DFT Relations for Upsampling and Downsampling

EE264 Session Notes -Jon Dattorro

Ref: [Oppenheim/Schafer, *Discrete–Time Signal Processing*, Prentice-Hall, 1989]

$$x[n] \leftrightarrow X[k], \text{ a length-}N\text{ DFT.}$$

Upsampling

L>1, the upsampling factor. Here we have,

$y_I[n] = x[n/L]$. From the text, pg.107, we derive the DFT of

$y_I[n]$ an NL-length sequence.

$$\begin{aligned}
Y_I[k] = \sum_{n=0}^{NL-1} y_I[n] W_{NL}^{nk} &= \sum_{n=0}^{NL-1} \left\{ \sum_{p=-\infty}^{\infty} x[p]\, \delta[n-pL] \right\} W_{NL}^{nk} \qquad ; k = 0 \to NL-1 \\
&= \sum_{p=-\infty}^{\infty} \left(\sum_{n=0}^{NL-1} x[p]\, \delta[n-pL] W_{NL}^{nk} \right) \\
&= \sum_{p=0}^{N-1} x[p]\, W_{NL}^{pLk} \\
&= \sum_{p=0}^{N-1} x[p]\, W_{N}^{pk} \\
&= X[k] \qquad \boldsymbol{;\ k = 0 \to NL-1}
\end{aligned}$$

Downsampling

M>1, the downsampling factor. Here we have,

$y_D[n] = x[nM]$. We derive the DFT of

$y_D[n]$ an N/M-length sequence assuming N is a multiple of M.

$$\begin{aligned}
Y_D[k] = \sum_{n=0}^{\frac{N}{M}-1} y_D[n] W_{\frac{N}{M}}^{nk} &= \sum_{n=0}^{\frac{N}{M}-1} \left(\sum_{p=-\infty}^{\infty} x[p]\, \delta[nM-p] \right) W_{\frac{N}{M}}^{nk} \qquad ; k = 0 \to \frac{N}{M}-1 \\
&= \sum_{\frac{p}{M}=0}^{\frac{N}{M}-1} \left(x[p]\, \frac{1}{M} \sum_{m=0}^{M-1} W_M^{-mp} \right) W_{\frac{N}{M}}^{\frac{p}{M}k} \\
&= \frac{1}{M} \sum_{m=0}^{M-1} \sum_{p=0}^{N-1} x[p]\, W_N^{pk}\, W_M^{-mp} \\
&= \frac{1}{M} \sum_{m=0}^{M-1} \sum_{p=0}^{N-1} x[p]\, W_N^{p(k-m\frac{N}{M})} \\
&= \frac{1}{M} \sum_{m=0}^{M-1} X\left(\left(k - m\frac{N}{M}\right)\right)_N \qquad \boldsymbol{;\ k = 0 \to \frac{N}{M}-1}
\end{aligned}$$

Downsampling followed by upsampling

$y_{DU}[n] = x[[n/L]M]$ having no cancellation possible in the argument. From these equations, the process of downsampling followed by upsampling where N is a multiple of M gives

$$Y_{DU}[k] = Y_D[k] \quad \text{for} \quad \boldsymbol{k = 0 \to (N/M)L - 1}.$$

Upsampling followed by downsampling

$y_{UD}[n] = x[nM/L]$. According to these equations, the process of upsampling followed by downsampling where NL is a multiple of M would result in

$$Y_{UD}[k] = \frac{1}{M}\sum_{m=0}^{M-1} X\left(\left(k - m\frac{NL}{M}\right)\right)_{NL} \qquad ; \boldsymbol{k = 0 \to \frac{NL}{M} - 1}$$

Repeating every sample of $\boldsymbol{x[n]}$ $\boldsymbol{L}$ times

$$Y_I[k]\sum_{l=0}^{L-1} W_{NL}^{kl}$$

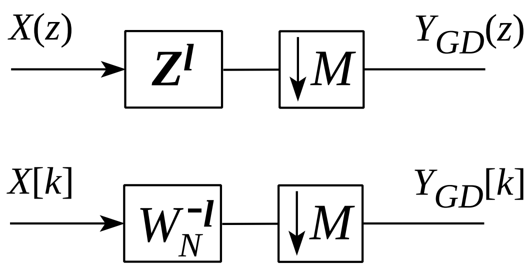
Generalized downsampling

that is, when there is an advance of l samples ahead of the downsampler.

$M>1$, the downsampling factor. Here we have,

$y_{GD}[n] = x[nM+l]$. We derive the DFT of

$y_{GD}[n]$ an N/M-length sequence again assuming that N is a multiple of M.



Note that these two operations are *not* equivalent; the latter corresponds to a circular advance in the time domain. It is the latter that we express below.

$$Y_{GD}[k] = \sum_{n=0}^{\frac{N}{M}-1} y_{GD}[n] W_{\frac{N}{M}}^{nk} = \sum_{n=0}^{\frac{N}{M}-1}\left(\sum_{p=-\infty}^{\infty} x((p+l))_N \;\delta[nM-p]\right) W_{\frac{N}{M}}^{nk} \qquad ; k = 0 \to \frac{N}{M} - 1$$

$$= \sum_{\frac{p}{M}=0}^{\frac{N}{M}-1}\left(x((p+l))_N \;\frac{1}{M}\sum_{m=0}^{M-1} W_M^{-mp}\right) W_{\frac{N}{M}}^{\frac{p}{M}k}$$

$$= \frac{1}{M}\sum_{m=0}^{M-1}\sum_{p=0}^{N-1} x((p+l))_N \; W_N^{pk}\; W_M^{-mp}$$

$$= \frac{1}{M}\sum_{m=0}^{M-1}\sum_{p=0}^{N-1} x((p+l))_N \; W_N^{p(k-m\frac{N}{M})}$$

$$= \frac{1}{M}\sum_{m=0}^{M-1} W_N^{-(k-m\frac{N}{M})l}\; X\left(\left(k - m\frac{N}{M}\right)\right)_N \qquad ; k = 0 \to \frac{N}{M} - 1$$