The Generalized Complex Sinusoid

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Definition of the Generalized Complex Sinusoid

$$A \quad e^{-t/\tau} \quad e^{j\phi} \quad e^{j\omega t} \;=\; Ae^{-t/\tau} \; e^{j(\omega t+\phi)}$$

Amplitude Term **Decay Term** **Phase Term** **Frequency Term**

Physical Variables: (all are real-valued)

A = Amplitude at t=0 ϕ = Phase at t=0

τ = Decay time constant ω = Angular frequency

Below is the definition of the Generalized Complex Sinusoid in *Mathematica*. **`GeneralizedSinusoid`** is a function of t, w, etc. Parameter variables to a function follow the name of the function in square brackets. Parameter names end in an underscore to signify variables (only use the underscore when *defining* a function). Optional values are indicated by following a variable with a ":". A function is defined with the delayed assignment operator, "**`:=`**". **`GenS`** is a shorthand version of **`GeneralizedSinusoid`**.

```
GeneralizedSinusoid@t_, ω_,  A_: 1, ϕ_: 0, τ_: ∞D :=
 Amplitude@AD Decay@t, τD Phase@ϕD Frequency@t, ωD

GenS@parameters___D := GeneralizedSinusoid@parametersD
```

```
Amplitude@A_D := A
Decay@t_, τ_D := ⅇ^(-t ê τ)
Phase@ϕ_D := ⅇ^(ⅈ ϕ)
Frequency@t_, ω_D := ⅇ^(ⅈ ω t)
```

Amplitude:

The magnitude, or absolute value of a complex sinusoid remains constant for all time. Here are a few checks of this fact at various times for a complex sinusoid with amplitude 6.6:

```
GeneralizedSinusoid@0.7, 2 Pi, 6.6D
```

```
-2.03951 - 6.27697 I
```

```
Abs@%D
```

```
6.6
```

```
Abs@GeneralizedSinusoid@-253.35, 2 Pi, 6.6DD
```

```
6.6
```

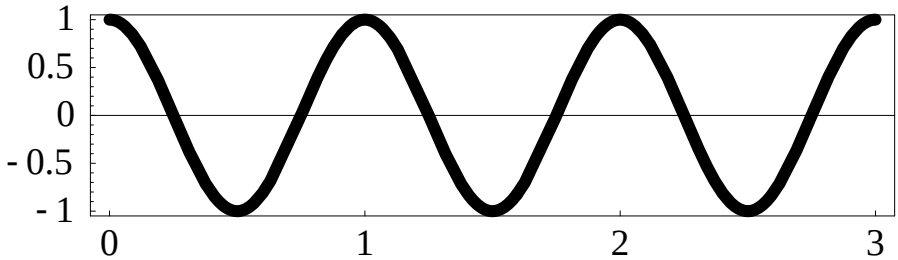
```
Abs@GeneralizedSinusoid@5642253.642, 2 Pi, 6.6DD
```

```
6.6
```

```
plotStyle2D = 8Frame → True, DefaultFont → 8"Times-Roman", 14<,
  FrameTicks → 8Range@0, 10D, Automatic, None, None<,
  PlotStyle → Thickness@0.015D, AspectRatio → 1 ê 4<;
```

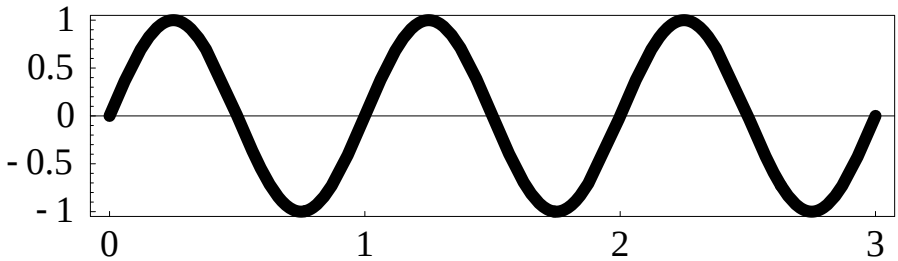
The real part of a complex sinusoid is equivalent to the cosine function. The peaks of the cosine always return to the same amplitude value, in this case 1:

```
Plot@Re@GenS@t, 2 πDD, 8t, 0, 3<, Evaluate@plotStyle2DDD;
```



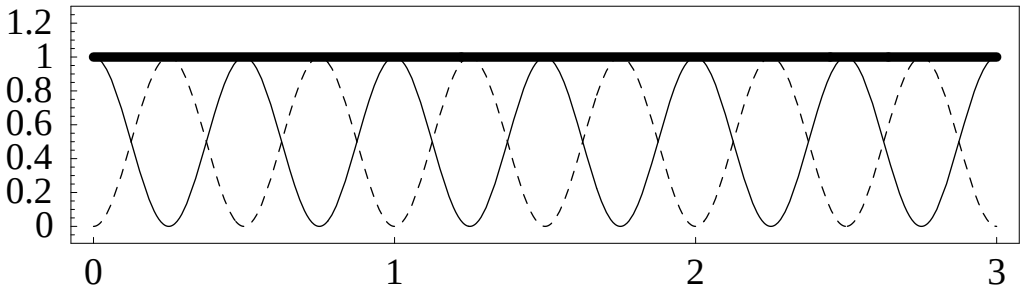
The imaginary part of a complex sinusoid is equivalent to the sine function:

```
Plot@Im@GenS@t, 2 πDD, 8t, 0, 3<, Evaluate@plotStyle2DDD;
```



Note that if you square each of the two curves above and add them together, you will get a straight line, as can be seen in the trignometric identity: cos^2 x + sin^2 x = 1.

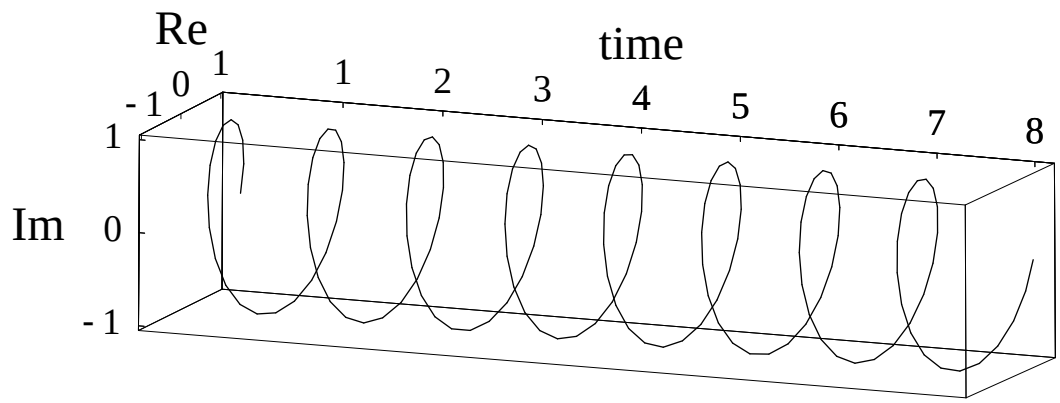
```
Show@Plot@Sin@2 π xD^2, 8x, 0, 3<,
  PlotRange -> 8-0.1, 1.3<, PlotStyle -> Dashing@80.01, 0.01<D,
  Evaluate@plotStyle2DD, PlotPoints -> 50D,
 Plot@Cos@2 π x D^2, 8x, 0, 3<, PlotPoints -> 50D, Plot@
  Sin@2 π xD^2 + Cos@2 π xD^2, 8x, 0, 3<, PlotStyle -> Thickness@0.01D D D;
```



Shown next is a plot of a steady-state complex sinusoid as it progresses in time.

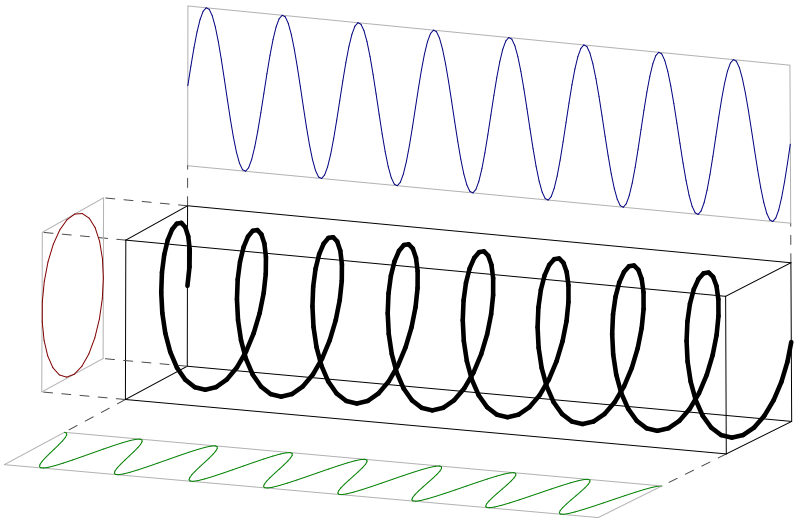
```
GenSPlot@8parameters___<, cycles_: 8, options___D :=
 ParametricPlot3D@8t,
   -Re@GenS@t, parametersDD, Im@GenS@t, parametersDD<, 8t, 0, cycles<,
  options, AxesLabel -> 8FontForm@"time", 8"Times-Bold", 18<D,
 FontForm@"Re", 8"Times-Bold", 18<D,
    FontForm@"Im", 8"Times-Bold", 18<D<,
  DefaultFont -> 8"Times-Roman", 14<, PlotPoints -> 150,
 Ticks -> 8Range@1, 100D,
    Transpose@8Range@-10, 10, 1D, Range@10, -10, -1D<D, Range@-10, 10, 1D<,
  AxesEdge → 88-1, 1<, 8-1, 1<, 8-1, 1<<,
 ViewPoint -> 8-10, -24, -6<, AspectRatio -> 1 ê 3D;
```

```
GenSPlot@82 π<D;
```

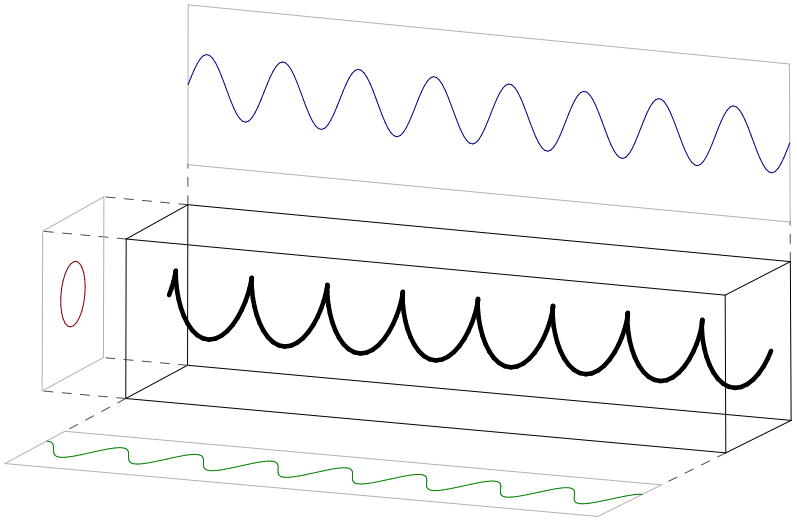


Below is shown a predefined function in the SCMTheory package to plot complex sinusoids. The black curve is the complex sinusoid. The blue curve at the top of the plot is the imaginary projection of the complex sinusoid. The green curve at the bottom of the plot is the imaginary projection of the complex sinusoid. The red curve to the left of the plot is a projection onto the complex plane of the complex sinusoid.

```
ComplexSinusoidPlot@0, 8, Amplitude -> 1D;
```



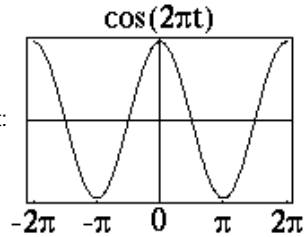
```
ComplexSinusoidPlot@0, 8, Amplitude -> 0.4, BoxScale → 81, 1<D;
```



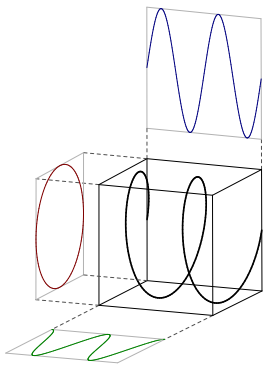
Phase:

no phase: $e^{j\omega t}$

real part:

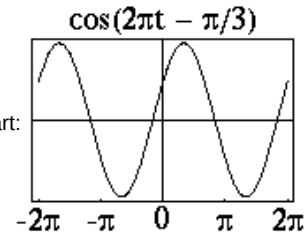



```
ComplexSinusoidPlot@-1, 1D;
```

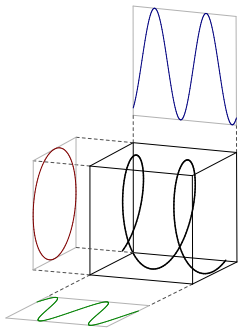


phase advance: $e^{j(\omega t-\phi)}$

real part:

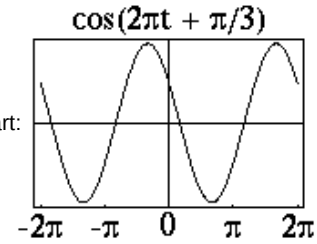



```
ComplexSinusoidPlot@-1, 1, Phase -> -π ê 3D;
```

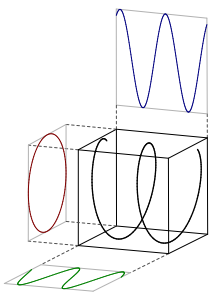


phase delay: $e^{j(\omega t+\phi)}$

real part:




```
ComplexSinusoidPlot@-1, 1, Phase -> π ê 3D;
```

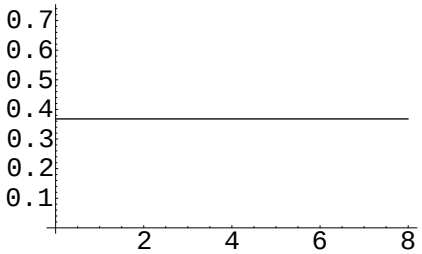


Decay:

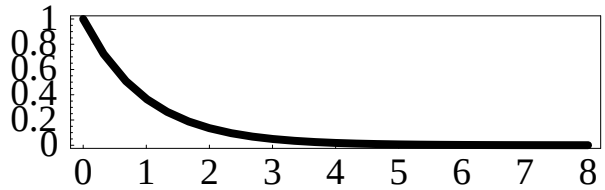
Exponential Decay Envelope

The decay time constant, t, controls an exponential amplitude envelope on the generalized complex sinusoid. t is the time for the envelope to decay by a factor of 1/„, or about 0.368. Look at the three plots below. The initial amplitude of the decay envelope is 1.0; therefore, the amplitude of the envelope at time t will by 0.37. Note below that the decayline plot crosses the amplitude envelope at time t.

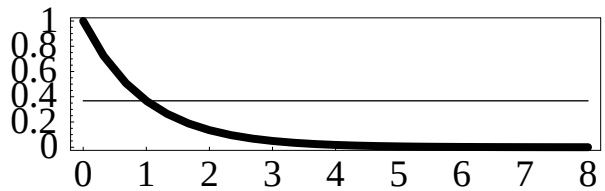
```
decayline = Plot@1 ê e, 8x, 0, 8<D;
```



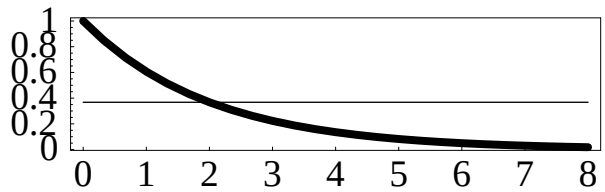
```
plotA = Plot@Decay@t, τ ê. τ → 1D, 8t, 0, 8<, PlotRange -> All,
  Evaluate@plotStyle2DDD;
```



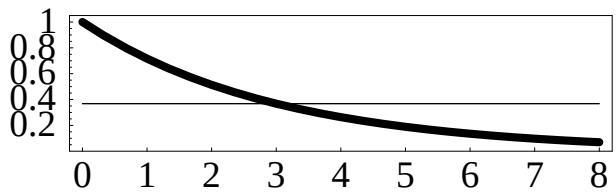

```
Show@plotA, decaylineD;
```




```
Show@Plot@Decay@t, τ ê. τ → 2D,
  8t, 0, 8<, PlotRange -> All, Evaluate@plotStyle2DDD,
 decaylineD;
```

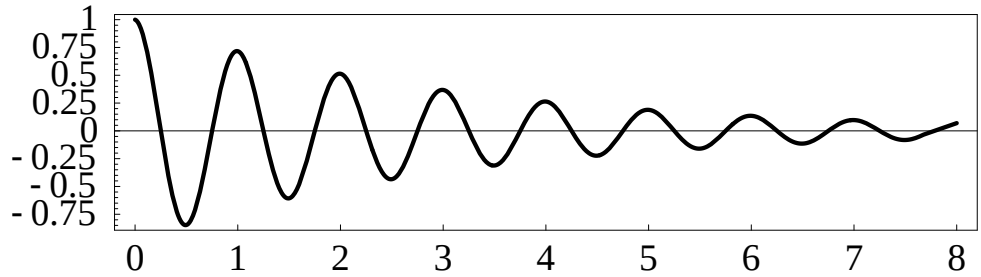



```
plotC = Plot@Decay@t, τ ê. τ → 3D,
  8t, 0, 8<, PlotRange -> 80, 1.05<, Evaluate@plotStyle2DDD;
Show@plotC, decaylineD;
```

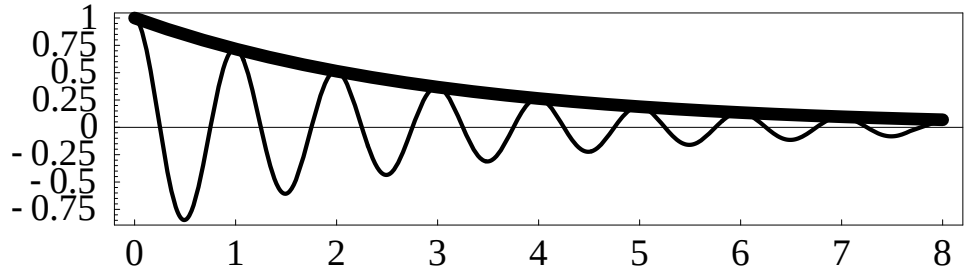


Exponential Decay Envelope on Sinusoids

```
plotD = Plot@Cos@2 π tD Decay@t, τ ê. τ → 3D, 8t, 0, 8<,
  PlotStyle -> Thickness@0.005D, Evaluate@plotStyle2DDD;
```

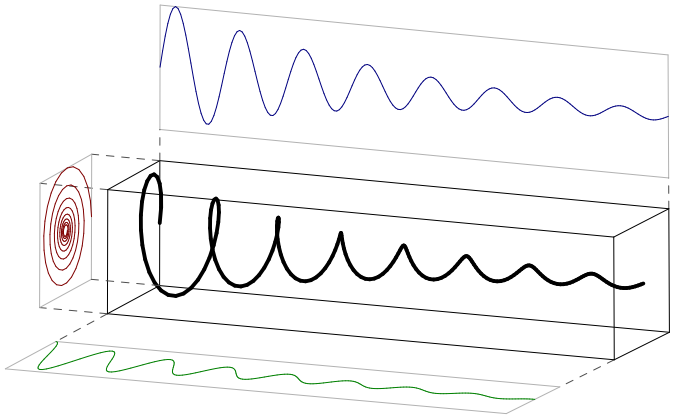



```
Show@plotD, plotCD;
```



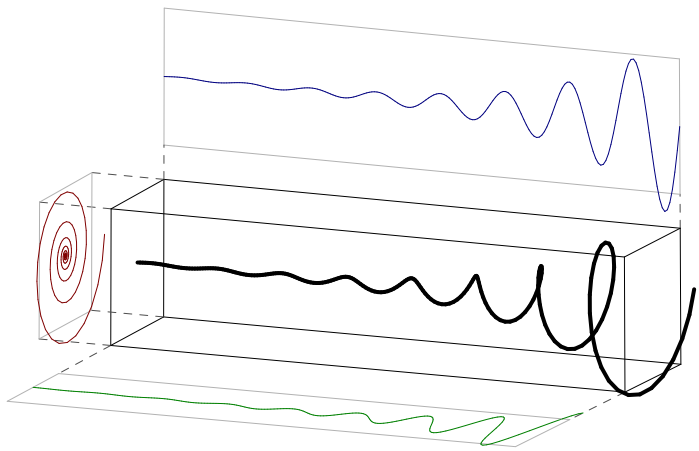
Exponential Decay on complex sinusoids works just the same as it does on real sinusoids. As time progresses, the magnitude of the steady-state complex sinusoid is multiplied by the exponential amplitude envelope.

```
ComplexSinusoidPlot@0, 8, Decay → 3D;
```

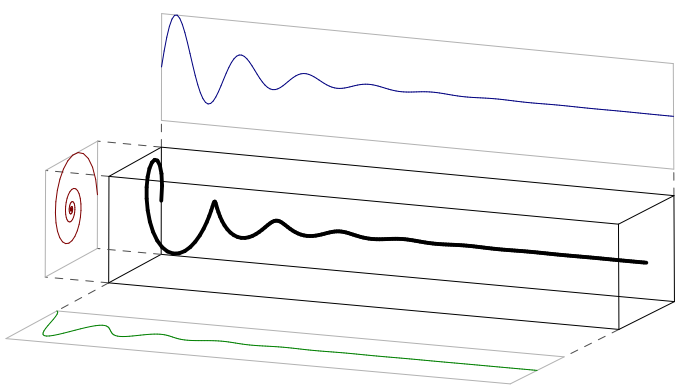


A negative time constant causes exponential growth rather than decay:

```
ComplexSinusoidPlot@0, 8, Decay → -1.6,
  Amplitude -> 0.01, BoxScale → 81, 1<D;
```



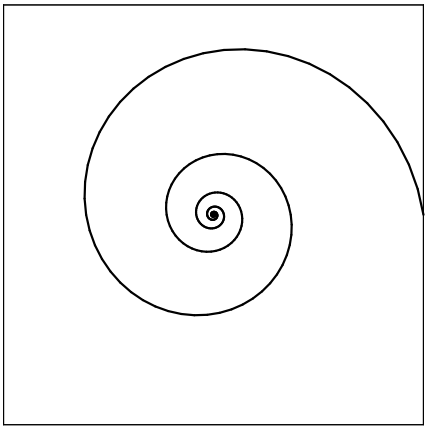
```
ComplexSinusoidPlot@0, 8, Decay → 1D;
```



An interesting viewpoint of the decay on the complex sinusoid can be seen on the complex plane projection which is shown as the red curve in the plot above. The plot below gives a better view of the curve shown in red above.

This curve is called a Logarithmic Sprial since it can be described in polar cordinates as log_b(r) = a q (or in exponential form at r = b^(aq)).

```
ComplexSinusoidPlot@0, 8, Decay -> 1, PlotPoints -> 400,
 ViewPoint → 8-1000, 0, 0<, ShowProjection -> False,
 BoxScale -> 81, 1<D;
```



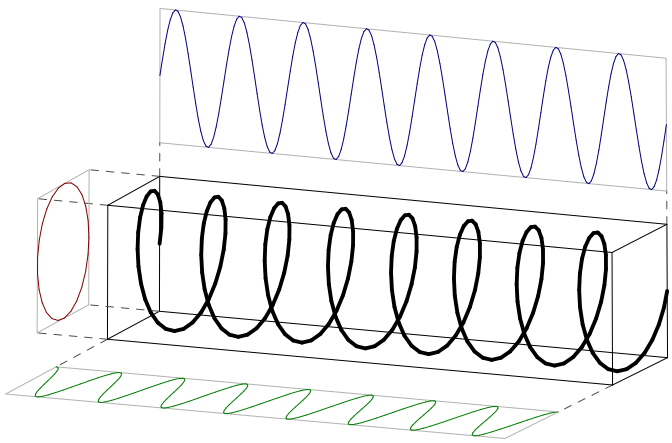
The logarithmic spiral was first discussed by Descartes in 1638; in 1698 its properties were further studied by the Swiss mathematician Jakob Bernoulli, who was particularly intergued by its *self-similarity*: if any portion is magnified or reduced, it is identical to any other portion of the curve.

T60

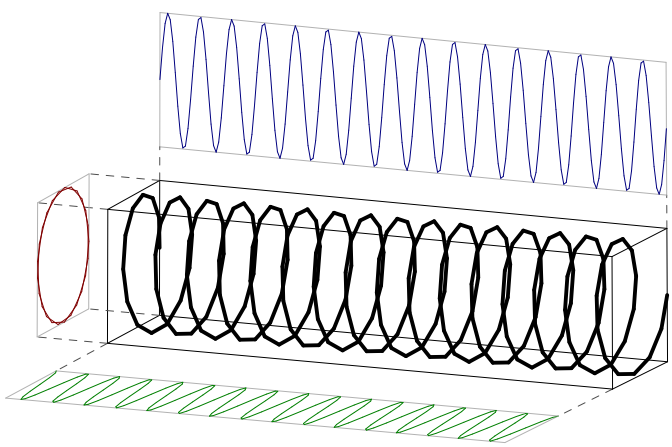
While t is the time it takes for the amplitude to decay to 0.37. T60 is the time it takes the amplitude to decay by 60 decibels or by an amplitude of 1000.

Frequency:

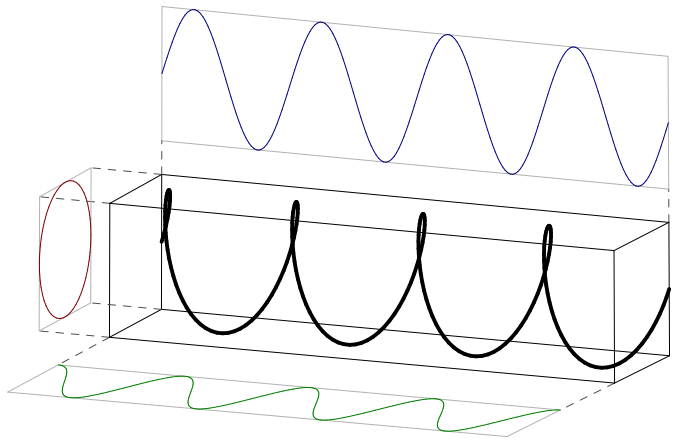
```
ComplexSinusoidPlot@0, 8, Frequency -> 1D;
```



```
ComplexSinusoidPlot@0, 8, Frequency -> 2D;
```

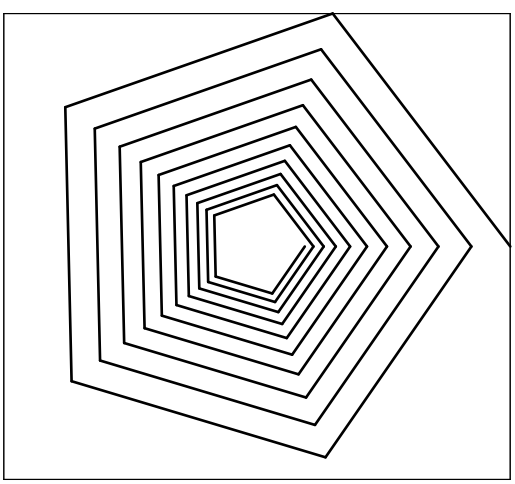


```
ComplexSinusoidPlot@0, 8, Frequency -> 0.5D;
```

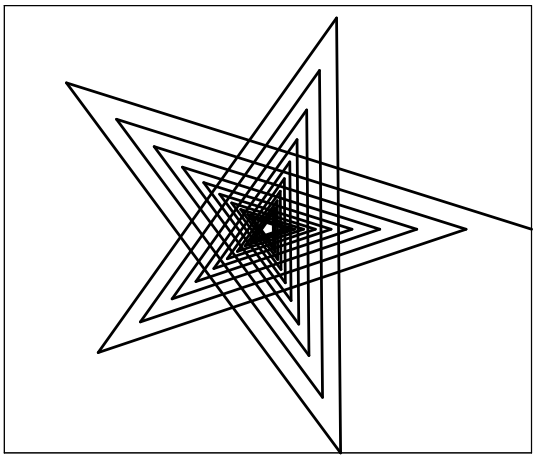


Neat pictures by aliasing

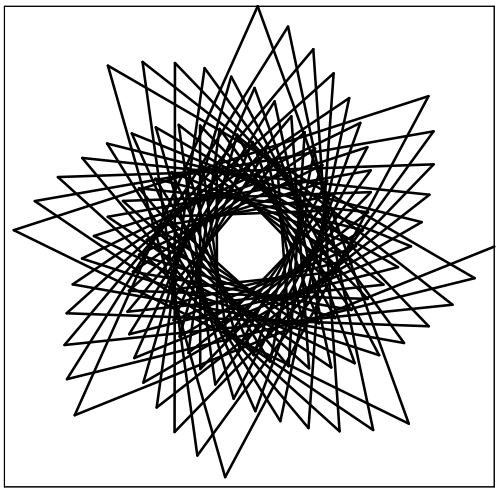
```
ComplexSinusoidPlot@0, 10, ShowProjection → False,
 Frequency -> 3601, PlotPoints -> 50, Decay → 6,
 ViewPoint -> 8-1000, 0, 0<D;
```



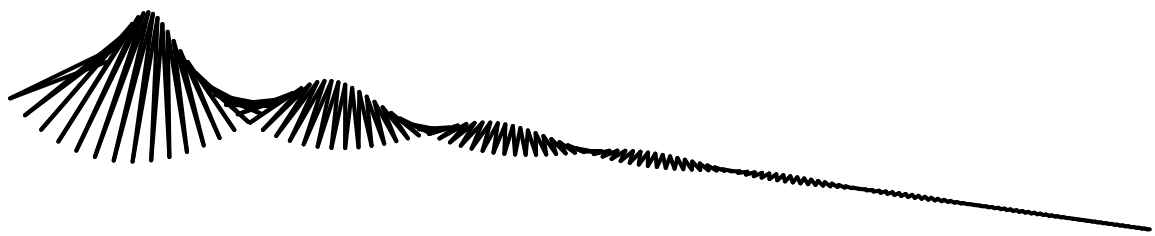
```
ComplexSinusoidPlot@0, 20, ShowProjection → False,
 Frequency -> 24301, PlotPoints -> 50, Decay → 7,
 ViewPoint -> 8-1000, 0, 0<D;
```



```
ComplexSinusoidPlot@0, 20, ShowProjection → False,
 Frequency -> 231.5467, PlotPoints -> 114, Decay → 19,
 ViewPoint -> 8-1000, 0, 0<D;
```



```
ComplexSinusoidPlot@0, 20, ShowProjection → False,
 Frequency -> 2511.5467, SinusoidStyle -> Thickness@0.003D,
 PlotPoints -> 311, Decay → 3, Boxed -> False,
 ViewPoint -> 8-1, -1, -.2<D;
```



```
ComplexSinusoidPlot@0, 20, ShowProjection → False,
 Frequency -> 2511.5467, SinusoidStyle -> Thickness@0.003D,
 PlotPoints -> 311, Decay → 3, Boxed -> False,
 ViewPoint -> 8-1000, 0, 0<D;
```