REMARKS TO MAURICE FRÉCHET’S ARTICLE “SUR LA DÉFINITION AXIOMATIQUE D’UNE CLASSE D’ESPACE DISTANCIÉS VECTORIELLEMENT APPLICABLE SUR L’ESPACE DE HILBERT

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1. Fréchet’s developments in the last section of his article suggest an elegant solution of the following problem.

Let

\[ a_{ik} = a_{ki} \quad (i \neq k ; i, k = 0, 1, \ldots, n) \]

be \( \frac{1}{2}n(n + 1) \) given positive quantities. What are the necessary and sufficient conditions that they be the lengths of the edges of an \( n \)-simplex \( A_0A_1 \cdots A_n \)? More general, what are the conditions that they be the lengths of the edges of an \( n \)-"simplex" \( A_0A_1 \cdots A_n \) lying in a euclidean space \( R_r \) (\( 1 \leq r \leq n \)) but not in \( R_{r-1} \)?

This problem is fundamental in K. Menger’s metric investigation of euclidean spaces ([6] and [7], particularly his third fundamental theorem in [7], pp. 737–743). It was solved by Menger by means of equations and inequalities involving certain determinants. Theorem 1 below furnishes a complete and independent solution of this problem. Theorem 2 solves the similar problem for spherical spaces previously treated by Menger’s methods by L. M. Blumenthal and G. A. Garrett ([1]) and Laura Klanfer ([5]); it may be conveniently applied (Theorems 3 and 3’) to prove and extend a theorem of K. Gödel ([4]). The method of Theorem 1 is finally applied to solve the corresponding problem for spaces with indefinite line element recently considered by A. Wald ([8]) and H. S. M. Coxeter and J. A. Todd ([2]).

Construction of simplexes of given edges in euclidean spaces

2. A complete answer to the questions stated above is given by the following theorem.

Theorem 1. A necessary and sufficient condition that the \( a_{ik} \) be the lengths of the edges of an \( n \)-"simplex" \( A_0A_1 \cdots A_n \) lying in \( R_r \), but not in \( R_{r-1} \), is that the quadratic form

\[ \sum a_{ik} x_i x_k = \text{constant} \]

\[ \text{for } i, k = 0, 1, \ldots, n \]
(1) \[ F(x_1, x_2, \ldots, x_n) = \sum_{i=1}^{n} a_{0,i}^2 x_i^2 + \sum_{i \neq k}^{n} (a_{0,i}^2 + a_{0,k}^2 - a_{i,k}^2) x_i x_k \]

\[ = \frac{1}{2} \sum_{i, k=1}^{n} (a_{0,i}^2 + a_{0,k}^2 - a_{i,k}^2) x_i x_k \]

(with \(a_{ik} = 0\) if \(i = k\))

be positive, i.e. always \(\geq 0\), and of rank \(r\).

The condition is necessary. Let \(A_0, A_1, \ldots, A_n\) be an \(n\)-"simplex" with \(A_i A_k = a_{ik}\). Let \(A_0 = 0\) be the origin of a \(R_n\) in which \(A_i\) has the cartesian coördinates \(\alpha_{i1}, \alpha_{i2}, \ldots, \alpha_{in}\). The point (in vector space notation)

\[ P = x_1 A_1 + x_2 A_2 + \cdots + x_n A_n = (\xi_1, \xi_2, \ldots, \xi_n) \]

has the coördinates

\[ \xi_v = x_1 \alpha_{v1} + x_2 \alpha_{v2} + \cdots + x_n \alpha_{vn}, \quad (v = 1, \ldots, n), \]

whence

\[ \overline{OP}^2 = \|P\|^2 = \sum_{v=1}^{n} \xi_v^2 = \sum_{v=1}^{n} (x_1 \alpha_{v1} + \cdots + x_n \alpha_{vn})^2 \]

\[ = \sum_{i=1}^{n} x_i^2 \sum_{v=1}^{n} \alpha_{i,v}^2 + 2 \sum_{i < k} x_i x_k \sum_{v=1}^{n} \alpha_{i,v} \alpha_{k,v}. \]

Since

\[ \sum_{v=1}^{n} \alpha_{i,v}^2 = \overline{OA_i}^2 = a_{0,i}^2, \]

\[ 2 \sum_{v=1}^{n} \alpha_{i,v} \alpha_{k,v} = \sum_{v=1}^{n} \alpha_{i,v}^2 + \sum_{v=1}^{n} \alpha_{k,v}^2 - \sum_{v=1}^{n} (\alpha_{i,v} - \alpha_{k,v})^2 = A_{0} \overline{A_i}^2 + A_{0} \overline{A_k}^2 - \overline{A_i A_k}^2 \]

\[ = a_{0,i}^2 + a_{0,k}^2 - a_{i,k}^2, \]

we have

(2) \[ \overline{OP}^2 = \|x_1 A_1 + \cdots + x_n A_n\|^2 = F(x_1, x_2, \ldots, x_n). \]

Hence \(F(x_1, \ldots, x_n)\) is positive. It follows furthermore from our assumptions that \(P = 0\), hence \(F = 0\), on a linear manifold of \(n - r\) dimensions in the variables \(x_1, \ldots, x_n\); hence \(F\) is of rank \(r\).

The condition is sufficient. Let us first assume \(F\) to be positive definite, i.e. \(r = n\). By means of a certain linear non-singular transformation

(3) \[ (y) = H(x) \]

we get the identity

(4) \[ F(x_1, \ldots, x_n) = y_1^2 + y_2^2 + \cdots + y_n^2. \]
Call $A_0$ the origin of the cartesian space of the variables $(y_1, \ldots, y_n)$ and $A_1, A_2, \ldots, A_n$, the $n$ points which in virtue of (3) correspond to

(5) $(x_1, x_2, \ldots, x_n) = (1, 0, \ldots, 0), (0, 1, \ldots, 0), \ldots, (0, 0, \ldots, 0, 1)$,

respectively. Their $y$-coördinates are readily found by (3). For their mutual distances we find by (3), (4) and (5),

\[
\begin{align*}
\overline{A_0A_i}^2 &= F(0, \ldots, 1, \ldots, 0) = a_{0i}^2, \\
\overline{A_iA_k}^2 &= F(0, \ldots, 1, \ldots, -1, \ldots, 0) = a_{0i}^2 + a_{0k}^2 - (a_{0i}^2 + a_{0k}^2 - a_{ik}^2) \\
&= a_{ik}^2, (i < k),
\end{align*}
\]

which show that $A_0A_1 \ldots A_n$ is precisely the $n$-simplex we are looking for. It is indeed an $n$-simplex because the points (5) are independent and (3) is non-singular.

If $r < n$, then (4) has to be replaced by

(6) $F(x_1, \ldots, x_n) = y_1^2 + y_2^2 + \cdots + y_r^2$.

The above procedure gives an $n$-simples $A_0A_1 \ldots A_n$, however the quantities

$F(1, 0, \ldots, 0) = a_{01}^2, \quad F(1, -1, 0, \ldots, 0) = a_{12}^2, \ldots$

are no more the squared lengths of the edges $\overline{A_0A_1}, \overline{A_1A_2}, \ldots$, but, viewing (6), the squared lengths of their projections on the sub-space $(y_1, \ldots, y_r)$, i.e., on the manifold $y_{r+1} = \ldots = y_n = 0$. Hence the projection $A'_0A'_1 \ldots A'_n$ on this manifold of the $n$-simplex $A_0A_1 \ldots A_n$ is an $n$-"simplex" of the type we are looking for, i.e. with $A'_iA'_k = a_{ik}$. This $n$-"simplex" $A'_0A'_1 \ldots A'_n$ is by construction contained in a $R_r$ but not in a $R_{r-1}$, as readily seen.

Remark. If the matrix $H$ of (3) is $H = ||h_{ik}||$, then the $y$-coördinates of the vertices $A_i$ and $A'_i$ are

$A_i = (h_{1i}, h_{2i}, \ldots, h_{ni}), \quad A'_i = (h_{1i}, h_{2i}, \ldots, h_{ri}, 0, \ldots, 0)$.

The actual construction (i.e. determination of the coördinates of its vertices) of an $n$-"simplex" of edges $a_{ik}$ is therefore carried out by a reduction of the quadratic form (1) to its canonical form (6). This is a problem of the second degree, for the transformation (3) is by no means required to be orthogonal.

As an illustration of this method let us construct a regular $n$-simplex with $a_{ik} = 1$. By (1) we have

$F(x_1, \ldots, x_n) = \sum_{i=1}^{n} x_i^2 + \sum_{i<k} x_ix_k$. 
The identity
\[ F(x_1, \ldots, x_n) = \sum_{i=1}^{n} \frac{i + 1}{2i} \left( x_i + \frac{x_{i+1}}{i+1} + \frac{x_{i+2}}{i+1} + \frac{x_{i+3}}{i+1} + \cdots \right)^2, \]
\[ (x_i = 0, \text{ if } i > n), \]
shows that \( F \) is positive definite, hence the existence of our regular \( n \)-simplex is insured. The coördinates of the vertices of one such simplex may be read off from this last identity: one vertex is \( A_0 = (0, \ldots, 0) \) while the coördinates of \( A_i \) (\( \nu = 1, \ldots, n \)) are
\[
\frac{1}{\sqrt{2 \cdot 1 \cdot 2}}, \frac{1}{\sqrt{2 \cdot 2 \cdot 3}}, \frac{1}{\sqrt{2 \cdot 3 \cdot 4}}, \ldots, \frac{1}{\sqrt{2(n - 1)n}}, \sqrt{\frac{\nu + 1}{2\nu}}, 0, \ldots, 0.
\]

**Construction of simplexes of given edges in spherical spaces**

3. Denote by \( S_r^\nu \) the \( r \)-dimensional spherical space
\[ x_1^2 + x_2^2 + \cdots + x_{r+1}^2 = \rho^2 \]
immersed in a \( R_{r+1} \). The problem is as follows.

Given \( \binom{n}{2} \) positive quantities \( \alpha_{ik} \) (\( i \neq k; i, k = 1, 2, \ldots, n \)) and a positive \( \rho \), to decide whether there exist, on some \( S_r^\nu \), \( n \) points \( A_1, A_2, \ldots, A_n \), such that their spherical distances \( \widehat{AA}_k = \alpha_{ik} \).

According to a remark of J. von Neumann this problem may be reduced to the preceding one regarding the construction of simplexes in euclidean spaces.\(^3\) Combining his remark with Theorem 1 we get the following theorem which solves completely the problem stated above.

**Theorem 2.** Let \( \alpha_{ik} = \alpha_{ki} \) (\( i \neq k; i, k = 1, 2, \ldots, n \)) be \( \binom{n}{2} \) given positive quantities. Necessary and sufficient conditions that there be, on some spherical manifold of radius \( \rho \), \( n \) points \( A_1, A_2, \ldots, A_n \), of mutual spherical distances equal to the \( \alpha_{ik} \), i.e. \( \widehat{AA}_k = \alpha_{ik} \), are the inequalities.
\[
(7) \quad \alpha_{ik} \leq \pi \rho,
\]
together with the condition that the quadratic form
\[
(8) \quad \Phi(x_1, x_2, \ldots, x_n) = \sum_{i, k=1}^{n} \cos (\alpha_{ik}/\rho) x_i x_k \quad (\alpha_{ik} = 0, \text{ if } i = k)
\]
be positive. If \( r \geq 1 \) is the rank of \( \Phi \), then we can find such points in \( S_r^\nu \), but not in \( S_r^{\nu-2} \) (which is undefined if \( r = 1 \)).

\(^3\) After Prof. von Neumann's verbal communication I noticed that the same reduction has already been used by Laura Klanfer ([5]) to carry over Menger's results from euclidean spaces to spherical spaces.
The meaning of the inequalities (7) is obvious viewing the fact that no distance on a sphere of radius $\rho$ can exceed $\pi \rho$. Suppose there are required points $A_1, \ldots, A_n$ on some $S^m_\rho (m \geq 1)$. Call $A_0$ the sphere's center. Then $A_0A_1 \ldots A_m$ is an $n$-"simplex" in $R_{m+1}$, the lengths of its edges being

(9) $A_0A_1 = \rho = a_{0i}, \; A_iA_k = 2\rho \sin \frac{\alpha_{ik}}{2\rho} = a_{ik} \quad (i, k = 1, \ldots, n; i \neq k)$.

From Theorem 1 we know that the construction of such a "simplex" amounts to the investigation of the quadratic form

$$F = \frac{1}{2} \sum_{i,k=1}^{n} (a_{0i}^2 = a_{0k}^2 - a_{ik}^2) x_i x_k = \rho^2 \sum_{i,k=1}^{n} \left(1 - 2 \sin^2 \frac{\alpha_{ik}}{2\rho}\right) x_i x_k$$

$$= \rho^2 \sum_{i,k=1}^{n} \cos \left(\frac{\alpha_{ik}}{\rho}\right) x_i x_k = \rho^2 \Phi.$$

Its positivity is necessary and sufficient for the existence of $A_0A_1 \ldots A_n$ with the properties (9). Its rank $r$ indicates that $A_0A_1 \ldots A_n$ is contained in $R_\rho$, but not in $R_{r-1}$, hence $A_1A_2 \ldots A_n$ with the desired properties, i.e. $A_iA_k = \alpha_{ik}$, is contained in $S^r_\rho$ but not in $S^{r-2}_\rho$.

4. The set of quantities $\alpha_{ik}$ in Theorem 2 could be thought of as the edges of an abstractly defined $(n-1)$-simplex (in Menger's terminology it is a semi-metric space composed of $n-1$ points). Theorem 2 answers the question whether or not this abstract simplex can be immersed isometrically, i.e. by congruence, in a spherical space of given radius.

An interesting consequence of Theorem 2 is the following theorem.

**THEOREM 3.** Let $\sigma_{n-1}$ be a $(n-1)$-simplex of a $S^e_{n-1}$; there exists a radius $\rho_1 \leq \rho_0$ such that $\sigma_{n-1}$ can be immersed isometrically in $S^e_{n-2}$. 

Thus for $n = 3$ we get the following geometrically obvious statement: Any ordinary spherical triangle of a $S^e_2$ can be placed isometrically on a circumference of suitable radius $\rho_1 \leq \rho_0$.

We note first that if $\sigma_{n-1}$ can be immersed in $S^e_{n-2}$, which happens when the rank of

(10) $\Phi(x; \rho) = \sum_{i,k=1}^{n} \cos \left(\frac{\alpha_{ik}}{\rho}\right) x_i x_k$

is $\leq n - 1$ for $\rho = \rho_1$, our theorem is proved with $\rho_1 = \rho_0$. Let us now assume $\Phi(x; \rho_0)$ to be of rank $n$, hence

$\Phi(x; \rho_0)$ positive definite and $\frac{\alpha_{ik}}{\pi} \leq \rho_0$,

by Theorem 2. Note that $\Phi(x; \rho)$ can not be positive definite for all $\rho$ with $0 < \rho \leq \rho_0$, for it fails to be so if e.g. $\rho = \alpha_{12}/\pi$ since the first principal minor of
order 2 of the discriminant of \( \Phi(x; \alpha_{ik}/\pi) \) vanishes. Call \( \rho_1 \) the greatest lower bound of the values \( \sigma \) with the property that \( \Phi(x; \rho) \) is positive definite if \( \sigma \leq \rho \leq \rho_0 \). By a previous remark necessarily

\[
\alpha_{ik} \leq \pi \rho_1.
\]

Now \( \Phi(x; \rho) \) can not be positive definite if \( \rho = \rho_1 \) for it would still be so (by continuity) for all values \( \rho \) sufficiently close to \( \rho_1 \) in contradiction to the definition of \( \rho_1 \). But \( \Phi(x; \rho_1) \) is necessarily positive, as the limit of positive definite forms \( \Phi(x; \rho) \), for \( \rho \to \rho_1 + 0 \). Hence \( \Phi(x; \rho_1) \) is positive and of rank \( < n \). Now the proof is completed by (11) and Theorem 2.\(^4\)

5. We shall now extend Theorem 3 to cover the case when \( \rho_0 = \infty \), that is when \( \sigma_{n-1} \) is in \( R_{n-1} \). We assume \( \sigma_{n-1} \), of edges \( \alpha_{ik} \), to be a \( (n - 1) \)-simplex of \( R_{n-1} \), i.e.

\[
\frac{1}{2} \sum_{i, k=2}^{n} \left( \alpha_{1,i}^2 + \alpha_{1,k}^2 - \alpha_{i,k}^2 \right) x_i x_k \text{ positive definite.}
\]

Let us prove that \( \sigma_{n-1} \) can be immersed isometrically in \( S^{n-1}_n \), provided \( \rho \) is sufficiently large. This is proved if we can show that

\[
\Phi(x; \rho) = \sum_{i, k=1}^{n} \cos \left( \frac{\alpha_{ik}}{\rho} \right) x_i x_k
\]

is positive definite if \( \rho \) is sufficiently large. A well known criterion states that a quadratic form is positive definite if and only if all the \( n \) principal minors of its discriminant chosen as follows

\[
\begin{vmatrix}
1 & \cos \frac{\alpha_{1k}}{\rho} \\
\cos \frac{\alpha_{11}}{\rho} & \cos \frac{\alpha_{1k}}{\rho} \\
& & \ddots
\end{vmatrix}
\]

are positive (see Dickson [3], §40). If in the matrix of coefficients

\[
\begin{vmatrix}
1 & \cos \frac{\alpha_{1k}}{\rho} \\
\cos \frac{\alpha_{11}}{\rho} & \cos \frac{\alpha_{1k}}{\rho} \\
& & \ddots
\end{vmatrix}
\]

\((i, k = 2, \ldots, n)\)

of \( \Phi(x; \rho) \) we subtract the first line from all the other lines and then the first column from all the other columns we get the symmetric matrix

\[
\begin{vmatrix}
1 & \cos \frac{\alpha_{1k}}{\rho} - 1 \\
\cos \frac{\alpha_{11}}{\rho} - 1 & \cos \frac{\alpha_{1k}}{\rho} - \cos \frac{\alpha_{11}}{\rho} - \cos \frac{\alpha_{1k}}{\rho} + 1
\end{vmatrix}
\]

\(^{4}\) Note that \( \rho = \rho_1 \) is the first value \( < \rho_0 \) which is a root of the transcendental equation \( \det \| \cos \left( \frac{\alpha_{ik}}{\rho} \right) \| = 0 \). It would be interesting to decide whether \( \rho = \rho_1 \) is necessarily a simple root of this equation.
which, as a result of the above criterion, will be the matrix of a positive definite form if and only if $\Phi(x; \rho)$ is positive definite itself. Noting that (13) can be written as follows

\[
\begin{vmatrix}
1 & -\frac{\alpha_{1}^{2}}{2\rho^{2}} + O\left(\frac{1}{\rho^{4}}\right) \\
-\frac{\alpha_{1}^{2}}{2\rho^{2}} + O\left(\frac{1}{\rho^{4}}\right) & 1 + \frac{1}{2\rho^{2}} (\alpha_{1,i}^{2} + \alpha_{1,k}^{2} - \alpha_{1,ik}^{2}) + O\left(\frac{1}{\rho^{4}}\right)
\end{vmatrix},
\quad (\rho \to \infty),
\]

we see that the $\nu^\text{th}$ ($\nu > 1$) principal minor of (13) is $\approx \rho^{-2(\nu-1)}$ times the $(\nu - 1)^\text{st}$ principal minor of the discriminant of (12), plus a remainder $O(\rho^{-2})$. By (12) all these minors are positive if $\rho$ is sufficiently large, hence $\Phi(x; \rho)$ is positive definite and $\sigma_{n-1}$ can be immersed in $S_{n-1}^{*}$. For any such $\rho = \rho_{0}$. Theorem 3 proves the existence of $S_{n-2}^{*}$, with $\rho_{1} < \rho_{0}$, in which $\sigma_{n-1}$ can be immersed. We have thus proved the following

**Theorem 3' (of Gödel).** If $\sigma_{n}$ is a $n$-simplex of $R_{n}$, then there always exists a $S_{n-1}^{*}$ in which $\sigma_{n}$ can be immersed isometrically.\textsuperscript{5}

**The case of indefinite spaces**

6. Consider the space of real variables $(y_{1}, \ldots, y_{m})$ with the property that the square of the distance $PP'$ of two points is given by the formula

\[
PP'^{2} = \sum_{i=1}^{n} \epsilon_{\nu} (y_{\nu} - y'_{\nu})^{2},
\]

with $\epsilon_{\nu} = +1$ for $\nu = 1, \ldots, p$, $\epsilon_{\nu} = -1$ for $\nu = p + 1, \ldots, p + q (= m)$. We denote this space by $R_{p,q}$; thus $R_{m} = R_{m,0}$. The linear geometry of $R_{p,q}$ is obviously the same as that of $R_{p+q} = R_{m}$.

Let now $\frac{1}{2}n(n + 1)$ real numbers $c_{ik}(c_{ii} = 0, c_{ik} = c_{ki}; i, k = 0, \ldots, n)$ be given. Are there $n + 1$ points $A_{0}, A_{1}, \ldots, A_{n}$ in some space $R_{p,q}$ such that $A_{i}A_{k}^{2} = c_{ik}$, and what is the space $R_{p,q}$ of the least number of dimensions in which there are such points? A complete answer is furnished by the following theorem.

**Theorem 1'.** Consider the quadratic form

\[
F(x_{1}, x_{2}, \ldots, x_{n}) = \frac{1}{2} \sum_{i, k=1}^{n} (c_{0i} + c_{0k} - c_{ik}) x_{i}x_{k}.
\]

\textsuperscript{5} A heuristic proof of this theorem for $n = 3$ is as follows. Think of the edges of $\sigma_{2}$ to be made of flexible strings; place in the interior of $\sigma_{2}$ a small sphere which is gradually inflated. This sphere will reach a certain definite size when it will become totally packed within the 6 strings (edges) of $\sigma_{2}$. Note that in the rigorous proof above a very large sphere was used which was gradually deflated to its proper size.
Let it be of type \((p, q)\). The necessary and sufficient conditions that there be \(n + 1\) points \(A_0, A_1, \ldots, A_n\) in \(R_{p', q'}\) with \(\overline{A_iA_k}^2 = c_{ik}\), are the inequalities
\[
p' \geq p, \quad q' \geq q.
\]
Thus \(R_{p, q}\) is the least space in which there are such points.

The condition is necessary. Let the points \(A_0 = 0, A_1, \ldots, A_n\) in \(R_{p', q'}\) have the required property and let \(R_{p, q}\) be the least linear subspace containing these points. We know that \(p \leq p', q \leq q', p + q \leq n\). Let \(p + q = m\) and let \(A_i = (\alpha_{i1}, \ldots, \alpha_{im})\) be the coördinates of \(A_i\) in \(R_{p, q}\) with respect to an orthogonal coördinate system. For the point
\[
P = x_1A_1 + \cdots + x_nA_n = (\xi_1, \ldots, \xi_m)
\]
of coördinates \(\xi_r = x_1\alpha_{r1} + \cdots + x_n\alpha_{rn}\) we find as in section 2 the identity
\[
\overline{OP}^2 = \sum_{r=1}^{m} \epsilon_r^2 = \sum_{r=1}^{m} \epsilon_r(x_1\alpha_{r1} + \cdots + x_n\alpha_{rn})^2 = F(x_1, \ldots, x_n).
\]
Viewing our assumption that the matrix of the \(\alpha_{pr}\) is of rank \(m\) and the law of inertia (Dickson, [3], p. 72), we see that \(F(x)\) is of type \((p, q)\).

The condition is sufficient. Assume first \(p + q = n\). By a non-singular transformation
\[(3') \quad (y) = H(x)\]
we get the identity
\[
F(x_1, \ldots, x_n) = y_1^2 + \cdots + y_p^2 - y_{p+1}^2 - \cdots - y_n^2.
\]
Consider in the space \(R_{p, q}\) of the variables \((y_1, \ldots, y_n)\) the points whose \(x\)-coördinates are given by (5). We find as in section 2 \(\overline{A_iA_k}^2 = c_{ik}\) and the theorem is proved, for \(R_{p, q}\) can be considered as a subspace of \(R_{p', q'}\), if \(p' \geq p, q' \geq q\).

If \(p + q = m < n\), then we get
\[
F(x_1, \ldots, x_n) = y_1^2 + \cdots + y_p^2 - y_{p+1}^2 - \cdots - y_m^2.
\]
To get the desired points we have to project the points \(A_0, \ldots, A_n\) on the manifold \(y_{m+1} = \cdots = y_n = 0\), which is a \(R_{p, q}\).

7. It should be remarked that \(F\) defined by (14) is the most general real quadratic form in \(n\) variables. We thus have the following

**Corollary.** Let
\[
(15) \quad F = \sum_{k=1}^{n} b_{ik} x_i x_k
\]

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\(^6\) That is of index \(p\) and rank \(p + q\). See Dickson [3], p. 71.
be a non-degenerate real quadratic form of type \((p, q)\). If by means of
\[(y) = H(x)\]
we have
\[(16) \quad F = y_1^2 + \cdots + y_p^2 - y_{p+1}^2 - \cdots - y_n^2,\]
then the columns of the matrix
\[
H = \begin{vmatrix}
  h_{11} & \cdots & h_{1n} \\
  \vdots & \ddots & \vdots \\
  h_{n1} & \cdots & h_{nn}
\end{vmatrix}
\]
are the \(y\)-coordinates in \(R_{p, q}\) of \(n\) points \(A_1, \ldots, A_n\), which together with \(A_0 = (0)\) have the property \(A_i A_k^2 = c_{ik}\), where
\[
c_{0i} = b_{ii}, \quad c_{ik} = b_{ii} + b_{kk} - 2b_{ik} \quad (i, k > 0).
\]

A geometric interpretation of the reduction of (15) to the canonical form (16) by means of an orthogonal linear transformation is well known from the theory of quadrics. The above Corollary furnishes a geometric interpretation of this, reduction by any linear non-singular transformation.

Probably the most concise description of the result of Theorems 1 and 1' is as follows. If the squares of the edges of a simplex \(A_0 A_1 \cdots A_n\) are given real numbers, \(A_i A_k^2 = c_{ik}\), then this defines uniquely a (indefinite) space which, if referred to the coördinate unit-vectors \(A_0 A_1, A_0 A_2, \ldots, A_0 A_n\), has the line element
\[
ds^2 = \frac{1}{2} \sum_{i, k=1}^{n} (c_{0i} + c_{0k} - c_{ik}) x_i x_k.
\]

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