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Fractional Sampling using the Asynchronous Shah with application to LINEAR PHASE FIR FILTER DESIGN

Abstract

We investigate the fundamental process of sampling using an impulse train, called the *shah* function [Bracewell], that is skewed in time by a constant offset relative to absolute time 0. We observe that when an originating analog signal is bandlimited, there is little difference between fractionally time-shifting the corresponding sequence, and synchronously sampling the time-shifted analog signal. We use these results to explain a curious parallel in the theoretical analysis of Types II and IV linear phase FIR (finite impulse response) filters that is manifest by an apparent 4π-periodicity in the "generalized" [O&S] linear amplitude and phase.

1. The Continuous-Time View of Sampling

$s(t)$ is the absolute-time synchronous *shah* function; [O&S,pg.83,Equ.(3.6)] [Bracewell]

$$s(t) = \sum_{n=-\infty}^{\infty} \delta(t - nT)$$

$$S(\Omega) = \frac{2\pi}{T} \sum_{k=-\infty}^{\infty} \delta\left(\Omega - k\frac{2\pi}{T}\right) \qquad ; \; \Omega = 2\pi f$$

The asynchronous shah (asynchronous to time zero) is similarly written

$$s_\tau(t) = \sum_{n=-\infty}^{\infty} \delta(t - nT + \tau)$$

$$S_\tau(\Omega) = \frac{2\pi}{T} \sum_{k=-\infty}^{\infty} e^{jk2\pi\tau/T} \; \delta\left(\Omega - k\frac{2\pi}{T}\right)$$

for T the sampling period, and for $0 \le \tau < T$. Sampling using the asynchronous shah results in

$$\begin{aligned} x_{s_\tau}(t) &= x(t)\, s_\tau(t) \\ &= x(t) \sum_n \delta(t - nT + \tau) \;=\; \sum_n x(nT - \tau)\, \delta(t - nT + \tau) \\ &\Leftrightarrow \\ X_{S_\tau}(\Omega) &= \frac{1}{2\pi} X(\Omega) * S_\tau(\Omega) \;=\; \frac{1}{T} \sum_{k=-\infty}^{\infty} X\left(\Omega - k\frac{2\pi}{T}\right) e^{jk2\pi\tau/T} \end{aligned} \tag{1}$$

From the point of view of the continuous-time domain, this equation says that when we sample a signal asynchronously to absolute-time 0, each replication in the frequency domain becomes multiplied by a complex constant $e^{jk2\pi\tau/T}$ that is dependent upon the replication number k and the asynchrony τ. When $\tau = 0$ then the complex constant becomes equal to 1 for all k and we have conventional synchronous sampling. A particularly noteworthy case is when $\tau = T/2$, for then the complex constant alternates between ± 1 at every replication. This leads to $4\pi/T$ periodicity in the frequency domain.

2. The Sequence-Domain View of Sampling

The interpretation of asynchronous sampling that we have from Equ.(1) is sufficient for many purposes. But we can write Equ.(1) equivalently as follows and then derive another interpretation that is intuitively appealing.

$$X_{S_\tau}(\Omega) = e^{j\Omega\tau}\,\frac{1}{T}\sum_{k=-\infty}^{\infty} X\!\left(\Omega - k\frac{2\pi}{T}\right) e^{-j(\Omega - k\frac{2\pi}{T})\tau} \qquad (2)$$

With a change of variable, we momentarily mix the discrete and continuous-time representations;

$$X_{S_\tau}(\Omega) \;=\; e^{j\Omega\tau}\left[\frac{1}{T}\sum_{k=-\infty}^{\infty} X\!\left(\frac{\omega}{T} - k\frac{2\pi}{T}\right) e^{-j(\frac{\omega}{T} - k\frac{2\pi}{T})\tau}\right] \quad ;\ \Omega = \frac{\omega}{T} \qquad (2A)$$

In the case that $X(\Omega)$ in Equ.(2) is bandlimited to the Nyquist frequency, then we may say

$$X_{S_\tau}(\Omega) \;=\; e^{j\Omega\tau}\left[X\!\left(e^{j\omega}\right) e^{-j\omega\tau/T}\right] \quad ;\ \text{if bandlimited} \qquad (2B)$$

where

$$X\!\left(e^{j\omega}\right) \;=\; \frac{1}{T}\sum_{k=-\infty}^{\infty} X\!\left(\frac{\omega}{T} - k\frac{2\pi}{T}\right) \quad \Leftrightarrow \quad \sum_n x(nT)\,\delta(t - nT)$$

In Equ.(2B) it is necessary to realize that $e^{j\Omega\tau}$ does not cancel $e^{-j\omega\tau/T}$ because the argument of the former is not periodic; the argument of the discrete-time linear phase term $e^{-j\omega\tau/T}$ *is* 2π-periodic in ω. $X(e^{j\omega})$ corresponds to the conventionally sampled discrete-time domain signal, and is always 2π-periodic.[1] [O&S,pg.87,Equ.(3.20)] Now if we separate out the permanently 2π-periodic part from Equ.(2A), we get the Fourier transform of a bandlimited sequence;

$$X\!\left(e^{j\omega}\right) e^{-j\omega\tau/T} \;=\; \frac{1}{T}\sum_{k=-\infty}^{\infty} X\!\left(\frac{\omega}{T} - k\frac{2\pi}{T}\right) e^{-j(\frac{\omega}{T} - k\frac{2\pi}{T})\tau}$$

$$\Leftrightarrow \qquad (3)$$

$$x(t-\tau)\sum_n \delta(t - nT) \;=\; \sum_n x(nT - \tau)\,\delta(t - nT)$$

[1]Note from Equ.(1) that $X_{S_\tau}(\Omega)$ is $2\pi/T$-periodic only when τ is any multiple of T. Then only in that case does it follow that $X_{S_\tau}(\Omega) = X\!\left(e^{j\omega}\right)$; $\tau = qT$, $\Omega = \frac{\omega}{T}$, for q an integer. [O&S,pg.87,Equ.(3.18)] This is true whether or not $X(\Omega)$ is bandlimited.

2.1 Interpretation

Comparing the continuous-time domain signal in Equ.(1) to that in Equ.(3), we see that the only discrepancy is the absolute time location of the samples as determined by the shah. The sample values are the same in both equations. But the sequence $x[n]$ is derived from the synchronous sampling in Equ.(3) as the sample values of the time-shifted continuous-time signal;

$$x[n] \equiv x(nT - \tau) \qquad (4)$$

Implicit is the convention that a "sequence" is always associated with synchronous sampling, regardless of the actual time origin of the sample values. From the point of view of the sequence domain, the Fourier transform is 2π-periodic as indicated by Equ.(3). In fact, the Fourier transform of all sequences is 2π-periodic by definition, regardless of the underlying and perhaps unknown shah synchronicity. This is in sharp contrast to the continuous-time transform Equ.(1) that is not necessarily $2\pi/T$-periodic.

We must remember to interpret the discrete-time phase as 2π-periodic; that is to say, $e^{-j\omega\tau/T}$ on the left-hand side of Equ.(3) must be adjusted at every replication as indicated on the right-hand side.[2]

> *Because we can design a digital filter having a frequency response arbitrarily close to $e^{-j\omega\tau/T}$, then from Equ.(3) we may conclude that when the original analog signal is bandlimited, there is little difference between fractionally time-shifting a sequence, and synchronously sampling the corresponding time-shifted analog signal.*

In other words, from the point of view of the sequence domain, both of the aforementioned operations are identical. It is therefore possible to design a digital filter for the purpose of delaying any sequence by a fraction of a sample. What actually becomes delayed, in the steady state, is the bandlimited analog signal that uniquely corresponds to that sequence.[3]

In the case that $X(\Omega)$ were not bandlimited, the discrete-time linear phase term $e^{-j\omega\tau/T}$ on the left-hand side of Equ.(3) becomes invalid because the phase term could not be so simply derived from the right-hand side. Time shifting and sampling are no longer interchangeable, and the sequence $x[n]$ now corresponds to some other bandlimited analog signal that can be determined by frequency-domain aliasing of the original analog signal.

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[2]This 2π-periodic adjustment is the conventional specification of linear phase in DSP.

[3]The fact that there is associated with any sequence a unique bandlimited continuous-time signal, is a consequence of *Whittaker's Theorem.*

3. Linear Phase

As it turns out, we may use the framework of the asynchronous shah to explain a curious 4π–periodicity phenomenon that arises commonly in fundamental FIR filter design. We begin with a simple example and then generalize the results.

Example 1

$$H(z) = 1 + z^{-1}$$

This transfer function corresponds to a sequence;

$$h[n] = \delta[n] + \delta[n-1]$$

Thus we know a priori that its Fourier transform

$$H(e^{j\omega}) = 1 + e^{-j\omega} \qquad ; |\omega| < \pi$$

is a 2π-periodic function of frequency ω. Each term of $H(e^{j\omega})$ is also 2π-periodic due to the assumption of linearity. $H(e^{j\omega})$ can be written equivalently as

$$H(e^{j\omega}) = 2\cos(\omega/2)\, e^{-j\omega/2} \qquad ; |\omega| < 2\pi \qquad (4A)$$

An important point here is to recognize that the exponential term has been expanded in the frequency domain by a factor of 2 as in Figure 1; i.e., the argument of $e^{-j\omega/2}$ is 4π-periodic as is $\cos(\omega/2)$.

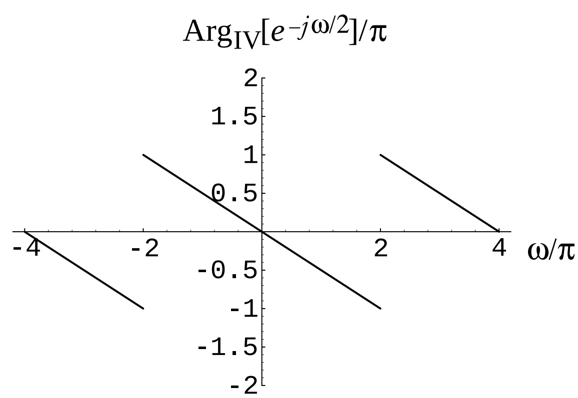


Figure 1. Phase of $e^{-j\omega}$ expanded in ω by factor of 2.

Equ.(4A) is not exactly in the form of Equ.(3) because neither $\cos(\omega/2)$ nor the argument of $e^{-j\omega/2}$ are 2π-periodic. That could be easily remedied by taking the absolute value of the cosine term thus forcing 2π-periodicity into the argument of the delay term. Notwithstanding, $H(e^{j\omega})$ remains 2π-periodic overall as required by Equ.(3), and by our interpretation of Equ.(3) there exists a unique bandlimited analog signal that we may associate with $H(e^{j\omega})$. Hence, the corresponding bandlimited analog signal $h(t-T/2)$ that becomes synchronously sampled to yield $h[n]$ as in Equ.(4), is calculated below and illustrated in Figure 2.

$$H(e^{j\omega}) \Leftrightarrow h(t-\tau)\sum_n \delta(t-nT)$$

$$h(t-\tau) \;=\; h_e(t-T/2) \;=\; \frac{1}{2\pi}\int_{-\pi}^{\pi} 2\cos\!\left(\frac{\omega}{2}\right) e^{-j\omega/2}\, e^{j\omega t/T}\, d\omega$$

$$= \frac{\sin(\pi t/T)}{(\pi t/T)(1-t/T)}$$

$$h[n] = h(nT - T/2) \qquad\qquad ;\ \tau = \frac{T}{2}$$

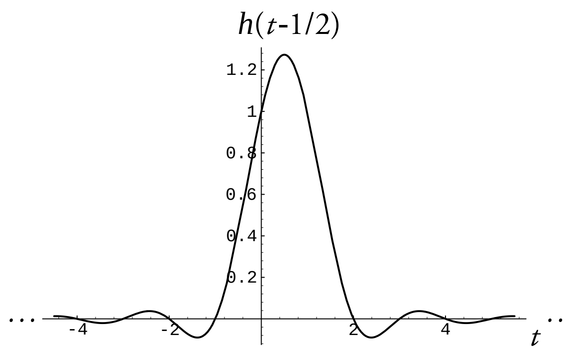


Figure 2. $h[n]$ results when synchronously sampled. T=1.

From [O&S,pg.255,Equ.(5.135)] we have the expression for generalized linear phase in a discrete-time signal or system;

$$H(e^{j\omega}) = A(e^{j\omega})\; e^{-j\omega\alpha + j\beta} \tag{5.135}$$

where $A(e^{j\omega})$ is real (including negative real), α is a unitless constant but can be considered a real number of samples delayed, and the constant phase shift β has units of radians.[4] Because $H(e^{j\omega})$ corresponds to a sequence, it is necessarily 2π-periodic. $A(e^{j\omega})$ is <u>not</u> necessarily 2π–periodic, however. The argument of $e^{-j\omega\alpha}$ has the same periodicity as $A(e^{j\omega})$, and the periodicity is 2π when α is an integer. In Equ.(4A) we saw that when α=1/2, then the periodicity is 4π. Because of these facts, we prefer a notation that reminds us that the periodicity is related to α ;

$$H(e^{j\omega}) = A(e^{j\omega\alpha})\; e^{-j\omega\alpha + j\beta} \qquad ;\; A(e^{j\omega\alpha}) = A(e^{j\omega}) \tag{5}$$

Sequence symmetry is a sufficient condition for generalized linear phase. There are only two cases of symmetry with regard to $h[n]$ that are of interest; symmetric and anti-symmetric. A symmetric sequence (FIR Type I, II) takes time-symmetric samples about αT of continuous functions of the form shown in Figure 3(a), while an anti-symmetric sequence (FIR Type III, IV) takes time-symmetric samples about αT of functions such as that depicted in Figure 3(b).

(a)

(b)

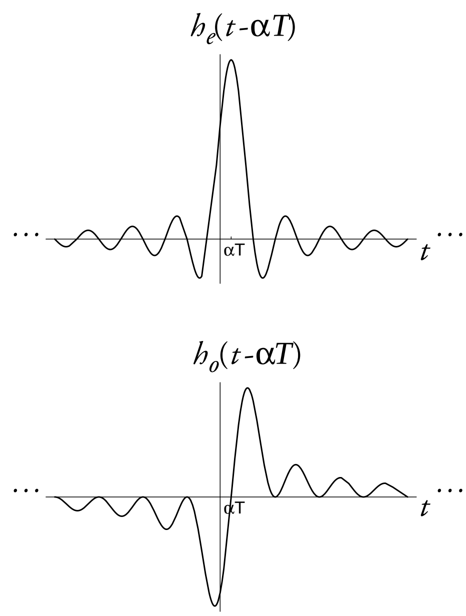


Figure 3. (a) Typical symmetric impulse response centered about αT. β=0.
(b) Typical anti-symmetric impulse response centered about αT. $\beta=\pi/2$.

[4]The choice of β is not arbitrary under generalized linear phase. If it were arbitrary, it would result in phase distortion as opposed to "shift".

Sequence symmetry is a sufficient condition for linear phase, but not a necessary condition. Any sequence having a frequency response that can be put into the general form of Equ.(5) will be considered linear in phase. When α=0, the Fourier transform of Figure 3(a) is pure real (zero phase,β=0) whereas the transform of Figure 3(b) is pure imaginary (constant phase shift,$\beta=\pi/2$). Assuming that the continuous-time impulse responses in Figure 3 each have a corresponding bandlimited frequency response,[5] <u>any value of α and any amount of shah asynchrony τ will result in a linear phase sequence if we take an infinite number of samples above the Nyquist rate</u>. [O&S,ch.5.7.1] The sequence symmetry condition only becomes <u>necessary</u> for linear phase when we demand <u>finite</u> length impulse response; i.e., FIR design. That necessity then limits the available choices of τ given α.

As mentioned early on, the shah asynchrony cases $\tau = 0$ and $\tau = T/2$ are noteworthy. We further distinguish these two cases because they represent the only cases of shah time-symmetry about absolute time 0. As sequence symmetry is of interest, so are these two cases of asynchrony. But as we already learned from Equ.(3) which is the point of view of the sequence domain, it makes no difference whether we time-shift a sequence, or synchronously sample the corresponding time-shifted bandlimited analog signal. Hence without loss of generality, we choose τ to be zero because no cases of sequence symmetry will be lost as a consequence.

Nonetheless, we discover a strong bond between asynchronous sampling, Equ.(1) and Equ.(3), and the generalized linear phase description, Equ.(5). The relationship may be expressed,

$$A(e^{j\omega\alpha})\, e^{j\beta} \quad \Leftrightarrow \begin{cases} h_e(t) \displaystyle\sum_n \delta(t - nT + \alpha T) & ;\ \beta = 0 \\ h_o(t) \displaystyle\sum_n \delta(t - nT + \alpha T) & ;\ \beta = \dfrac{\pi}{2} \end{cases} \tag{6}$$

$$H(e^{j\omega}) = A(e^{j\omega\alpha})\, e^{-j\omega\alpha + j\beta} \Leftrightarrow \begin{cases} h_e(t - \alpha T) \displaystyle\sum_n \delta(t - nT) & ;\ \beta = 0 \\ h_o(t - \alpha T) \displaystyle\sum_n \delta(t - nT) & ;\ \beta = \dfrac{\pi}{2} \end{cases} \tag{7}$$

$$h(t - \alpha T) \ = \ \frac{1}{2\pi} \int_{-\pi}^{\pi} A(e^{j\omega\alpha})\, e^{-j\omega\alpha + j\beta}\ e^{j\omega t/T}\, d\omega$$

where αT now assumes the earlier role of τ. Equ.(6) is not necessarily 2π-periodic, while Equ.(7) is; that is precisely the same relationship as Equ.(1) to Equ.(3). Like τ in Equ.(1), αT in Equ.(6) controls the periodicity of the generalized amplitude $A(e^{j\omega\alpha})$. With τ set to 0, there remain only two choices of α which produce sequence symmetry, hence linear phase; α=0 and α=1/2. The

[5]The explanation of this requirement for bandlimiting is just like the interpretation of Equ.(3).

pure realness or imaginariness of $A(e^{j\omega\alpha})\ e^{j\beta}$ is then determined solely by the $h(t)$ waveform symmetry. The desire for linear phase is that which demands sequence symmetry as well. Hence, it is more difficult to design fractional delay FIR filters for other values of delay, α.

When the sample values $h[n]$ are then taken as in Equ.(4),

$$h[n] \equiv h_e(nT) \qquad ;\ \text{Type } \mathbf{I}\text{:}\ \ \alpha = 0,\ \beta = 0$$

then we have what is called the Type I linear phase FIR filter. [O&S,pg.257] Similarly, the three other types of linear phase FIR filters that arise because of sequence symmetry can be found;

$$h[n] \equiv h_e(nT - \alpha T) \qquad ;\ \text{Type } \mathbf{II}\text{:}\ \ \alpha = \frac{1}{2},\ \beta = 0$$

$$h[n] \equiv h_o(nT) \qquad ;\ \text{Type } \mathbf{III}\text{:}\ \ \alpha = 0,\ \beta = \frac{\pi}{2}$$

$$h[n] \equiv h_o(nT - \alpha T) \qquad ;\ \text{Type } \mathbf{IV}\text{:}\ \ \alpha = \frac{1}{2},\ \beta = \frac{\pi}{2}$$

The characteristics of both the sequence and its transform are summarized in Table 1 along with the values of the other related parameters. We see from the table that α=1/2 will result in a 4π-periodic generalized amplitude (and generalized linear phase). The same also holds true when α equals any odd multiple of 1/2.

Table 1. Symmetry and Synchronicity

FIR Type	**I**	**II**	**III**	**IV**
$\boldsymbol{h[n]}$	sym.	sym.	anti-sym.	anti-sym.
$\boldsymbol{A(e^{j\omega\alpha})}$	Sym.	Sym.	Anti-Sym.	Anti-Sym.
Periodicity	2π	4π	2π	4π
$\boldsymbol{A(e^{j\omega\alpha})}$	Real	Real	Imag.	Imag.
α	0	1/2	0	1/2
β	0	0	$\pi/2$	$\pi/2$

Example 2

$$H(e^{j\omega}) = e^{-j\omega/2} \qquad ; |\omega| < \pi \qquad (8)$$

The magnitude $|1|$ and phase $-\omega/2$ of this frequency response are 2π–periodic by definition. The corresponding bandlimited analog signal $h_e(t\text{-}T/2)$ that becomes synchronously sampled to yield $h[n]$ as in Equ.(4), is calculated below and shown in Figure 4.

$$H(e^{j\omega}) \Leftrightarrow h(t-\alpha T)\sum_n \delta(t-nT)$$

$$h(t-\alpha T) = h_e(t-T/2) = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{-j\omega/2}\, e^{j\omega t/T}\, d\omega$$

$$= \frac{\sin(\pi(t/T-1/2))}{\pi(t/T-1/2)}$$

$$h[n] = h(nT-T/2) = \frac{\sin(\pi(n-1/2))}{\pi(n-1/2)} \quad ;\ \alpha = \frac{1}{2},\ [\text{O\&S}, (5.128); \omega_c = \pi]$$

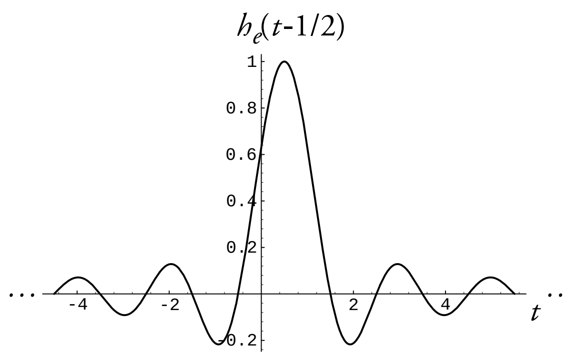


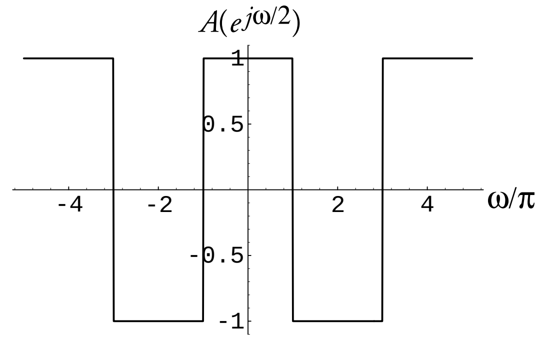
Figure 4. $h[n]$ results when synchronously sampled. T=1.

From our discussion of linear phase, we see that $H(e^{j\omega})$ in Equ.(8) can be expressed in the generalized form of Equ.(5) with $\beta=0$;

$$H(e^{j\omega}) = A(e^{j\omega/2})\ e^{-j\omega/2} \qquad ;\ |\omega| < 2\pi \qquad (9)$$

When written in this form, generalized amplitude and linear phase, we know that $A(e^{j\omega/2})$ is 4π-periodic as is $-\omega/2$ the generalized linear phase, because $\alpha=1/2$ in Equ.(6). This periodicity is illustrated in Figure 5. $H(e^{j\omega})$ remains 2π–periodic.

(a)



(b)

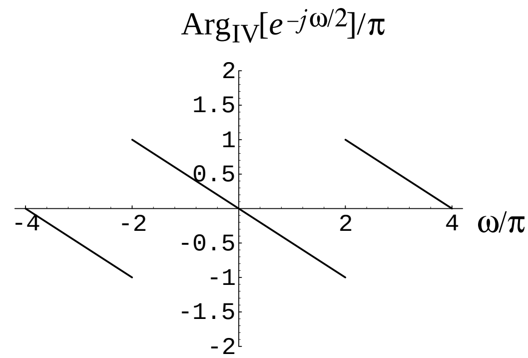


Figure 5. Generalized amplitude (a) and phase (b) for Example 2 are 4π–periodic.

That $A(e^{j\omega/2})$ must alternate between ± 1 with a period of 4π can be verified by shifting the generalized linear phase form Equ.(9) where no assumption is made regarding the periodicity of its individual terms;

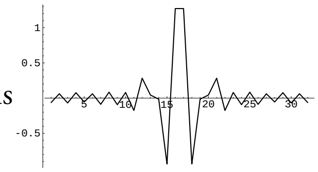
$$H(e^{j\omega}) = A(e^{j\omega/2})\ e^{-j\omega/2} = H(e^{j(\omega+2\pi)}) = A(e^{j(\omega+2\pi)/2})\ e^{-j(\omega+2\pi)/2}$$

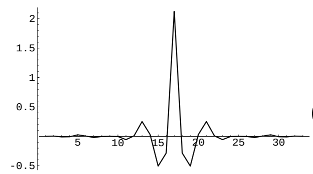
where $e^{-j(\omega+2\pi)/2} = -e^{-j\omega/2}$.

4 Application:

LINEAR PHASE FIR FILTER DESIGN by IDFT

A Cookbook

The technique of frequency-domain sampling and IDFT is useful when it is desired to make a filter having arbitrary frequency response. The result is a discrete impulse response that looks like this 

(order M odd), or like this 

(order M even). At the risk of leading one down a garden path, we present a method of solution based upon the concepts presented earlier; generalized amplitude and linear phase. We point out that while this may be an intuitive approach in one sense, there is an easier approach, based on the familiar concept of magnitude and phase, presented at the end of this section.

We consider only the case that the desired frequency response $A(e^{j\omega})$ is **pure real** and **even**;[6] i.e., FIR Types I and II. We make the impulse response corresponding to $A(e^{j\omega})$ causal via the delay

$$\tau = \frac{M}{2}$$

where ***M*** = desired filter order. If we forget to incorporate the delay term, then we will always get an impulse response that looks like the "M even" case but with an extra sample tacked on one end when M is actually odd. That extra sample destroys the time-domain symmetry and the linear phase. (Implicit in this method is some underlying continuous-time sinc() that is being sampled in one of two ways depending upon the order M.)

The IDTFT is

$$\begin{aligned} h[n] &= \frac{1}{2\pi}\int_{-\pi}^{\pi} H(e^{j\omega})\, e^{j\omega n}\, d\omega \\ &= \frac{1}{2\pi}\int_{-\pi}^{\pi} \left(A(e^{j\omega})\, e^{-j\omega\tau}\right) e^{j\omega n}\, d\omega \end{aligned}$$

Recall that when $h[n]$ is finite length, the DFT is simply the DTFT evaluated in frequency by some simple function of k.

$$H[k] \equiv H(e^{j\omega})\big|_{\omega=f(k)}$$

Then we may approximate $h[n]$ via the IDFT. It is critical that the IFFT input buffer $H[k]$ be aligned with absolute frequency 0. Anticipating that, we write the IDTFT in the equivalent form:

$$h[n] = \frac{1}{2\pi}\int_{0}^{\pi} A(e^{j\omega})\, e^{-j\omega\tau}\, e^{j\omega n}\, d\omega + \frac{1}{2\pi}\int_{\pi}^{2\pi} A\left(e^{j(\omega-2\pi)}\right) e^{-j(\omega-2\pi)\tau}\, e^{j\omega n}\, d\omega \qquad (1)$$

The second integral in (1) holds the left half of the first periodic-replication[7] of $H(e^{j\omega})$.

[6] FIR filter Types III and IV produce phase distortion because of their constant $\pi/2$ phase shift.

[7] We assume that $A(e^{j(\omega-2\pi)}) = A(e^{j\omega})$. Hence the delay term in τ must be coerced into periodicity via the subtraction of 2π.

Let the number of frequency domain samples be equal to $\boldsymbol{N} = M+1$, the IDFT length; the length of our approximation to $h[n]$. As in the time domain, we also consider two methods of sampling in the frequency domain, and we consider M even or odd under each method separately. For linear phase to occur, the frequency domain samples must also be symmetrical. But in the discrete frequency domain, the symmetry we are concerned with is circular; i.e., the resulting pattern of samples when viewed along the unit circle, not along the ω-axis. [Rabiner/Gold,pg.113]

$$h[n] = \frac{1}{2\pi}\int_0^{\pi} A\left(e^{j\omega}\right) e^{-j\omega\tau}\, e^{j\omega n}\, d\omega + \frac{1}{2\pi}\int_{\pi}^{2\pi} A\left(e^{j(\omega-2\pi)}\right) e^{-j(\omega-2\pi)\tau}\, e^{j\omega n}\, d\omega \qquad (1)$$

DC-Aligned frequency domain sampling:

$$\omega \to \frac{2\pi k}{N}$$

Case 1, M even, N odd: Type I linear phase FIR.
From equation (1),

$$\begin{cases} H[k] = A[k]\, e^{-j\frac{2\pi k}{N}\frac{M}{2}} = A[k]\, e^{-j\frac{\pi k M}{N}} & ;\ k = 0 \to \dfrac{N-1}{2} \\ H[k] = A[k-N]\, e^{-j\frac{2\pi k}{N}\frac{M}{2}}\, e^{j2\pi\frac{M}{2}} = A[k]\, e^{-j\frac{\pi k M}{N}} & ;\ k = \dfrac{N+1}{2} \to N-1 \end{cases}$$

Note that $A[k-N] = A[k]$.

$$\boxed{\Rightarrow\ H[k] = A[k]\, e^{-j\frac{\pi k M}{N}} \qquad ;\ k = 0 \to N-1}$$

Another useful representation results when we set $k\text{-}N=q$. For then we get,

$$\begin{cases} H[k] = A[k]\, e^{-j\frac{\pi k M}{N}} & ;\ k = 0 \to \dfrac{N-1}{2} \\ H[q+N] = A[q]\, e^{-j\frac{\pi q M}{N}} & ;\ q = -\dfrac{N-1}{2} \to -1 \end{cases}$$

$\omega \to 2\pi q/N$. This formulation allows us to think in terms of the two-sided spectrum.

$$h[n] = \frac{1}{2\pi}\int_0^{\pi} A\left(e^{j\omega}\right) e^{-j\omega\tau}\, e^{j\omega n}\, d\omega + \frac{1}{2\pi}\int_{\pi}^{2\pi} A\left(e^{j(\omega-2\pi)}\right) e^{-j(\omega-2\pi)\tau}\, e^{j\omega n}\, d\omega \qquad (1)$$

DC-Aligned frequency domain sampling: *(cont.)*

$$\omega \to \frac{2\pi k}{N}$$

Case 2, M odd, N even: Type II linear phase FIR.
From equation (1),

$$\begin{cases} H[k] = A[k]\, e^{-j\frac{2\pi k}{N}\frac{M}{2}} = A[k]\, e^{-j\frac{\pi k M}{N}} & ;\ k = 0 \to \frac{N}{2} - 1 \\ H\left[\frac{N}{2}\right] = 0 & \\ H[k] = A[k-N]\, e^{-j\frac{2\pi k}{N}\frac{M}{2}}\, e^{j2\pi\frac{M}{2}} = -A[k]\, e^{-j\frac{\pi k M}{N}} & ;\ k = \frac{N}{2} + 1 \to N - 1 \end{cases}$$

Note that $A[k\text{-}N] = A[k]$. The zero is demanded by the desired symmetry of the impulse response. [Oppenheim/Schafer,*D-TSP*,pg.265]

An alternate representation for the two-sided spectrum is obtained by setting $k\text{-}N\text{=}q$:

$$\begin{cases} H[k] = A[k]\, e^{-j\frac{\pi k M}{N}} & ;\ k = 0 \to \frac{N}{2} - 1 \\ H\left[\frac{N}{2}\right] = 0 & \\ H[q+N] = A[q]\, e^{-j\frac{\pi q M}{N}} & ;\ q = -\frac{N}{2} + 1 \to -1 \end{cases}$$

$\omega \to 2\pi q/N$. This formulation allows us to think in terms of the two-sided spectrum.

$$h[n] = \frac{1}{2\pi}\int_0^{\pi} A\left(e^{j\omega}\right) e^{-j\omega\tau}\, e^{j\omega n}\, d\omega \;+\; \frac{1}{2\pi}\int_{\pi}^{2\pi} A\left(e^{j(\omega-2\pi)}\right) e^{-j(\omega-2\pi)\tau}\, e^{j\omega n}\, d\omega \qquad (1)$$

Non-DC-Aligned frequency domain sampling:

$$\omega \rightarrow \frac{2\pi\left(k+\frac{1}{2}\right)}{N}$$

<u>Case 3</u>, <u>M even</u>, <u>N odd</u>: Type I linear phase FIR.
From equation (1),

$$\begin{cases} H[k] = A[k]\, e^{-j\frac{2\pi\left(k+\frac{1}{2}\right)}{N}\frac{M}{2}} = A[k]\, e^{-j\frac{\pi\left(k+\frac{1}{2}\right)M}{N}} & ;\; k = 0 \rightarrow \dfrac{N-1}{2} \\ H[k] = A[k-N]\, e^{-j\frac{2\pi\left(k+\frac{1}{2}\right)}{N}\frac{M}{2}}\, e^{j2\pi\frac{M}{2}} = A[k]\, e^{-j\frac{\pi\left(k+\frac{1}{2}\right)M}{N}} & ;\; k = \dfrac{N+1}{2} \rightarrow N-1 \end{cases}$$

Note that $A[k\text{-}N] = A[k]$.

$$\left[\boxed{\Rightarrow H[k] = e^{-j\frac{\pi M}{2N}}\left\{A[k]\, e^{-j\frac{\pi k M}{N}}\right\} \quad ;\; k = 0 \rightarrow N-1}\right]$$

Another useful representation results when we set $k\text{-}N\text{=}q$. For then we get,

$$\begin{cases} H[k] \qquad = e^{-j\frac{\pi M}{2N}}\, A[k]\, e^{-j\frac{\pi k M}{N}} & ;\; k = 0 \rightarrow \dfrac{N-1}{2} \\ H[q+N] = e^{-j\frac{\pi M}{2N}}\, A[q]\, e^{-j\frac{\pi q M}{N}} & ;\; q = -\dfrac{N-1}{2} \rightarrow -1 \end{cases}$$

$\omega \rightarrow 2\pi(q + 1/2)/N$. This formulation allows us to think in terms of the two-sided spectrum.

The evaluation of the IDTFT kernel $e^{j\omega n}$ at the specified frequencies demands that

$$h[n] \leftarrow e^{j\frac{\pi n}{N}}\, h[n]$$

$$h[n] = \frac{1}{2\pi}\int_0^{\pi} A\left(e^{j\omega}\right) e^{-j\omega\tau}\, e^{j\omega n}\, d\omega \;+\; \frac{1}{2\pi}\int_{\pi}^{2\pi} A\left(e^{j(\omega-2\pi)}\right) e^{-j(\omega-2\pi)\tau}\, e^{j\omega n}\, d\omega \qquad (1)$$

Non-DC-Aligned frequency domain sampling: *(cont.)*

$$\omega \rightarrow \frac{2\pi\left(k+\frac{1}{2}\right)}{N}$$

Case 4, M odd, N even: Type II linear phase FIR.
From equation (1),

$$\begin{cases} H[k] = A[k]\, e^{-j\frac{2\pi\left(k+\frac{1}{2}\right)}{N}\frac{M}{2}} = A[k]\, e^{-j\frac{\pi\left(k+\frac{1}{2}\right)M}{N}} & ;\ k = 0 \rightarrow \frac{N}{2}-1 \\ H[k] = A[k-N]\, e^{-j\frac{2\pi\left(k+\frac{1}{2}\right)}{N}\frac{M}{2}}\, e^{j2\pi\frac{M}{2}} = -A[k]\, e^{-j\frac{\pi\left(k+\frac{1}{2}\right)M}{N}} & ;\ k = \frac{N}{2} \rightarrow N-1 \end{cases}$$

Note that $A[k\text{-}N] = A[k]$.

$$\begin{cases} H[k] = e^{-j\frac{\pi M}{2N}}\ \left\{\ A[k]\, e^{-j\frac{\pi k M}{N}}\right\} & ;\ k = 0 \rightarrow \frac{N}{2}-1 \\ H[k] = e^{-j\frac{\pi M}{2N}}\ \left\{-A[k]\, e^{-j\frac{\pi k M}{N}}\right\} & ;\ k = \frac{N}{2} \rightarrow N-1 \end{cases}$$

An alternate representation for the two-sided spectrum is obtained by setting $k\text{-}N\text{=}q$:

$$\begin{cases} H[k] \qquad = e^{-j\frac{\pi M}{2N}}\ \left\{A[k]\, e^{-j\frac{\pi k M}{N}}\right\} & ;\ k = 0 \rightarrow \frac{N}{2}-1 \\ H[q+N] = e^{-j\frac{\pi M}{2N}}\ \left\{A[q]\, e^{-j\frac{\pi q M}{N}}\right\} & ;\ q = -\frac{N}{2} \rightarrow -1 \end{cases}$$

$\omega \rightarrow 2\pi(q + 1/2)/N$. This formulation allows us to think in terms of the two-sided spectrum.

The evaluation of the IDTFT kernel $e^{j\omega n}$ at the specified frequencies demands that

$$h[n] \leftarrow e^{j\frac{\pi n}{N}}\, h[n]$$

4.1 An Easier Way

The foregoing cookbook was prepared from the point of view of generalized amplitude and linear phase. From our study *Phase Response* (on this site) we learn that it is in fact easier to solve the problem using the point of view of magnitude and phase for all the cases. Use $\boldsymbol{N} = M+1$ points and the IDFT; not a power-of-2 type program.

Case 1) M is even. Just sample the frequency domain using the magnitude and phase representation. The phase is always periodic in 2π. The baseband and first replication of phase are respectively $-\omega M/2$ and $-(\omega - 2\pi)M/2$. Hence their difference $M\pi$ is a multiple of 2π at the discontinuity.

Case 2) M is odd. Again sample the frequency domain using the magnitude and phase representation. The phase difference $M\pi$ is an odd multiple of π at the phase discontinuity. That accounts for the required sign inversion. Remember that for linear phase and M odd, the sample at $z=-1$ must be zero.

Case 3) M is even. The main difference here is that the frequency samples miss DC. The phase term is $e^{-j\omega M/2} = e^{-j2\pi(k+1/2)/N\, M/2} = e^{-j\pi(k+1/2)M/N}$. Again, the phase difference at the phase discontinuity is a multiple of 2π. The problem with this method is that the IDFT buffer, which requires absolute frequency alignment, is loaded with left shifted (by 1/2 bin) frequency domain DTFT samples. So after IDFT, we must compensate in the time domain via the modulation term $e^{j\pi n/N}$.

Case 4) M is odd. The frequency sample at $z=-1$ is missed, so not of concern. The phase difference is an odd multiple of π at the phase discontinuity, which accounts for the required sign inversion. Time domain compensation is again required because of the frequency shift into the IDFT buffer.

Closed-form solutions can be found in [McClellan,pp.251-252].

References

[O&S] Alan V. Oppenheim, Ronald W. Schafer, *Discrete-Time Signal Processing*, 1989, Prentice-Hall, Inc., Englewood Cliffs, NJ 07632 USA, Internet: http://www.prenhall.com/

[Bracewell] Ronald N. Bracewell, *The Fourier Transform and Its Applications*, Second Edition, Revised, 1986, McGraw–Hill

[McClellan] McClellan, Burrus, Oppenheim, et al., *Signal Processing using Matlab 5*, Prentice Hall, 1998

[Rabiner/Gold] Rabiner, Gold, *Theory and Applications of Digital Signal Processing*, Prentice Hall, 1975