Fourier Series

Two of our homework problems regard Fourier series, and going in and out of discrete time from continuous time. Let's review Fourier series as presented on page 17 of Boaz Porat's DSP book.

Let's call the sample period T and then use the analysis equation, Porat (2.40), to derive the result for discrete time. Assume that $x(t)$ is periodic in t, $-\infty < t < \infty$, and the period of repetition for the continuous-time waveform is T_o.

$$\sum_n x(t)\,\delta(t-nT) \equiv \sum_k X^s(k)\, e^{j\frac{2\pi k}{T_o}t} \sum_n \delta(t-nT)$$

where $X^S(k)$ is the Fourier series coefficient for the *continuous*-time waveform. Equation (2.39) in Porat is the defining synthesis equation stating that a signal can be expressed as a sum of sinusoids. Here we restate it for the discrete sequence.

$$x(nT) = \sum_k X^s(k)\; e^{j\frac{2\pi k}{T_o}nT} \qquad (2.39a)$$

$$\begin{aligned} X^s(k) &= \frac{1}{T_o}\int_0^{T_o} x(t)\, e^{-j\frac{2\pi k}{T_o}t}\, dt \\ &= \frac{T}{T_o}\int_0^{T_o}\left[\sum_{n=-\infty}^{\infty} x(t)\,\delta(t-nT)\right] e^{-j\frac{2\pi k}{T_o}t}\, dt \qquad ;\ T_o \neq NT \\ &= \frac{T}{T_o}\sum_{n=0}^{N-1} x(nT)\, e^{-j\frac{2\pi k}{T_o}nT} \qquad ;\ -\infty < k < \infty \qquad (1) \end{aligned}$$

for positive integer N. The integral limits the range of n in (1). The multiplication by T compensates for the spectral scaling of a sampled signal as in Porat (3.9). Eq. (1) is true because the aliased spectral lines do not interfere with those of the continuous time signal. Note that (1) is not necessarily periodic in k.

$$N = \left\lfloor \frac{T_o}{T} \right\rfloor \text{ in general.}$$

Under this circumstance, what we have is a continuous-time Fourier series then sampled in time.

Let's try to understand more why (1) is not necessarily digitally periodic in k. Define

$$x(t) = \sum_{k=-\infty}^{\infty} \tilde{x}(t-kT_o) = \tilde{x}(t) * \sum_k \delta(t-kT_o) \qquad ;\ \text{where } \tilde{x}(t) \text{ is one period of } x(t).$$

Then we may write

$$\boldsymbol{F}\left\{\left[\tilde{x}(t) * \sum_k \delta(t - kT_o)\right]\sum_n \delta(t - nT)\right\} = \boldsymbol{F}\left\{\sum_k X^s(k)\, e^{j\frac{2\pi k}{T_o}t} \sum_n \delta(t - nT)\right\}$$

$$= \frac{1}{T}\sum_k \frac{1}{T_o}\tilde{X}\left(\frac{k}{T_o}\right)\sum_n \delta\left(f - \frac{n}{T} - \frac{k}{T_o}\right) = \frac{1}{T}\sum_k X^s(k) \sum_n \delta\left(f - \frac{n}{T} - \frac{k}{T_o}\right) \qquad (2)$$

$$= \frac{1}{T}\sum_n \frac{1}{T_o}\tilde{X}\left(f - \frac{n}{T}\right)\sum_k \delta\left(f - \frac{n}{T} - \frac{k}{T_o}\right)$$

Equation (2) says that the Fourier transform of a sampled signal is periodic in frequency f at the sample rate. Equation (2) also makes plain the aliasing of the spectral lines of $\tilde{X}(f)$ due to sampling in time.

The sampled transform $\tilde{X}\left(\frac{k}{T_o}\right)$ is related to the Fourier coefficients $X^S(k)$ by

$$X^s(k) = \frac{1}{T_o}\tilde{X}\left(\frac{k}{T_o}\right) \qquad (2a)$$

So (1) is *not* periodic in k primarily because it corresponds to samples in frequency of the Fourier transform of one period of the *continuous*-time waveform, and secondarily because the sampled waveform is *not* digitally periodic.

Discrete Fourier Series

In the special circumstance that $TN = T_o$, the discrete signal becomes digitally periodic in n with period N. Making the appropriate substitutions in (1) we get

$$\frac{1}{N}X[k] = \frac{1}{N}\sum_{n=0}^{N-1} x[n]\, e^{-j\frac{2\pi k}{N}n} \qquad ; \; -\infty < k < \infty \qquad (3)$$

$$= \frac{1}{N}\mathrm{DTFT}\{\tilde{x}(nT)\}\Big|_{f=\frac{k}{T_o}} \qquad ; \; T_o = NT$$

$$= \frac{1}{NT}\sum_n \tilde{X}\left(f - \frac{n}{T}\right)\Big|_{f=\frac{k}{T_o}}$$

We can no longer use the same notation in (3) as in (1), (that is, $X^S(k)$) because now the aliased spectral lines really do interfere with those of the continuous time signal. (Recall that $X^S(k)$ denotes the Fourier series coefficients for the continuous-time waveform.) So we must use the new notation $X[k]$ which is recognized as the expression for the DFT when we cancel the $1/N$ factor on both sides of (3). The normalized DFT, we conclude, is somehow related to the Fourier series. Indeed, we will adopt the name, *discrete* Fourier series (**DFS**) in the special circumstance that $TN = T_o$.

The discrete Fourier series *is* digitally periodic in frequency; i.e., periodic in k. Hence by the IDFS (the substitution of (3) into (2.39a) here), the reformulation of $x[n]$ requires only N samples of $X[k]$.

Let's find the relationship between the DFT and the DFS. When the discrete-time signal becomes digitally periodic with period $T\,N = T_o$, it can be shown that Eq. (2) reduces to

$$\frac{1}{T}\sum_k \left\{ \frac{1}{T_o}\sum_n \tilde{X}\left(\frac{k}{T_o} - \frac{n}{T}\right) \right\} \delta\left(f - \frac{k}{T_o}\right) = \frac{1}{T}\sum_k \frac{X[k]}{N}\,\delta\left(f - \frac{k}{T_o}\right) \qquad (4)$$

$$\Rightarrow\; X[k] = \frac{N}{T_o}\sum_n \tilde{X}\left(\frac{k}{T_o} - \frac{n\,N}{N\,T}\right) = N\sum_n X^s(k - n\,N)$$

Equation (4) says that the Fourier coefficients now include a summation of aliased spectral components due to sampling; i.e., replications of Eq. (2a). When that happens, we call those new coefficients the discrete Fourier series coefficients, $X[k]$. Eq. (4) is obviously of period N in k as also shown in Porat (4.66).

In the case that the periodic signal is bandlimited to the Nyquist frequency, $X[k]$ in Eq. (4) reduces to

$$\frac{1}{N}X[k] = X^s(k) \qquad ;\; x(t) \text{ bandlimited}$$