- Vector Notation
- Spaces
- Projection
- Orthogonality
- Signal Measures
- Basis Vectors

- Vector Notation

- **Spaces**

• Inner Product Space

Inner Product:

$$\langle x \;\; y \rangle \triangleq \sum_{n=0}^{N-1} x(n)\overline{y(n)} \qquad\qquad \text{where } x,\ y \in \mathbf{C}^N$$

some Inner Product properties:

$$1\text{: } \langle x - y, z \rangle = \langle x, z \rangle - \langle y, z \rangle$$
$$2\text{: } \langle \alpha x, y \rangle = \alpha \langle x, y \rangle$$
$$3\text{: } \langle x, \alpha y \rangle = \overline{\alpha} \langle x, y \rangle$$

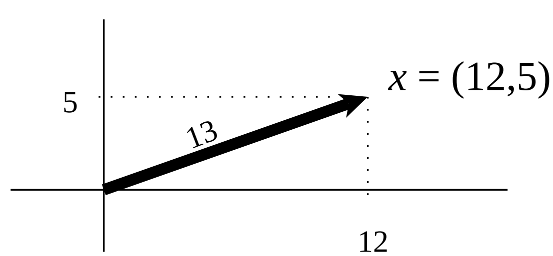
Inner Product is linear.

Norm:

$$\|x\|^2 \triangleq \langle x \;\; x \rangle$$

example: $\|x\| = \sqrt{12^2 + 5^2} = 13$

The Norm of a vector is its "length".



Properties of the Norm:

1: $\|x\| = 0 \iff x = 0$
2: $\|x + y\| \leq \|x\| + \|y\|$
3: for $c \in \mathbf{C}$, $\|cx\| = |c| \cdot \|x\|$
4: $\|x\| \cdot \|y\| \geq |\langle x, y \rangle|$ (Schwartz inequality)

- **Orthogonality**

$$x \perp y \iff \langle x, y \rangle = 0$$

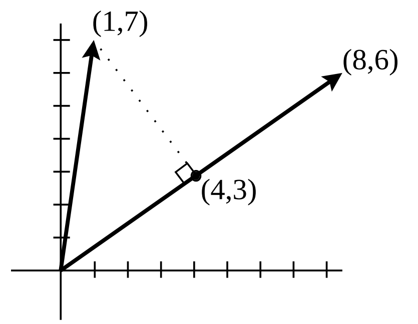
• Projection

orthogonal projection:

$$\mathcal{P}_x(y) = x_y \triangleq \left(\frac{\langle y, x\rangle}{\|x\|^2}\right) x$$

example:

$$\begin{aligned}
\mathcal{P}_x(y) &= \frac{\langle y, x\rangle}{\|x\|^2}\, x \\
&= \frac{\langle (1,7),(8,6)\rangle}{\|(8,6)\|^2}\,(8,6) \\
&= \frac{8+42}{64+36}\,(8,6) \\
&= \frac{1}{2}\,(8,6) \\
&= \boxed{(4,3)}
\end{aligned}$$



• Signal Measures

Mean (average):

$$\mu_x \triangleq \frac{1}{N}\sum_{n=0}^{N-1} x_n \qquad\qquad x \in \mathbf{C}^N$$

Energy:

$$\mathcal{E}_x \triangleq \langle x \;\; x\rangle = \|x\|^2 = \sum_{n=0}^{N-1} |x|^2$$

Power:

$$P_x \triangleq \frac{1}{N}\mathcal{E}_x = \frac{1}{N}\sum_{n=0}^{N-1} |x_n|^2$$

- **also called the mean square**
- **represents the average energy per sample**

Variance:

$$\sigma_x^2 \triangleq P_x - \mu_x^2$$

"mean square" – "squared mean"

$$\sigma_x^2 = \frac{1}{N}\sum_{n=0}^{N-1}(x_n - \mu_x)^2$$

– Removes "DC" component of signal – gives a better feel for the signal level.

• Basis Vectors

A set of mutually orthogonal vectors

"Natural" Basis (Catesian Basis)
– Basis set for samples in a signal

$$\hat{e}_0 = [1,\, 0,\, 0] \qquad \hat{e}_1 = [0,\, 1,\, 0] \qquad \hat{e}_2 = [0,\, 0,\, 1]$$

"Sinusoidal" Basis
– Basis set for samples in a spectrum

$$\hat{s}_k(n) \triangleq e^{j\omega_k nT} = e^{j2\pi kn/N}$$

3D-example:

$$\hat{s}_0 = e^{j2\pi 0n}\Big|_{n=0}^{3} = [1,\, 1,\, 1]$$

$$\hat{s}_1 = e^{j2\pi 1n}\Big|_{n=0}^{3} = [1,\, e^{j2\pi/3},\, e^{j4\pi/3}] = [1,\, \frac{-1+j\sqrt{3}}{2},\, \frac{-1-j\sqrt{3}}{2}]$$

$$\hat{s}_2 = e^{j2\pi 2n}\Big|_{n=0}^{3} = [1,\, e^{j4\pi/3},\, e^{j8\pi/3}] = [1,\, \frac{-1-j\sqrt{3}}{2},\, \frac{-1+j\sqrt{3}}{2}]$$

$$\hat{s}_0 \perp \hat{s}_1 \perp \hat{s}_2$$

Orthonormal Basis

The natural basis vectors all have a norm of 1. However, the sinusoidal basis vectors all have a norm of N.

Orthonormal Sinusoidal Basis:

$$\tilde{s}_k \triangleq \frac{\hat{s}_k}{\|\hat{s}_k\|}$$