Near-Optimal Discretization of the Brachistochrone Problem

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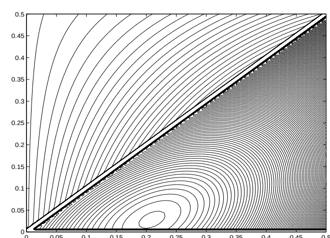

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Abstract

The analytical solution to the Brachistochrone problem is well known. [Thomas] [Strang] In this digital age, what is less-well studied is the proper discretization[1] of the problem so as to arrive at an optimal numerical solution. The problem thus posed asks, given a fixed number of samples with which to discretize a numerical solution, what is the optimal spacing of those samples along the abscissa? It is intuitively clear that the answer is not equi-spacing, because the proper spacing must somehow be related to curvature. Indeed, the study of non-uniform discretization is prompted by observations like this. [Marks] But an analytical solution to the optimal discretization problem is not what we seek here. What we develop is a simple numerical algorithm using a piecewise-linear fit to find the best discretization of the Brachistochrone problem for a fixed given <u>number</u> of samples. We conclude by speculating as to the best discretization using a fit of any order.

[1] A *discrete* one-dimensional signal conventionally refers to a continuous signal sampled with infinite precision along both axes. A *digital* signal is a discrete signal, but quantized along the ordinate. Both discrete and digital signals are said to be *sampled* signals.

1 What exactly <u>is</u> a cycloid?

We defer the reader to an excellent statement of the physical problem given in [Thomas].[2] "Among all smooth curves joining two given points, find that one along which a bead might slide, subject only to the force of gravity, in the shortest time[3]." Thomas' problem statement applies to any two arbitrary points (t,x), one implicitly higher than the other. The analytical solution of this arbitrary problem is most accurately described by him as an arc of a cycloid. A *cycloid* is a specific trajectory that is maintained by a strict relationship of the abscissa t with the ordinate x, each in the variable ϕ and the parameter a;

$$t = a(\phi - \sin(\phi)), \qquad x = a(1 + \cos(\phi)) \tag{1}$$

This means, for example, that scaling the ordinate without similarly scaling the abscissa, produces a trajectory which is not, strictly speaking, a cycloid. The utility of this rigid definition of the cycloid comes when we describe the solution to this Brachistochrone problem as "an arc" of a cycloid. That is to say, the solution for arbitrary points is not necessarily a half-cycle of a cycloid; the solution will always be some portion of a half-cycle of the cycloid described by Eq.(1). But the solution can never be a half-cycle of a cycloid expanded or contracted (scaled) in either coordinate alone.

In this paper we restrict our attention to a corresponding normalized mathematical problem, illustrated in <u>Figure 1</u>. The normalization is such that we restrict the range x and domain t of the function describing the path travelled by the bead, such that

$$x \in [0, 1], \qquad t \in \left[0, \frac{\pi}{2}\right] \tag{2}$$

and we specify the two "given points" as the boundaries of these two intervals. The purpose of this normalization is to make our exposition less abstract, the tradeoff being some loss of generality. The given start and end points are chosen such that the expected solution is precisely one half-cycle of the cycloid Eq.(1) having parameter $a = 1/2$, the half-cycle occurring when $\phi = \pi$.

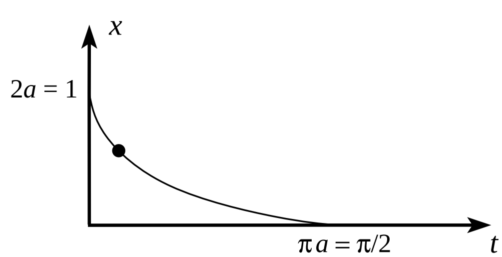

Figure 1. A bead travels along the "normalized" path shown, motivated only by gravity.

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[2]Neither [Thomas] nor [Strang] present the method of solution, known as the calculus of variations. The method <u>and</u> the solution are presented in [Bliss].

[3]Brachistochrone is Greek for "shortest time".

1.1 A Good Guess

This Brachistochrone problem is unusual in so far as we have a good obvious guess for the solution, which is not too far from the optimal solution; a straight line between the two normalized points,

$$x_o = 1 - \frac{2}{\pi} t \tag{3}$$

Eq.(3) will serve as the initialization of the curve x in all our subsequent experiments.

1.2 The Analytical Solution

The continuous optimal solution to the Brachistochrone problem is the curve x that minimizes the following integral:

$$T(x) = \frac{1}{2k} \int_0^{\frac{\pi}{2}} \sqrt{\frac{1 + \left(\frac{dx}{dt}\right)^2}{1 - x}}\, dt \qquad ; \; k \triangleq \sqrt{1 + \left(\frac{\pi}{2}\right)^2} \tag{4}$$

A lower bound $\mathrm{T}_{\min}$ on the value of the integral is provided by the cycloid trajectory described by the parametric Eq.(1) over one half-cycle;

$$\mathrm{T}_{\min} = T(x)\Big|_{\text{Eq.}(1), a=\frac{1}{2}} = \frac{1}{2k} \int_0^{\pi} \sqrt{\frac{\left(\frac{dt}{d\phi}\right)^2 + \left(\frac{dx}{d\phi}\right)^2}{1 - x}}\,\Bigg|_{\text{Eq.}(1), a=\frac{1}{2}} d\phi = \frac{\pi}{2k} = 0.843563 \tag{5}$$

The constant k is chosen for convenience such that $T(x_o)=1$ for x_o from Eq.(3).

2 A Strategy for Discrete Solution

We will numerically approximate x by finding a sampled, linearly interpolated version of it that minimizes Eq.(4). We change the meaning of our nomenclature by now designating x to represent a vector of sample values of the continuous function in Eq.(2). Likewise, we hereinafter designate the vector t to represent the corresponding instants in time.

$$\begin{aligned} & x \in \Re^{\mathrm{N}}, && t \in \Re^{\mathrm{N}} \\ & x(i) \in [0, 1], && t(i) \in \left[0, \frac{\pi}{2}\right] \\ & x(1) \triangleq 1,\; x(\mathrm{N}) \triangleq 0, && t(1) \triangleq 0,\; t(\mathrm{N}) \triangleq \frac{\pi}{2} \end{aligned} \tag{6}$$

We may precisely calculate the integral $T(x)$ piece-meal; over the intervals of time between instants t. Given the sample values and corresponding time instants, we rewrite Eq.(4) in these terms, approximating the continuity of points between samples by linear interpolation.

$$T_{\mathrm{D}}(x) \;=\; \frac{1}{2k}\sum_{i=1}^{\mathrm{N}-1}\int_{t(i)}^{t(i+1)}\sqrt{\frac{1+\left(\frac{x(i+1)-x(i)}{t(i+1)-t(i)}\right)^2}{1-x(i)-(\tau-t(i))\left(\frac{x(i+1)-x(i)}{t(i+1)-t(i)}\right)}}\;d\tau \qquad (7)$$

Since x is interpolated between samples, $T_{\mathrm{D}}(x)$ is only an approximation to $T(x)$ Eq.(4), having the properties

$$\lim_{\mathrm{N}\to\infty} T_{\mathrm{D}}(x) \;=\; T(x)$$
$$T_{\mathrm{D}}(x_o) \;=\; T(x_o) \;=\; 1 \qquad (8)$$

Eq.(8) is true independently of the scheme used to space samples in t.[4] It is of interest to minimize the required number of samples N to achieve a given level of accuracy with respect to the ideal solution Eq.(1). Alternatively, we wish to know the optimal spacing of samples Δt given a fixed number of samples N. We explore the latter.

Eq.(7) can be simplified significantly;[5]

$$T_{\mathrm{D}}(x,\Delta t) \;=\; \frac{1}{k}\sum_{i=1}^{\mathrm{N}-1}\sqrt{1+\left(\frac{\Delta t(i)}{x(i+1)-x(i)}\right)^2}\;\left(\sqrt{1-x(i+1)}-\sqrt{1-x(i)}\right)$$
$$\Delta t(i) \triangleq t(i+1)-t(i) \qquad (9)$$

Assuming that the intervals of integration $\Delta t(i)$ were known, we wish to minimize T_{D} in Eq.(9) with respect to the vector variable x. If instead we allow the intervals $\Delta t(i)$ to become the unknown variables, then we may minimize T_{D} with respect to those intervals. Our proposed strategy for discrete solution is to iterate the alternation of minimizing T_{D} first with respect to x and then with respect to Δt. We will use the strategy of fixing x to find the gradient of $T_{\mathrm{D}}(x,\Delta t)$ with respect to Δt,[6] and vice versa.

[4]We will not prove these last two assertions, but they are true intuitively so long as the samples are distributed well over the entire domain.

[5]This analytical simplification is the principal advantage to employing linear interpolation as opposed to some more sophisticated spline method.

[6]For this reason, we do not express dependence of x upon Δt in our notation throughout, even though the vector variable x is implicitly a function of Δt. The gradient of $T_{\mathrm{D}}(x(\Delta t),\Delta t)$ with respect to Δt is, generally, unknown unless we have some analytical expression for the curve $x(\Delta t)$ purely in terms of Δt, or unless x is momentarily fixed (momentarily independent of Δt).

2.1 The Alternating-Problem Statement

The alternating problems in their simplest form are expressed,

$$
\begin{array}{lll}
\textbf{Problem 1} & \min_{x} & T_{\mathrm{D}}(x,\Delta t) \\
& \ni & x(1) = 1 \\
& & x(\mathrm{N}) = 0 \\
& & \\
\textbf{Problem 2} & \min_{\Delta t} & T_{\mathrm{D}}(x,\Delta t) \\
& \ni & e^{\mathrm{T}}\Delta t = \dfrac{\pi}{2}
\end{array}
\tag{10}
$$

The fact that the solutions to these two problems are not simultaneously found suggests that any complete solution $(x, \Delta t)$ found via this method may be sub-optimal. We shall not prove that such an alternation converges, but, for the Brachistochrone problem, the observed convergence is quite slow when close to the optimal value of the objective.

2.1.1 Equivalent Problem 1

Problem 1 transforms easily into its equivalent unconstrained form,

$$
\textbf{Problem 1} \qquad \min_{\zeta} \; T_{\mathrm{D}}(\zeta,\Delta t), \qquad x \triangleq \begin{pmatrix} 1 \\ \zeta \\ 0 \end{pmatrix}, \;\; \zeta \in \Re^{\mathrm{N}-2} \tag{11}
$$

Because of the trivial nature of the equality constraints in Problem 1 Eq.(10), it is straightforward to show that the gradient of the constrained problem is identical to the gradient of the equivalent unconstrained problem if we ignore the first and last element of the former. The unconstrained gradient is then,

$$
\nabla_{\zeta} T_{\mathrm{D}}(\zeta,\Delta t) \; = \; \left.\begin{pmatrix} \nabla_x T_{\mathrm{D}}(x,\Delta t)_2 \\ \vdots \\ \nabla_x T_{\mathrm{D}}(x,\Delta t)_{\mathrm{N}-1} \end{pmatrix}\right|_{x = \left(\begin{smallmatrix} 1 \\ \zeta \\ 0 \end{smallmatrix}\right)} \tag{12}
$$

2.1.2 Equivalent Problem 2

We transform Problem 2 into an equivalent unconstrained problem using a nullspace method. First we fix Δt_o to any solution of the equality constraint, say

$$
\Delta t_o \triangleq \frac{\pi/2}{\mathrm{N}-1}\, e \;\in\; \Re^{\mathrm{N}-1} \tag{13}
$$

Then Problem 2 becomes, equivalently,

$$
\textbf{Problem 2} \qquad \min_{\xi} \; T_{\mathrm{D}}(x,\xi), \qquad \Delta t \triangleq \Delta t_o + Z\xi, \;\; Z \in \Re^{\mathrm{N}-1\times\mathrm{N}-2} \tag{14}
$$

where

$$Z = \begin{pmatrix} -1 & -1 & \cdots & -1 \\ 1 & & & 0 \\ & 1 & & \\ & & \ddots & \\ 0 & & & 1 \end{pmatrix} \qquad \text{and} \qquad e^{\mathrm{T}} Z = 0 \tag{15}$$

The gradient in terms of ξ is found from the gradient in terms of Δt as

$$\nabla_{\xi} T_{\mathrm{D}}(x, \xi) = Z^{\mathrm{T}} \nabla_{\Delta t} T_{\mathrm{D}}(x, \Delta t) \Big|_{\Delta t = \Delta t_o + Z\xi} \tag{16}$$

2.2 The Gradients for Trust-Region

$T_{\mathrm{D}}(x,\Delta t)$ in Eq.(9) will serve as the objective function in a minimization process. As we employ a trust-region method of numerical solution, [Prac.Opt.] we shall need analytical expressions for the gradients with respect to both x and Δt.[7]

For Problem 1,

$$\nabla_x T_{\mathrm{D}}(x, \Delta t)_i = \frac{1}{k}\left(\frac{q(i)}{\gamma(i)} - \frac{q(i-1)}{\gamma(i-1)} + \frac{\gamma(i) - \gamma(i-1)}{2\sqrt{1 - x(i)}}\right) \qquad ;\ i = 2\ldots\mathrm{N}-1 \tag{17}$$

where

$$q(i) \triangleq \frac{\Delta t(i)^2 \left(\sqrt{1 - x(i)} - \sqrt{1 - x(i+1)}\right)}{\gamma(i)\ \left(x(i) - x(i+1)\right)^3} \tag{18}$$

$$\gamma(i) \triangleq \sqrt{1 + \left(\frac{\Delta t(i)}{x(i+1) - x(i)}\right)^2} \tag{19}$$

For Problem 2,

$$\nabla_{\Delta t} T_{\mathrm{D}}(x, \Delta t)_i = \frac{1}{k} \frac{\Delta t(i)\left(\sqrt{1 - x(i+1)} - \sqrt{1 - x(i)}\right)}{\gamma(i)\ \left(x(i+1) - x(i)\right)^2} \qquad ;\ i = 1\ldots\mathrm{N}-1 \tag{20}$$

[7] These expressions were found via *Mathematica*. [Wolfram]

3 Numerical Solution via Trust-Region

Figure 2 shows the first few alternations towards the solution. Each plot depicts the samples $(\, x_{k+1},\ \Sigma_i\, \Delta t_k(i)\,);\ k = 0 \ldots 7$, then linearly interpolated. The first plot represents the solution of Problem 1 having $\Delta t = \Delta t_o$ from Eq.(13). The higher curve in each plot represents the ideal solution Eq.(1) sampled at equal angles of rotation ϕ, then linearly interpolated.

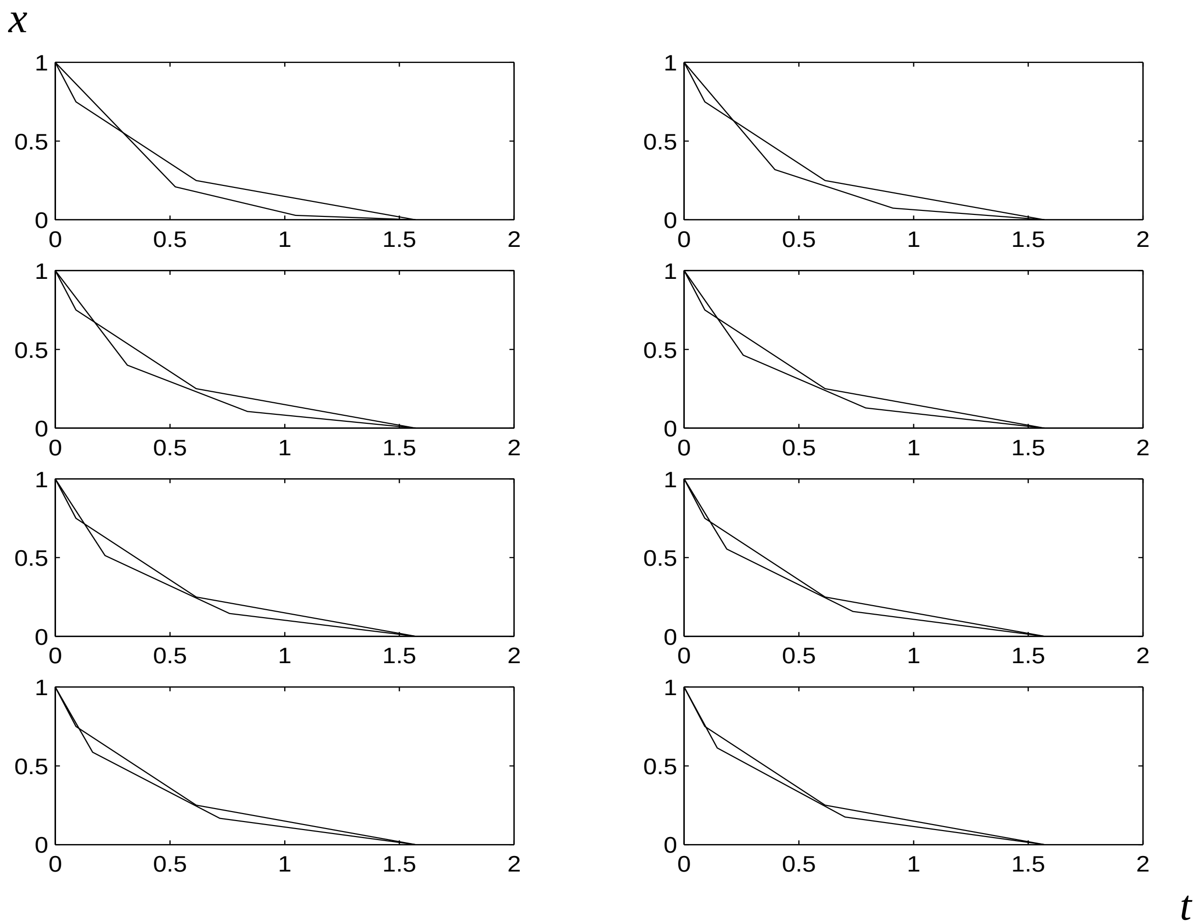

Figure 2. The first eight alternations with N = 4. Read left to right.

Figure 3 shows our near-optimal solution to the discrete Brachistochrone problem, given only a few samples. As before, the higher curve is the sampled ideal solution. All our data and solutions are obtained using the `fminunc()` function from the Matlab Optimization Toolbox v2.0 (R11). We provide that function with the gradients in Eq.(17) and Eq.(20). The number of iterations required is high because of our alternating-problem scheme; the pertinent solution-data can be found in Table 1.

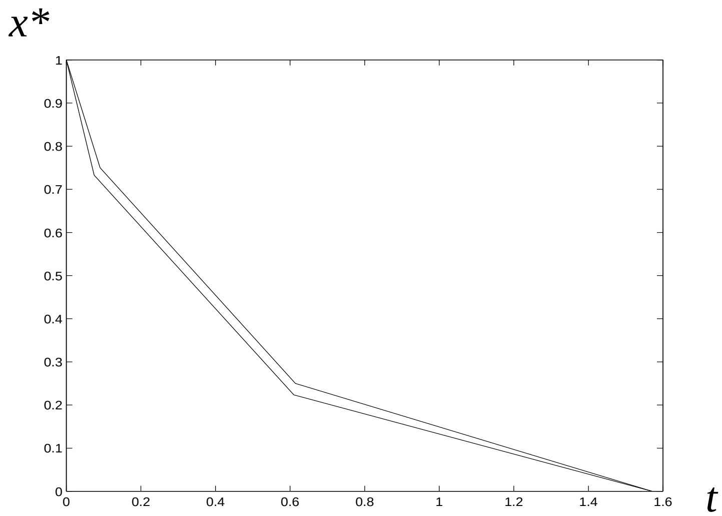

Figure 3. Best discrete approximation by linear interpolation at N=4. $T_D(x^*,\Delta t^*) = 0.853656$.

Table 1. Solution Data for the Alternating-Problem Scheme

Objective	Iterations	N	Execution Time[8]	Function Evaluations	MFlop
0.853656	148	4	12 sec.	148	0.717
0.844698	496	10	57	496	5.43
0.843884	439	25	89	439	22.3
0.843761	267	50	122	267	56.4
0.843716	132	100	223	132	128
0.843671	117	150	538	117	296

Recall from Eq.(5) that the lower bound on the optimal objective is $T_{min} = \underline{0.843563}$. A plausible explanation for the decreasing number of iterations with higher N in Table 1, is that there becomes less required adjustment of Δt as N increases. In contrast, Table 2 shows the solution data for the case that Δt is fixed throughout at Δt_o from Eq.(13).

Table 2. Solution Data for Problem 1 Alone

Objective	Iterations	N	Execution Time	Function Evaluations	MFlop
0.873230	6	4	1 sec.	6	0.0277
0.852035	8	10	1	8	0.0846
0.846545	10	25	2	10	0.543
0.844987	10	50	5	10	2.17
0.844257	11	100	16	11	9.89
0.844022	11	150	39	11	23.2

[8]The execution time is for a 300MHz Pentium II CPU running on a lap-top computer.

The settings of the various parameters required by `fminunc()` for all our trials are given in Table 3. These settings are stored in the Matlab structure called "`options`". The notation `[]` indicates that the default value of the parameter will be used, if relevant.

Table 3. `fminunc()` Parameters

```
    ActiveConstrTol: []
    DerivativeCheck: 'off'
        Diagnostics: 'off'
      DiffMaxChange: 0.1
      DiffMinChange: 1e-008
            Display: 'iter'
  GoalsExactAchieve: []
         GradConstr: []
            GradObj: 'on'
            Hessian: 'off'
           HessMult: []
        HessPattern: 'sparse(ones(numberOfVariables))'
         HessUpdate: 'gillmurray'
           Jacobian: []
          JacobMult: []
       JacobPattern: []
         LargeScale: 'on'

 LevenbergMarquardt: []
     LineSearchType: 'quadcubic'
        MaxFunEvals: '100*numberOfVariables'
            MaxIter: 400
         MaxPCGIter: 'max(1,floor(numberOfVariables/2))'
      MeritFunction: []
          MinAbsMax: []
     Preconditioner: []
   PrecondBandWidth: 0
   ShowStatusWindow: []
             TolCon: []
             TolFun: 1e-006
             TolPCG: 0.1
               TolX: 1e-006
           TypicalX: 'ones(numberOfVariables,1)'
```

4 Contour Plots of Objective

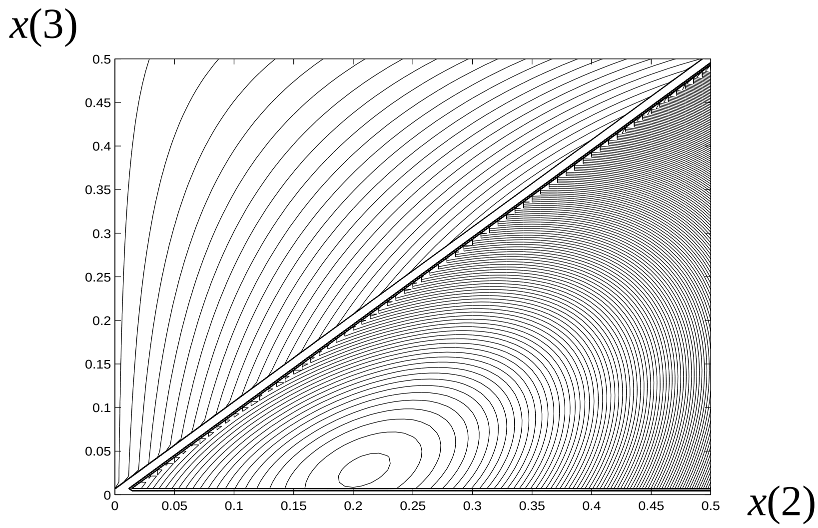

Figure 4. Contour plot for Problem 1 at $N = 4$ and $\Delta t = \Delta t_o$.

Regarding only Problem 1, Figure 4 shows some contour lines of $T_D(x,\Delta t)$ Eq.(9). The diagonal in the plot forms a blade of infinity. We see from Figure 4 that the initial condition on x Eq.(3) is critical to finding the desired solution. The desired solution is encircled by the smallest ellipsoid in the region where $x(2) > x(3)$. Problem 2 Eq.(14) is much better behaved as evidenced by Figure 5.

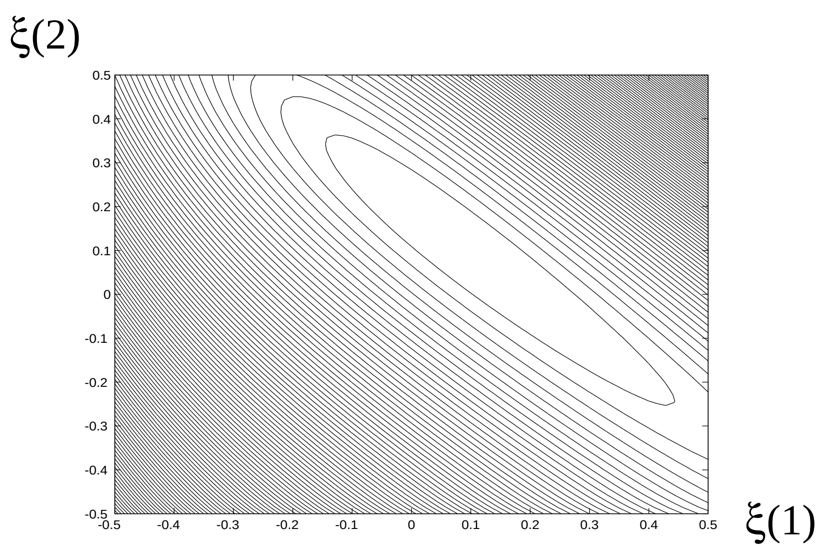

Figure 5. Contour plot for Problem 2 at $N = 4$ and $x = (1, 0.3, 0.03, 0)^T$.

5 Conclusions

The continuous Brachistochrone problem can be fit well by a piecewise-linear discrete approximation. But Eq.(8) tells us that there is error built into our fit which only disappears as $\mathrm{N} \to \infty$. It is obvious that the most error will occur for low N, which is why we concentrated there. What is not obvious is the best spacing of samples Δt given any finite N. We observed that with no optimization of the sample spacing, our linear fit makes its worst errors at locations of high curvature. That observation motivated a look into optimization of the sample-spacing vector Δt. Because of the proximity of our near-optimal solution for $\mathrm{N} = 4$ to the sampled ideal solution, illustrated in Figure 3, we speculate that the best sample spacing corresponds to equal angles of rotation ϕ in Eq.(1). We recommend that spacing independent of the particular order fit used to solve the discrete Brachistochrone problem. The data observed in Table 1 tells us that as N is increased, the sample spacing becomes less important. In any case, we developed a numerical algorithm which calculates, in the second phase of two alternating minimization problems, the best sample spacing for our chosen piecewise-linear fit.

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