

- Notation
- Linearity
- Reversal
- Symmetry
- Shift
- Stretch
- Decimation
- Convolution
- Correlation
- Power

- Definitions, Notation Conventions

Let $y(n)$ be a signal of length N . This corresponds to a spectrum $Y(k)$ also of length N .

signal	$y(n), \quad n = 0, 1, 2, \dots, N - 1$
spectrum	$Y(k), \quad k = 0, 1, 2, \dots, N - 1$

The relationship between $y(n)$ and $Y(k)$ is:

$$\text{DFT}_n(y) = Y(k) \triangleq \sum_{n=0}^{N-1} y(n) e^{-j2\pi nk/N} = \sum_{n=0}^{N-1} y(n) e^{-j\omega_k nT}$$

Standard style: time-domain signals are lowercase and frequency-domain related spectrum is in uppercase. Notation showing relation between a signal and its spectrum is an arrow with signal on the left:

$$y \leftrightarrow Y$$

- **Linearity Theorem:**

given constants α and β , the following relationship between domains is true:

$$\boxed{\alpha y_1 + \beta y_2 \leftrightarrow \alpha Y_1 + \beta Y_2}$$

Proof:

$$\begin{aligned} \text{DFT}(\alpha y_1 + \beta y_2) &= \sum_{n=0}^{N-1} (\alpha y_1 + \beta y_2) e^{-j2\pi nk/N} \\ &= \alpha \sum y_1(n) e^{-j2\pi nk/N} + \beta \sum y_2(n) e^{-j2\pi nk/N} \\ &= \alpha Y_1 + \beta Y_2 \end{aligned}$$

- **Reversal (Flip) Theorem:**

Reversal \leftrightarrow $\text{Flip}(Y)$,

$$\text{Flip}(\bar{y}) \leftrightarrow \bar{Y} \quad \text{or} \quad \text{Flip}(y) \leftrightarrow \text{Flip}(Y)$$

proof:

$$\text{DFT}(\text{FLIP}(\bar{y})) = \sum_{n=0}^{N-1} \overline{y(N-n)} e^{-j\omega n}$$

Let $m = N - n$

$$\begin{aligned} &= \sum_{m=0}^{N-1} \overline{y(m)} e^{-j\omega_k(N-m)} \\ &= \sum_{m=0}^{N-1} \overline{y(m)} e^{j\omega_k m} \\ &= \frac{1}{N-1} \sum_{m=0}^{N-1} y(m) e^{-j\omega_k m} \\ &= \frac{\text{DFT}_k(y(m))}{N-1} \\ &= \overline{Y(k)} \end{aligned}$$

- Symmetry Theorems

Symmetry:

- y real $\Leftrightarrow Y(-k) = \overline{Y(k)}$ i.e., Y is Hermitian
- y even $\Leftrightarrow Y$ even,
- y odd $\Leftrightarrow Y$ odd,
- y real/even $\Leftrightarrow Y$ real/even,
- y real/odd $\Leftrightarrow Y$
- y imaginary/even $\Leftrightarrow Y$ imaginary/even,
- y imaginary/odd $\Leftrightarrow Y$ real/odd,

- Shift Theorem

Shift:

$\leftrightarrow e^{-j\omega_k l} Y(k)$

(Delay in the time domain corresponds to
an additive linear phase term in the frequency domain)

Proof:

$$\begin{aligned}
 \text{DFT}(y(n-l)) &= \sum_{n=0}^{N-1} y(n-l) e^{-j\omega_k n} && \text{let } m = n - l \\
 &= \sum_{m=-l}^{N-1-l} y(m) e^{-j\omega_k (m+l)} \\
 &= e^{-j\omega_k l} \sum_{m=0}^{N-1} y(m) e^{-j\omega_k m} \\
 &= e^{-j\omega_k l} Y(k)
 \end{aligned}$$

Similarly:

$$e^{j\omega_k l} y(n) \leftrightarrow Y(k-l)$$

- Stretch Theorem

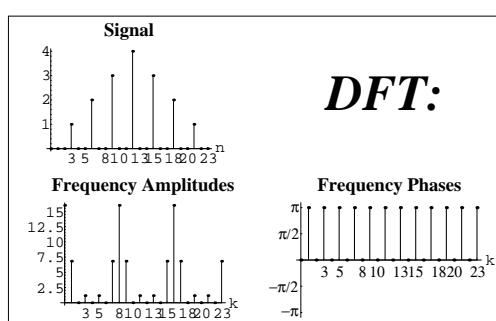
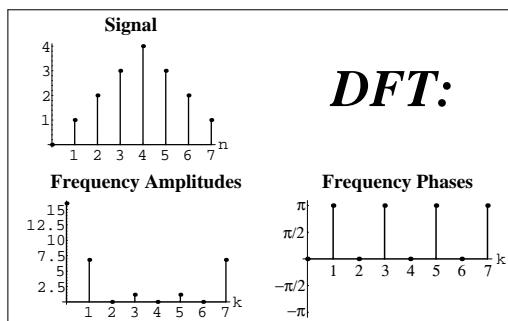
Stretch Theorem

Stretching \rightarrow Redata (Y)

Proof:

Let $y(n) = \text{STRETCH}_{L,n}(x)$, where $x(n)$ is length M , and y is length $N = L \cdot M$. Then,

$$\begin{aligned}
 Y(k) &= \sum_{n=0}^{N-1} y(n) e^{-j\omega_k n} && \text{let } m=n/L \\
 &= \sum_{m=0}^{M-1} x(m) e^{-j\omega_k mL} \\
 &= \sum_{m=0}^{M-1} x(m) e^{-j2\pi kmL/N} \\
 &= \sum_{m=0}^{M-1} x(m) e^{-j2\pi km/M} \\
 &= X(k)
 \end{aligned}$$

Example of the Stretch Theorem:


- **Decimation Theorem**

Decimation: $\text{Dec}_M(Y) \stackrel{!}{=} \text{Alias}_M(Y)$

Proof of the Decimation Theorem:

Let $X(l) = \text{ALIAS}_{M,l}(Y)$
 X has length L , Y has length N , $N=LM$. Then,

$$\begin{aligned} X(l) &= \sum_{k=0}^{M-1} Y(l+kL) \\ &= \sum_{k=0}^{M-1} \sum_{n=0}^{N-1} y(n) e^{-j\omega_l n + j2\pi kn/M} \\ &= \sum_{n=0}^{N-1} y(n) \sum_{k=0}^{M-1} e^{-j2\pi ln/N} e^{-j2\pi kn/M} \\ &= \sum_{n=0}^{N-1} y(n) e^{-j2\pi ln/N} \sum_{k=0}^{M-1} e^{-j2\pi kn/M} \\ &= \sum_{n=0}^{N-1} y(n) e^{-j\omega_ln} \begin{cases} M, & n \text{ a multiple of } M \text{ or } L \\ 0, & \text{otherwise} \end{cases} \end{aligned}$$

let $n = pM$

$$\begin{aligned} &= M \sum_{p=0}^{L-1} y(p) e^{-j\omega_lpM} \\ &= M \sum_{p=0}^{L-1} y(p) e^{-j2\pi p \frac{lM}{ML}} \\ &= M \sum_{p=0}^{L-1} y(p) e^{-j2\pi lp/L} \\ &= M \cdot \text{DFT}_l(\text{DEC}_M(y)) \end{aligned}$$

- Convolution Theorem

Convolution:

Similarly:

Proof:

$$\begin{aligned}
 x * y &= \sum_{m=0}^{N-1} x(m)y(n-m) \\
 \text{DFT}_k(x * y) &= \sum_{n=0}^{N-1} \sum_{m=0}^{N-1} x(m)y(n-m) e^{-j\omega_k n} \\
 &= \sum_{m=0}^{N-1} x(m) \sum_{n=0}^{N-1} y(n-m) e^{-j\omega_k n} \\
 &= \sum_{m=0}^{N-1} x(m) e^{-j\omega_k m} Y(k) \quad \text{by the shift theorem} \\
 &= X(k) Y(k)
 \end{aligned}$$

- Correlation Theorem

Correlation:

- **Power**

$$\text{Power} \triangleq \frac{1}{N} \langle X, Y \rangle$$

Proof:

$$\begin{aligned}\langle x, y \rangle &\triangleq \sum_{n=0}^{N-1} x(n) \overline{y(n)} \\&= (y * x)(0) \\&= \text{IDFT}_0(X \cdot \bar{Y}) \\&= \frac{1}{N} \sum_{k=0}^{N-1} X(k) \overline{Y(k)} \\&= \frac{1}{N} \langle X, Y \rangle\end{aligned}$$

Rayleigh Theorem:

$$\|x\|^2 = \frac{1}{N} \|X\|^2$$

$$\sum_{n=0}^{N-1} |x(n)|^2 = \frac{1}{N} \sum_{n=0}^{N-1} |X(k)|^2$$

– special case of the Power Theorem