DFT Relations for Upsampling and Downsampling

EE264 Session Notes - Jon Dattorro

$x[n] \leftrightarrow X[k]$, a length-$N$ DFT.

**Upsampling**

$L > 1$, the upsampling factor. Here we have,

$y_1[n] = x[n/L]$. From the text, pg.107, we derive the DFT of

$y_1[n]$ an $NL$-length sequence.

$$Y_1[k] = \sum_{n=0}^{NL-1} y_1[n] W_{NL}^{nk} = \sum_{n=0}^{NL-1} \left( \sum_{p=-\infty}^{\infty} x[p] \delta[n-pL] \right) W_{NL}^{nk} \quad ; \quad k = 0 \rightarrow NL - 1$$

$$= \sum_{p=-\infty}^{\infty} \left( \sum_{n=0}^{NL-1} x[p] \delta[n-pL] W_{NL}^{nk} \right)$$

$$= \sum_{p=0}^{N-1} x[p] W_{NL}^{pLk}$$

$$= \sum_{p=0}^{N-1} x[p] W_N^{pk}$$

$$= X[k] \quad ; \quad k = 0 \rightarrow NL - 1$$

**Downsampling**

$M > 1$, the downsampling factor. Here we have,

$y_D[n] = x[nM]$. We derive the DFT of

$y_D[n]$ an $N/M$-length sequence assuming $N$ is a multiple of $M$.

$$Y_D[k] = \sum_{n=0}^{N-1} y_D[n] W_N^{nk} = \sum_{n=0}^{N-1} \left( \sum_{p=0}^{\infty} x[p] \delta[nM-p] \right) W_N^{nk} \quad ; \quad k = 0 \rightarrow \frac{N}{M} - 1$$

$$= \sum_{p=0}^{N-1} \left( x[p] \frac{1}{M} \sum_{m=0}^{M-1} W_M^{-mp} \right) W_N^{pk}$$

$$= \frac{1}{M} \sum_{m=0}^{M-1} \sum_{p=0}^{N-1} x[p] W_N^{pk} W_M^{-mp}$$

$$= \frac{1}{M} \sum_{m=0}^{M-1} \sum_{p=0}^{N-1} x[p] W_N^{p(k-m\frac{N}{M})}$$

$$= \frac{1}{M} \sum_{m=0}^{M-1} X\left(\left(k - m \frac{N}{M}\right)\right) \quad ; \quad k = 0 \rightarrow \frac{N}{M} - 1$$
Downsampling followed by upsampling

\[ y_{DU}[n] = x[n/L]M \] having no cancellation possible in the argument. From these equations, the process of downsampling followed by upsampling where \( N \) is a multiple of \( M \) gives

\[ Y_{DU}[k] = Y_D[k] \quad \text{for} \quad k = 0 \rightarrow (N/M)L - 1. \]

Upsampling followed by downsampling

\[ y_{UD}[n] = x[nM/L]. \] According to these equations, the process of upsampling followed by downsampling where \( NL \) is a multiple of \( M \) would result in

\[ Y_{UD}[k] = \frac{1}{M} \sum_{m=0}^{M-1} X\left(\frac{k-mNL}{M}\right)_{NL} \quad ; \quad k = 0 \rightarrow NL - 1. \]

Repeating every sample of \( x[n] \) \( L \) times

\[ Y_I[k] \sum_{l=0}^{L-1} W_{NL}^{kl}. \]

Generalized downsampling

that is, when there is an advance of \( l \) samples ahead of the downsampler. \( M>1 \), the downsampling factor. Here we have,

\[ y_{GD}[n] = x[nM+l]. \] We derive the DFT of

\[ y_{GD}[n] \] an \( N/M \)-length sequence again assuming that \( N \) is a multiple of \( M \).

\[ X(z) \xrightarrow{Z^l} Y_{GD}(z) \]

\[ X[k] \xrightarrow{W_N^{-l}} Y_{GD}[k] \]

Note that these two operations are not equivalent; the latter corresponds to a circular advance in the time domain. It is the latter that we express below.

\[ Y_{GD}[k] = \sum_{n=0}^{N-1} y_{GD}[n] W_N^{nk} = \sum_{n=0}^{N-1} \left( \sum_{p=-\infty}^{\infty} x((p+l))_N \delta(nM-p) \right) W_N^{nk} \quad ; \quad k = 0 \rightarrow N - 1 \]

\[ = \sum_{n=0}^{N-1} \left( \sum_{p=0}^{M-1} x((p+l))_N \frac{1}{M} \sum_{m=0}^{M-1} W_{M}^{-mp} \right) W_N^{nk} \]

\[ = \frac{1}{M} \sum_{m=0}^{M-1} \sum_{p=0}^{N-1} x((p+l))_N W_N^{pk} W_M^{-mp} \]

\[ = \frac{1}{M} \sum_{m=0}^{M-1} \sum_{p=0}^{N-1} x((p+l))_N W_N^{p(k-m)} \]

\[ = \frac{1}{M} \sum_{m=0}^{M-1} W_N^{-(k-mN)} I x \left( \left( k - m \frac{N}{M} \right)_N \right) \quad ; \quad k = 0 \rightarrow N - 1 \]