	3-lattice of sensors and anchors for localization example	258
102	4-lattice of sensors and anchors for localization example	259
103	5-lattice of sensors and anchors for localization example	260
104	Uncertainty ellipsoids orientation and eccentricity	261
105	2-lattice localization solution	263
106	3-lattice localization solution	264
107	4-lattice localization solution	264
108	5-lattice localization solution	265
109	10-lattice localization solution	265
110	100 randomized noiseless sensor localization	266
111	100 randomized sensors localization	267
112	Nonnegative spectral factorization	270
113	Regularization curve for convex iteration	272
114	1-norm heuristic	274
115	Sparse sampling theorem	277
116	Signal dropout	280
117	Signal dropout reconstruction	281
118	Simplex with intersecting line problem in compressed sensing	283
119	Geometric interpretations of sparse-sampling constraints	285
120	Permutation matrix column-norm and column-sum constraint	291
121	MAX CUT problem	297
122	Shepp-Logan phantom	301
123	MRI radial sampling pattern in Fourier domain	304
124	Aliased phantom	305
125	Neighboring-pixel stencil on Cartesian grid	307
126	Differentiable almost everywhere	308
127	<i>Eternity II</i>	310
128	<i>Eternity II</i> game-board grid	312
129	<i>Eternity II</i> demo-game piece illustrating edge-color ordering	313
130	<i>Eternity II</i> vectorized demo-game-board piece descriptions	314
131	<i>Eternity II</i> difference Δ and boundary coefficient β construction	315
132	<i>Eternity II</i> composite variable matrix sparsity pattern	317
133	<i>Eternity II</i> problem visualization in three dimensions	322

134	<i>Eternity</i> II permutation polyhedron vertices visualization on sphere	323
135	<i>Chimera</i> topology for D:Wave 1152-qubit chip	325
136	D:Wave <i>Chimera</i> chip layout	327
137	MIT logo	333
138	One-pixel camera	333
139	One-pixel camera - compression estimates	334
140	Convergence of Singular Value Decomposition by Convex Iteration	339
141	Straight line through three direction vectors by midpoint fit	341
5	Euclidean Distance Matrix	343
142	Convex hull of three points	344
143	Complete dimensionless <i>EDM graph</i>	346
144	Fifth Euclidean metric property	347
145	<i>Fermat point</i>	354
146	Arbitrary hexagon in \mathbb{R}^3	355
147	Kissing number	356
148	<i>Trilateration</i>	360
149	This EDM graph provides unique isometric reconstruction	363
150	Two sensors \bullet and three anchors \circ	363
151	Two discrete linear trajectories of sensors	364
152	Coverage in cellular telephone network	367
153	Contours of equal signal power	367
154	Depiction of molecular conformation	368
155	Square diamond	374
156	Orthogonal complements in \mathbb{S}^N abstractly oriented	376
157	Elliptope \mathcal{E}^3	390
158	Elliptope \mathcal{E}^2 interior to \mathbb{S}_+^2	391
159	Smallest eigenvalue of $-V_N^T D V_N$	395
160	Some entrywise EDM compositions	395
161	Map of United States of America	404
162	Largest ten eigenvalues of $-V_N^T O V_N$	406
163	<i>Relative-angle inequality tetrahedron</i>	411
164	Nonsimplicial pyramid in \mathbb{R}^3	414
6	Cone of distance matrices	417
165	Relative boundary of cone of Euclidean distance matrices	419
166	Example of $V_{\mathcal{X}}$ selection to make an EDM	423
167	Vector $V_{\mathcal{X}}$ spirals	425
168	Three views of translated negated elliptope	431
169	Halfline \mathcal{T} on PSD cone boundary	434
170	Vectorization and projection interpretation example	435
171	Intersection of EDM cone with hyperplane	437
172	Neighborhood graph	439
173	<i>Trefoil knot untied</i>	440
174	<i>Trefoil ribbon</i>	442
175	Orthogonal complement of geometric center subspace	446
176	EDM cone construction by flipping PSD cone	447
177	Decomposing member of polar EDM cone	450
178	Ordinary dual EDM cone projected on \mathbb{S}_h^3	454

7 Proximity problems	457
179 Pseudo-Venn diagram for EDM	459
180 Elbow placed in path of projection	459
181 Convex envelope	473
A Linear algebra	487
182 Geometrical interpretation of full SVD	510
B Simple matrices	517
183 Four fundamental subspaces for any dyad	518
184 Four fundamental subspaces for doublet	521
185 Four fundamental subspaces for elementary matrix	522
186 Antireflection	529
187 Gimbal	531
188 Arrow matrix	532
D Matrix calculus	549
189 Convex quadratic bowl in $\mathbb{R}^2 \times \mathbb{R}$	557
E Projection	573
190 Action of pseudoinverse	574
191 Nonorthogonal projection of $x \in \mathbb{R}^3$ on $\mathcal{R}(U) = \mathbb{R}^2$	579
192 Biorthogonal expansion of point $x \in \text{aff } \mathcal{K}$	587
193 Linear regression <i>versus</i> principal component analysis	591
194 Dual interpretation of projection on convex set	603
195 Projection on orthogonal complement	605
196 Projection on dual cone	607
197 Projection product on convex set in subspace	611
198 von Neumann-style projection of point b	613
199 Alternating projection on two halfspaces	614
200 Distance, feasibility, optimization	615
201 Alternating projection on nonnegative orthant and hyperplane	618
202 Geometric convergence of iterates in norm	618
203 Distance between PSD cone and iterate in \mathcal{A}	622
204 Dykstra's alternating projection algorithm	623
205 Polyhedral normal cones	624
206 Normal cone to ellipptope	625
207 Normal-cone progression	626