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REMARKS TO MAURICE FRÉCHET'S ARTICLE "SUR LA DÉFINITION AXIOMATIQUE D'UNE CLASSE D'ESPACE DISTANCIÉS VECTORIELLEMENT APPLICABLE SUR L'ESPACE DE HILBERT[1]

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1. Fréchet's developments in the last section of his article suggest an elegant solution of the following problem.

Let

$$a_{ik} = a_{ki} \qquad (i \neq k\,; i, k = 0, 1, \cdots, n)$$

be $\frac{1}{2}n(n+1)$ *given positive quantities. What are the necessary and sufficient conditions that they be the lengths of the edges of a n-simplex* $A_0A_1 \cdots A_n$? *More general, what are the conditions that they be the lengths of the edges of a n-"simplex"*[2] $A_0A_1 \cdots A_n$ *lying in a euclidean space* R_r $(1 \leqq r \leqq n)$ *but not in a* R_{r-1}?

This problem is fundamental in K. Menger's metric investigation of euclidean spaces ([6] and [7], particularly his third fundamental theorem in [7], pp. 737–743). It was solved by Menger by means of equations and inequalities involving certain determinants. Theorem 1 below furnishes a complete and independent solution of this problem. Theorem 2 solves the similar problem for spherical spaces previously treated by Menger's methods by L. M. Blumenthal and G. A. Garrett ([1]) and Laura Klanfer ([5]); it may be conveniently applied (Theorems 3 and 3′) to prove and extend a theorem of K. Gödel ([4]). The method of Theorem 1 is finally applied to solve the corresponding problem for spaces with indefinite line element recently considered by A. Wald ([8]) and H. S. M. Coxeter and J. A. Todd ([2]).

Construction of simplexes of given edges in euclidean spaces

2. A complete answer to the questions stated above is given by the following theorem.

Theorem 1. *A necessary and sufficient condition that the* a_{ik} *be the lengths of the edges of an n-"simplex"* $A_0A_1 \cdots A_n$ *lying in* R_r, *but not in* R_{r-1}, *is that the quadratic form*

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[1] These Annals, vol. 36 (1935), pp. 705–718.

[2] The quotation marks should indicate that the configuration may lie in a euclidean space of less than n dimensions.

$$(1) \qquad F(x_1, x_2, \cdots, x_n) = \sum_{i=1}^{n} a_{0\,i}^2\, x_i^2 + \sum_{\substack{i,k=1 \\ (i<k)}}^{n} (a_{0\,i}^2 + a_{0\,k}^2 - a_{i\,k}^2)\, x_i x_k$$

$$= \tfrac{1}{2} \sum_{i,k=1}^{n} (a_{0\,i}^2 + a_{0\,k}^2 - a_{i\,k}^2)\, x_i x_k$$

$$(\text{with } a_{ik} = 0 \text{ if } i = k)$$

be positive, i.e. *always* $\geqq 0$, *and of rank* r.

The condition is necessary. Let $A_0 A_1 \cdots A_n$ be an n-"simplex" with $A_i A_k = a_{ik}$. Let $A_0 = 0$ be the origin of a R_n in which A_i has the cartesian coördinates $\alpha_{i1}, \alpha_{i2}, \cdots, \alpha_{in}$. The point (in vector space notation)

$$P = x_1 A_1 + x_2 A_2 + \cdots + x_n A_n = (\xi_1, \xi_2, \cdots, \xi_n)$$

has the coördinates

$$\xi_\nu = x_1 \alpha_{1\nu} + x_2 \alpha_{2\nu} + \cdots + x_n \alpha_{n\nu} \qquad (\nu = 1, \cdots, n),$$

whence

$$\overline{OP}^2 = \| P \|^2 = \sum_{1}^{n} \xi_\nu^2 = \sum_{\nu=1}^{n} (x_1 \alpha_{1\nu} + \cdots + x_n \alpha_{n\nu})^2$$

$$= \sum_{i=1}^{n} x_i^2 \sum_{\nu=1}^{n} \alpha_{i\,\nu}^2 + 2 \sum_{i<k} x_i x_k \sum_{\nu=1}^{n} \alpha_{i\nu}\alpha_{k\nu}.$$

Since

$$\sum_{\nu=1}^{n} \alpha_{i\,\nu}^2 = \overline{OA}_i^2 = a_{0\,i}^2,$$

$$2 \sum_{\nu=1}^{n} \alpha_{i\nu}\alpha_{k\nu} = \sum_{\nu=1}^{n} \alpha_{i\,\nu}^2 + \sum_{\nu=1}^{n} \alpha_{k\,\nu}^2 - \sum_{\nu=1}^{n} (\alpha_{i\nu} - \alpha_{k\nu})^2 = \overline{A_0 A}_i^2 + \overline{A_0 A}_k^2 - \overline{A_i A}_k^2$$

$$= a_{0\,i}^2 + a_{0\,k}^2 - a_{i\,k}^2,$$

we have

$$(2) \qquad \overline{OP}^2 = \| x_1 A_1 + \cdots + x_n A_n \|^2 = F(x_1, x_2, \cdots, x_n).$$

Hence $F(x_1, \cdots, x_n)$ *is positive.* It follows furthermore from our assumptions that $P = 0$, hence $F = 0$, on a linear manifold of $n - r$ dimensions in the variables $x_1, \cdots, x_n$; hence F *is of rank* r.

The condition is sufficient. Let us first assume F to be positive definite, i.e. $r = n$. By means of a certain linear non-singular transformation

$$(3) \qquad (y) = H(x)$$

we get the identity

$$(4) \qquad F(x_1, \cdots, x_n) = y_1^2 + y_2^2 + \cdots + y_n^2.$$

Call A_0 the origin of the cartesian space of the variables $(y_1, \cdots, y_n)$ and

$$A_1, A_2, \cdots, A_n,$$

the n points which in virtue of (3) correspond to

$$(5)\quad (x_1, x_2, \cdots, x_n) = (1, 0, \cdots, 0), (0, 1, 0, \cdots, 0), \cdots, (0, 0, \cdots, 0, 1),$$

respectively. Their y-coördinates are readily found by (3). For their mutual distances we find by (3), (4) and (5),

$$\overline{A_0A}_i^2 = F(0, \cdots, \overset{(i)}{1}, \cdots, 0) = a_{0i}^2,$$

$$\overline{A_iA}_k^2 = F(0, \cdots, \overset{(i)}{1}, \cdots, \overset{(k)}{-1}, \cdots, 0) = a_{0i}^2 + a_{0k}^2 - (a_{0i}^2 + a_{0k}^2 - a_{ik}^2)$$
$$= a_{ik}^2, (i < k)$$

which show that $A_0A_1 \cdots A_n$ is precisely the n-simplex we are looking for. It is indeed an n-simplex because the points (5) are independent and (3) is non-singular.

If $r < n$, then (4) has to be replaced by

$$(6)\qquad F(x_1, \cdots, x_n) = y_1^2 + y_2^2 + \cdots + y_r^2.$$

The above procedure gives an n-simples $A_0A_1 \cdots A_n$, however the quantities

$$F(1, 0, \cdots, 0) = a_{01}^2, \qquad F(1, -1, 0, \cdots, 0) = a_{12}^2, \cdots$$

are no more the squared lengths of the edges $\overline{A_0A}_1^2, \overline{A_1A}_2^2, \cdots$, but, viewing (6), the squared lengths of their projections on the sub-space $(y_1, \cdots, y_r)$, i.e., on the manifold $y_{r+1} = \cdots = y_n = 0$. Hence the projection $A_0'A_1' \cdots A_n'$ on this manifold of the n-simplex $A_0A_1 \cdots A_n$ is an n-"simplex" of the type we are looking for, i.e. with $A_i'A_k' = a_{ik}$. This n-"simplex" $A_0'A_1' \cdots A_n'$ is by construction contained in a R_r but not in a R_{r-1}, as readily seen.

Remark. If the matrix H of (3) is $H = \| h_{ik} \|$, then the y-coördinates of the vertices A_i and A_i' are

$$A_i = (h_{1i}, h_{2i}, \cdots, h_{ni}), \qquad A_i' = (h_{i1}, h_{2i}, \cdots, h_{ri}, 0, \cdots, 0).$$

The actual construction (i.e. determination of the coördinates of its vertices) of an n-"simplex" of edges a_{ik} is therefore carried out by a reduction of the quadratic form (1) *to its canonical form* (6). This is a problem of the second degree, for the transformation (3) is by no means required to be orthogonal.

As an illustration of this method let us construct a regular n-simplex with $a_{ik} = 1$. By (1) we have

$$F(x_1, \cdots, x_n) = \sum_{i=1}^{n} x_i^2 + \sum_{i<k} x_ix_k.$$

The identity

$$F(x_1, \cdots, x_n) = \sum_{i=1}^{n} \frac{i+1}{2i}\left(x_i + \frac{x_{i+1}}{i+1} + \frac{x_{i+2}}{i+1} + \frac{x_{i+3}}{i+1} + \cdots\right)^2,$$

$$(x_i = 0\,,\ \text{if } i > n)\,,$$

shows that F is positive definite, hence the existence of our regular n-simplex is insured. The coördinates of the vertices of one such simplex may be read off from this last identity: one vertex is $A_0 = (0, \cdots, 0)$ while the coördinates of A_ν $(\nu = 1, \cdots, n)$ are

$$\frac{1}{\sqrt{2\cdot 1\cdot 2}}, \frac{1}{\sqrt{2\cdot 2\cdot 3}}, \frac{1}{\sqrt{2\cdot 3\cdot 4}}, \cdots, \frac{1}{\sqrt{2(\nu-1)\nu}}, \sqrt{\frac{\nu+1}{2\nu}}, \overbrace{0, \cdots, 0}^{n-\nu}.$$

Construction of simplexes of given edges in spherical spaces

3. Denote by S_r^ρ the r-dimensional spherical space

$$x_1^2 + x_2^2 + \cdots + x_{r+1}^2 = \rho^2$$

immersed in a R_{r+1}. The problem is as follows.

Given $\binom{n}{2}$ *positive quantities* α_{ik} $(i \neq k;\ i, k = 1, 2, \cdots, n)$ *and a positive* ρ, *to decide whether there exist, on some* S_r^ρ, n *points* $A_1, A_2, \cdots, A_n$, *such that their spherical distances* $\widehat{A_iA_k} = \alpha_{ik}$.

According to a remark of J. von Neumann this problem may be reduced to the preceding one regarding the construction of simplexes in euclidean spaces.[3] Combining his remark with Theorem 1 we get the following theorem which solves completely the problem stated above.

THEOREM 2. *Let* $\alpha_{ik} = \alpha_{ki}$ $(i \neq k;\ i, k = 1, 2, \cdots, n)$ *be* $\binom{n}{2}$ *given positive quantities. Necessary and sufficient conditions that there be, on some spherical manifold of radius* ρ, n *points* $A_1, A_2, \cdots, A_n$, *of mutual spherical distances equal to the* α_{ik}, i.e. $\widehat{A_iA_k} = \alpha_{ik}$, *are the inequalities.*

$$\alpha_{ik} \leqq \pi\rho\,, \tag{7}$$

together with the condition that the quadratic form

$$\Phi(x_1, x_2, \cdots, x_n) = \sum_{i,k=1}^{n} \cos(\alpha_{ik}/\rho)\, x_i x_k \qquad (\alpha_{ik} = 0,\ \text{if } i = k) \tag{8}$$

be positive. If r $(\geqq 1)$ *is the rank of* Φ, *then we can find such points in* S_{r-1}^ρ, *but not in* S_{r-2}^ρ *(which is undefined if* $r = 1$*).*

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[3] After Prof. von Neumann's verbal communication I noticed that the same reduction has already been used by Laura Klanfer ([5]) to carry over Menger's results from euclidean spaces to spherical spaces.

The meaning of the inequalities (7) is obvious viewing the fact that no distance on a sphere of radius ρ can exceed $\pi\rho$. Suppose there are required points $A_1, \cdots, A_n$ on some $S_m^\rho (m \geqq 1)$. Call A_0 the sphere's center. Then $A_0 A_1 \cdots A_m$ is an n-"simplex" in R_{m+1}, the lengths of its edges being

$$(9) \qquad A_0A_1 = \rho = a_{0i}, \quad A_iA_k = 2\rho \sin \frac{\alpha_{ik}}{2\rho} = a_{ik} \quad (i, k = 1, \cdots, n; i \neq k).$$

From Theorem 1 we know that the construction of such a "simplex" amounts to the investigation of the quadratic form

$$F = \frac{1}{2} \sum_{i,k=1}^{n} (a_{0i}^2 = a_{0k}^2 - a_{ik}^2)\, x_i x_k = \rho^2 \sum_{i,k=1}^{n} \left(1 - 2 \sin^2 \frac{\alpha_{ik}}{2\rho}\right) x_i x_k$$

$$= \rho^2 \sum_{i,k=1}^{n} \cos (\alpha_{ik}/\rho)\, x_i x_k = \rho^2 \,\Phi\,.$$

Its positivity is necessary and sufficient for the existence of $A_0 A_1 \cdots A_n$ with the properties (9). Its rank r indicates that $A_0 A_1 \cdots A_n$ is contained in R_r but not in R_{r-1}, hence $A_1 A_2 \cdots A_n$ with the desired properties, i.e. $\widehat{A_iA_k} = \alpha_{ik}$, is contained in S_{r-1}^ρ but not in S_{r-2}^ρ.

4. The set of quantities α_{ik} in Theorem 2 could be thought of as the edges of an abstractly defined $(n-1)$-simplex (in Menger's terminology it is a semi-metric space composed of $n-1$ points). Theorem 2 answers the question whether or not this abstract simplex can be immersed isometrically, i.e. by congruence, in a spherical space of given radius.

An interesting consequence of Theorem 2 is the following theorem.

THEOREM 3. *Let σ_{n-1} be a $(n-1)$-simplex of a $S_{n-1}^{\rho_0}$; there exists a radius $\rho_1 \leqq \rho_0$ such that σ_{n-1} can be immersed isometrically in $S_{n-2}^{\rho_1}$.*

Thus for $n = 3$ we get the following geometrically obvious statement: Any ordinary spherical triangle of a $S_2^{\rho_0}$ can be placed isometrically on a circumference of suitable radius $\rho_1 \leqq \rho_0$.

We note first that if σ_{n-1} can be immersed in $S_{n-2}^{\rho_0}$, which happens when the rank of

$$(10) \qquad \Phi(x; \rho) = \sum_{i,k=1}^{n} \cos (\alpha_{ik}/\rho)\, x_i x_k$$

is $\leqq n-1$ for $\rho = \rho_1$, our theorem is proved with $\rho_1 = \rho_0$. Let us now assume $\Phi(x; \rho_0)$ to be of rank n, hence

$$\Phi(x; \rho_0) \text{ positive definite and } \frac{\alpha_{ik}}{\pi} \leqq \rho_0\,,$$

by Theorem 2. Note that $\Phi(x; \rho)$ can not be positive definite for all ρ with $0 < \rho \leqq \rho_0$, for it fails to be so if e.g. $\rho = \alpha_{12}/\pi$ since the first principal minor of order 2 of the discriminant of $\Phi(x; \alpha_{12}/\pi)$ vanishes. Call ρ_1 the greatest lower bound of the values σ with the property that $\Phi\ (x;\ \rho)$ is positive definite if $\sigma \leqq \rho \leqq \rho_0$. By a previous remark necessarily

$$\alpha_{ik} \leqq \pi\rho_1. \tag{11}$$

Now $\Phi(x;\ \rho)$ can not be positive definite if $\rho = \rho_1$ for it would still be so (by continuity) for all values ρ sufficiently close to ρ_1 in contradiction to the definition of ρ_1. But $\Phi(x;\ \rho_1)$ is necessarily positive, as the limit of positive definite forms $\Phi(x\ ;\rho)$, for $\rho \to \rho_1 + 0$. Hence $\Phi(x;\rho_1)$ is positive and of rank $< n$. Now the proof is completed by (11) and Theorem 2.[4]

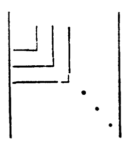
5. We shall now extend Theorem 3 to cover the case when $\rho_0 = \infty$, that is when σ_{n-1} is in R_{n-1}. We assume σ_{n-1}, of edges α_{ik}, to be a $(n-1)$-simplex of R_{n-1}, i.e.

$$\tfrac{1}{2} \sum_{i,k=2}^{n} (\alpha_{1i}^2 + \alpha_{1k}^2 - \alpha_{ik}^2)\, x_i x_k \text{ positive definite.} \tag{12}$$

Let us prove that σ_{n-1} *can be immersd isometrically in* S_{n-1}^{ρ}, *provided* ρ *is sufficiently large.* This is proved if we can show that

$$\Phi(x;\ \rho) = \sum_{i,k=1}^{n} \cos\ (\alpha_{ik}/\rho)\ x_i x_k$$

is positive definite if ρ is sufficiently large. A well known criterion states that a quadratic form is positive definite if and only if all the n principal minors of its discriminant chosen as follows



are positive (see Dickson [3], §40). If in the matrix of coefficients

$$\begin{Vmatrix} 1 & \cos\dfrac{\alpha_{1k}}{\rho} \\[2ex] \cos\dfrac{\alpha_{i1}}{\rho} & \cos\dfrac{\alpha_{ik}}{\rho} \end{Vmatrix} \qquad (i, k = 2, \cdots, n)$$

of $\Phi(x;\ \rho)$ we subtract the first line from all the other lines and then the first column from all the other columns we get the symmetric matrix

$$\begin{Vmatrix} 1 & \cos\dfrac{\alpha_{1k}}{\rho} - 1 \\[2ex] \cos\dfrac{\alpha_{i1}}{\rho} - 1 & \cos\dfrac{\alpha_{ik}}{\rho} - \cos\dfrac{\alpha_{i1}}{\rho} - \cos\dfrac{\alpha_{1k}}{\rho} + 1 \end{Vmatrix} \tag{13}$$

[4] Note that $\rho = \rho_1$ is the first value $< \rho_0$ which is a root of the transcendental equation $\det \|\cos\ (\alpha_{ik}/\rho)\| = 0$. It would be interesting to decide whether $\rho = \rho_1$ is necessarily a simple root of this equation.

which, as a result of the above criterion, will be the matrix of a positive definite form if and only if $\Phi\ (x;\, \rho)$ is positive definite itself. Noting that (13) can be written as follows

$$\left\| \begin{matrix} 1 & -\dfrac{\alpha_{1k}^2}{2\rho^2} + O\left(\dfrac{1}{\rho^4}\right) \\ -\dfrac{\alpha_{i1}^2}{2\rho^2} + O\left(\dfrac{1}{\rho^4}\right) & \dfrac{1}{2\rho^2}\,(\alpha_{1i}^2 + \alpha_{1k}^2 - \alpha_{ik}^2) + O\left(\dfrac{1}{\rho^4}\right) \end{matrix} \right\|, \quad (\rho \to \infty),$$

we see that the ν^{th} ($\nu > 1$) principal minor of (13) is = to $\rho^{-2(\nu-1)}$ times the $(\nu - 1)^{\text{st}}$ principal minor of the discriminant of (12), plus a remainder $O(\rho^{-2\nu})$. By (12) all these minors are positive if ρ is sufficiently large, hence $\Phi(x;\, \rho)$ is positive definite and σ_{n-1} can be immersed in S_{n-1}^{ρ}. For any such $\rho = \rho_0$. Theorem 3 proves the existence of $S_{n-2}^{\rho_1}$, with $\rho_1 < \rho_0$, in which σ_{n-1} can be immersed. We have thus proved the following

THEOREM 3′ (of Gödel). *If σ_n is a n-simplex of R_n, then there always exists a S_{n-1}^{ρ} in which σ_n can be immersed isometrically.*[5]

The case of indefinite spaces

6. Consider the space of real variables $(y_1, \cdots, y_m)$ with the property that the square of the distance PP' of two points is given by the formula

$$\overline{PP'}^2 = \sum_{\nu=1}^{m} \epsilon_\nu\, (y_\nu - y'_\nu)^2,$$

with $\epsilon_\nu = +1$ for $\nu = 1, \cdots, p$, $\epsilon_\nu = -1$ for $\nu = p + 1, \cdots, p + q\ (= m)$. We denote this space by $R_{p,q}$; thus $R_m = R_{m,0}$. The linear geometry of $R_{p,q}$ is obviously the same as that of $R_{p+q} = R_m$.

Let now $\frac{1}{2}n(n + 1)$ real numbers $c_{ik}(c_{ii} = 0,\, c_{ik} = c_{ki};\, i, k = 0, \cdots, n)$ be given. Are there $n + 1$ points $A_0, A_1, \cdots, A_n$ in some space $R_{p,q}$ such that $\overline{A_iA_k}^2 = c_{ik}$, and what is the space $R_{p,q}$ of the least number of dimensions in which there are such points? A complete answer is furnished by the following theorem.

THEOREM 1′. *Consider the quadratic form*

$$F(x_1, x_2, \cdots, x_n) = \tfrac{1}{2} \sum_{i,k=1}^{n} (c_{0i} + c_{0k} - c_{ik})\, x_i x_k. \tag{14}$$

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[5] A heuristic proof of this theorem for $n = 3$ is as follows. Think of the edges of σ_3 to be made of flexible strings; place in the interior of σ_3 a small sphere which is gradually inflated. This sphere will reach a certain definite size when it will become tightly packed within the 6 strings (edges) of σ_3. Note that in the rigorous proof above a very large sphere was used which was gradually deflated to its proper size.

Let it be of type (p, q).[6] *The necessary and sufficient conditions that there be* $n + 1$ *points* $A_0, A_1, \cdots, A_n$ *in* $R_{p', q'}$ *with* $\overline{A_iA}_k^2 = c_{ik}$, *are the inequalities*

$$p' \geqq p, \qquad q' \geqq q.$$

Thus $R_{p,q}$ *is the least space in which there are such points.*

The condition is necessary. Let the points $A_0 = 0, A_1, \cdots, A_n$ in $R_{p',q'}$ have the required property and let $R_{p,q}$ be the least linear subspace containing these points. We know that $p \leqq p'$, $q \leqq q'$, $p + q \leqq n$. Let $p + q = m$ and let $A_i = (\alpha_{i1}, \cdots, \alpha_{im})$ be the coördinates of A_i in $R_{p,q}$ with respect to an orthogonal coördinate system. For the point

$$P = x_1A_1 + \cdots + x_nA_n = (\xi_1, \cdots, \xi_m)$$

of coördinates $\xi_\nu = x_1\alpha_{1\nu} + \cdots + x_n\alpha_{n\nu}$ we find as in section 2 the identity

$$\overline{OP}^2 = \sum_{\nu=1}^{m} \epsilon_\nu \xi_\nu^2 = \sum_{\nu=1}^{m} \epsilon_\nu (x_1\alpha_{1\nu} + \cdots + x_n\alpha_{n\nu})^2 = F(x_1, \cdots, x_n).$$

Viewing our assumption that the matrix of the $\alpha_{\mu\nu}$ is of rank m and the law of inertia (Dickson, [3], p. 72), we see that $F(x)$ is of type (p, q).

The condition is sufficient. Assume first $p + q = n$. By a non-singular transformation

$$(y) = H(x) \tag{3'}$$

we get the identity

$$F(x_1, \cdots, x_n) = y_1^2 + \cdots + y_p^2 - y_{p+1}^2 - \cdots - y_n^2.$$

Consider in the space $R_{p,q}$ of the variables $(y_1, \cdots, y_n)$ the points whose x-coördinates are given by (5). We find as in section 2 $\overline{A_iA}_k^2 = c_{ik}$ and the theorem is proved, for $R_{p,q}$ can be considered as a subspace of $R_{p',q'}$, if $p' \geqq p$, $q' \geqq q$.

If $p + q = m < n$, then we get

$$F(x_1, \cdots, x_n) = y_1^2 + \cdots + y_p^2 - y_{p+1}^2 - \cdots - y_m^2.$$

To get the desired points we have to project the points $A_0, \cdots, A_n$ on the manifold $y_{m+1} = \cdots = y_n = 0$, which is a $R_{p,q}$.

7. It should be remarked that F defined by (14) is the most general real quadratic form in n variables. We thus have the following

COROLLARY. *Let*

$$F = \sum_{1}^{n} b_{ik}\, x_ix_k \tag{15}$$

[6] That is of *index* p and rank $p + q$. See Dickson [3], p. 71.

be a non-degenerate real quadratic form of type (p, q). *If by means of*

$$(y) = H(x) \tag{3''}$$

we have

$$F = y_1^2 + \cdots + y_p^2 - y_{p+1}^2 - \cdots - y_n^2, \tag{16}$$

then the columns of the matrix

$$H = \begin{Vmatrix} h_{11} & \cdots & h_{1n} \\ \vdots & & \vdots \\ h_{n1} & \cdots & h_{nn} \end{Vmatrix}$$

are the y-coördinates in $R_{p,q}$ *of n points* $A_1, \cdots, A_n$, *which together with* $A_0 = (0)$ *have the property* $\overline{A_iA_k}^2 = c_{ik}$, *where*

$$c_{0i} = b_{ii}, \qquad c_{ik} = b_{ii} + b_{kk} - 2b_{ik} \qquad (i, k > 0).$$

A geometric interpretation of the reduction of (15) to the canonical form (16) by means of an *orthogonal* linear transformation is well known from the theory of quadrics. The above Corollary furnishes a geometric interpretation of this reduction by any linear non-singular transformation.

Probably the most concise description of the result of Theorems 1 and 1′ is as follows. If the squares of the edges of a simplex $A_0A_1 \cdots A_n$ are given real numbers, $\overline{A_iA_k}^2 = c_{ik}$, then this defines uniquely a (indefinite) space which, if referred to the coördinate unit-vectors $A_0A_1, A_0A_2, \cdots, A_0A_n$, has the line element

$$ds^2 = \tfrac{1}{2} \sum_{i,k=1}^{n} (c_{0i} + c_{0k} - c_{ik})\, x_ix_k.$$

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