PERTURBATION THEORY AND ITS APPLICATION TO MUSICAL AIR COLUMN ADJUSTMENT

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WHAT IS THE WAVE EQUATION?

As long as density & bulk modulus are same everywhere
Rayleigh 1876 etc etc

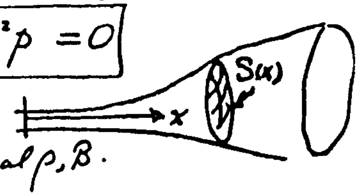
$$\begin{cases} \phi(x,r) = J_0\left[r\sqrt{\frac{\partial^2}{\partial x^2}+(\omega/c)^2}\right] F(x) \\ F(x) = \phi(x,0) \end{cases}$$

With shape of boundaries coming in via

$$-\frac{\partial\phi}{\partial x}\frac{\partial a}{\partial x}+\frac{\partial\phi}{\partial r}=0 \qquad r=a \text{ at wall}$$

Simplified version known for 200 years is

$$\boxed{p''+(S'/S)p'+k^2p=0}$$



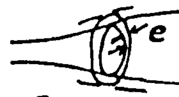
Note that Z depends on shape of horn, and on local ρ, B.

This is not too hard to solve, not too hard to "understand"
has many interesting kinds of systematic behavior

"TEXTBOOK PERTURBATION THEORY" (as in QM)

There are many nicer ways to get $\Delta\omega_n$ ... that don't depend on orthog. P_n's

$$\Delta\omega_n = \frac{\text{Const}}{\omega_n}\int S_0(x)\,p_n(x)\,Z(x)\,p_n(x)\,dx$$



$r(x) = a_0 + e$
$S_0 = \pi a_0^2$

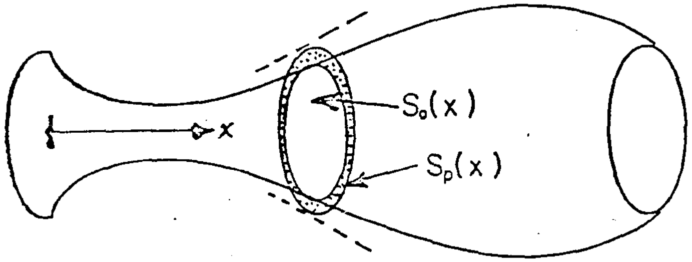
$$L\left[\frac{e'}{a_0}+\frac{e}{a_0}\,\frac{2a_0'}{a_0}\right]$$

THIS IS A MESS TO THINK WITH

IT ISNT EVEN RIGHT UNLESS YOU CAN PROPERLY SHOW THE ORTHOGONALITY OF THE P_n'S.

(Trouble arises from boundary conditions at ends where one usually gets $(P/p') = (\text{function of } \omega)$!)

A FRIENDLY PERTURBATION METHOD



CROSS SECTION CHANGE = $S_p(x)$

EHRENFEST'S ADIABATIC INVARIANCE

ΔE = WORK DONE AGAINST RADIATION PRESSURE AS WALLS ARE MOVED TO GIVE HORN NEW SHAPE

RADIATION PRESSURE = $U(x) - K(x)$

U, K = PE, KE DENSITIES

$$(\Delta\omega/\omega)_n = \frac{\int [K(x) - U(x)]_n S_p(x)\,dx}{\int [K(x) + U(x)]_n S_o(x)\,dx}$$

$$= \frac{\int [\text{LAGRANGIAN DENSITY}]\, S_p\,dx}{\int [\text{ENERGY DENSITY}]\, S_o\,dx}$$

$U_n(x) = (1/2B)\,p_n^2(x)$

$K_n(x) = (\rho/2)\,V_n^2(x) = (1/2B)(c/\omega)_n^2(\partial p/\partial x)_n^2$

FOR PRACTICAL PURPOSES

WE FIND IT USEFUL TO DEFINE THE PERTURBATION WEIGHT FUNCTION

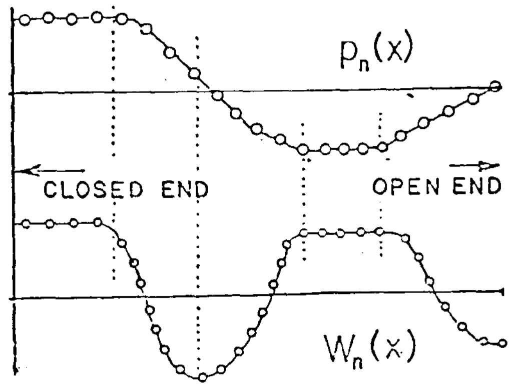
$W_n(x)$

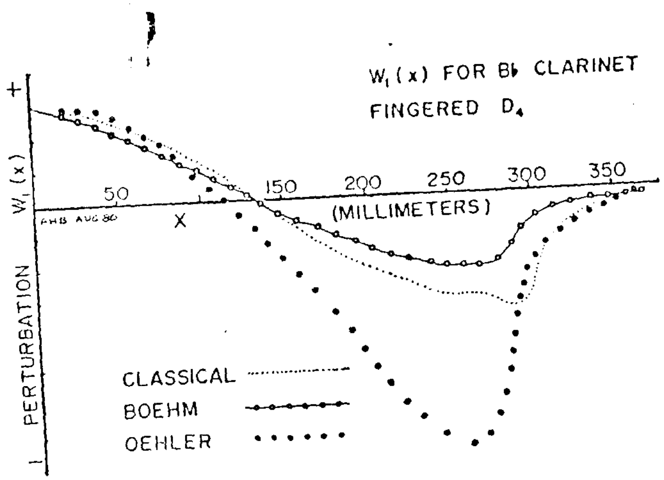
SUCH THAT

$$(\Delta F/F)_n = -\int W_n S_p\,dx$$

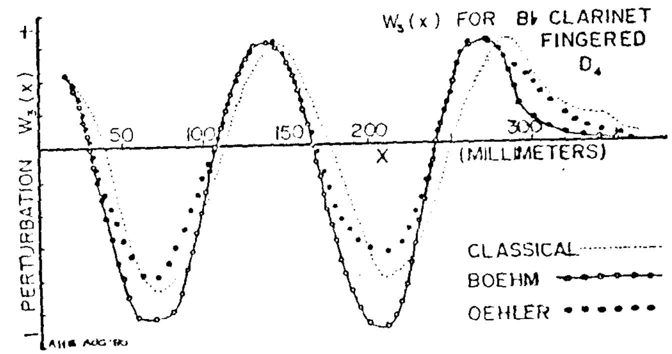
(A) ENLARGEMENT WHERE $p_n(x)$ IS LARGE —— LOWERS F_n

(B) ENLARGEMENT WHERE GRAD p IS LARGE —— RAISES F_n

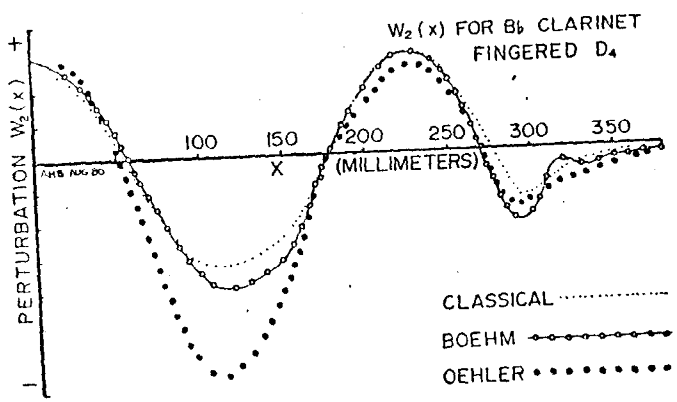




$W_1(x)$ FOR B♭ CLARINET
FINGERED D_4
+
-
PERTURBATION $W_1(x)$
50
150
200
250
300
350
X
(MILLIMETERS)
AHB AUG 80
CLASSICAL
BOEHM
OEHLER

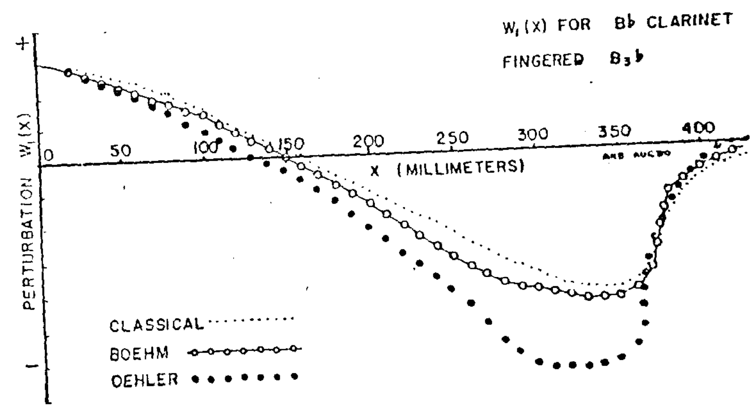


$W_3(x)$ FOR B♭ CLARINET
FINGERED
D_4
+
-
PERTURBATION $W_3(x)$
50
100
150
200
300
X
(MILLIMETERS)
AHB AUG 80
CLASSICAL
BOEHM
OEHLER



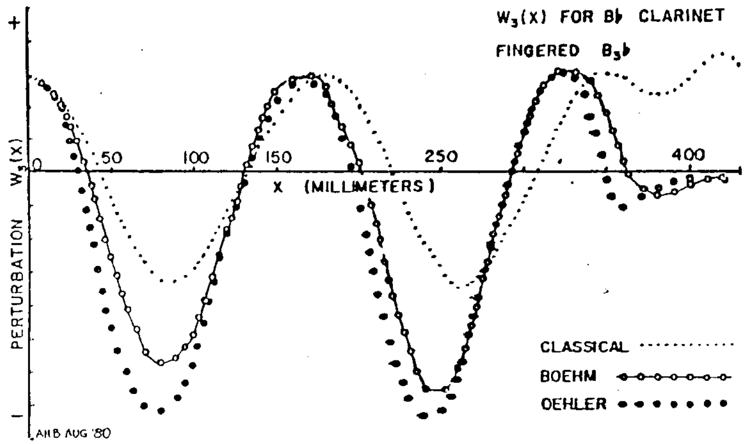
$W_2(x)$ FOR B♭ CLARINET
FINGERED D_4
+
-
PERTURBATION $W_2(x)$
100
150
200
250
300
350
X
(MILLIMETERS)
AHB AUG 80
CLASSICAL
BOEHM
OEHLER

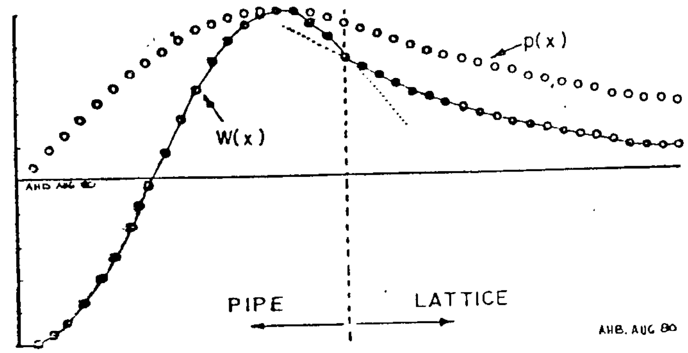
(4)

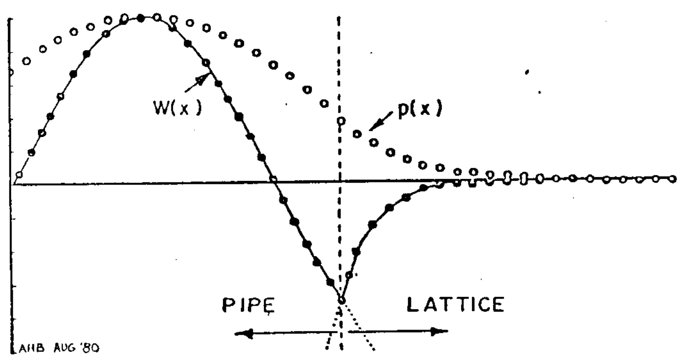


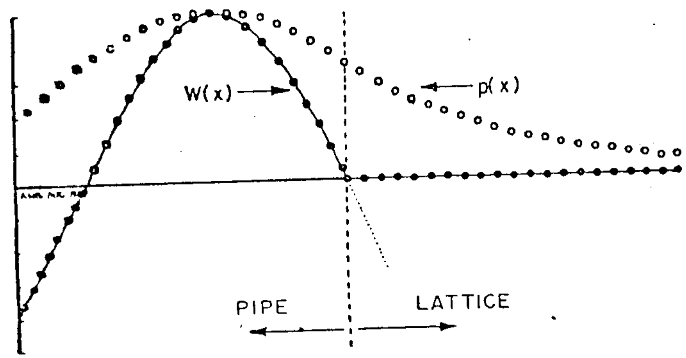
$W_1(x)$ FOR B♭ CLARINET
FINGERED B_3♭
+
-
PERTURBATION $W_1(x)$
0
50
100
150
200
250
300
350
400
X (MILLIMETERS)
AHB AUG 80
CLASSICAL
BOEHM
OEHLER

(5)


W₃(X) FOR B♭ CLARINET
FINGERED B₃♭
PERTURBATION W₃(X)
X (MILLIMETERS)
0
50
100
150
250
400
+
−
CLASSICAL
BOEHM
OEHLER
AHB AUG '80

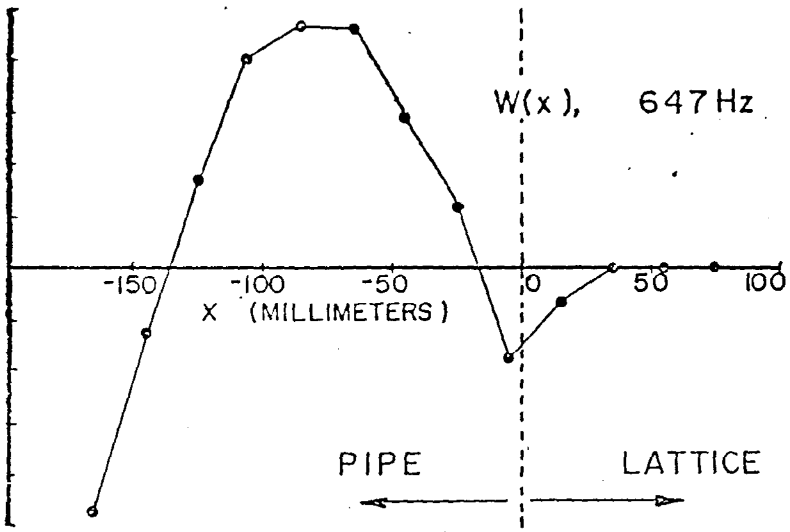

p(x)
W(x)
PIPE
LATTICE
AHB. AUG 80

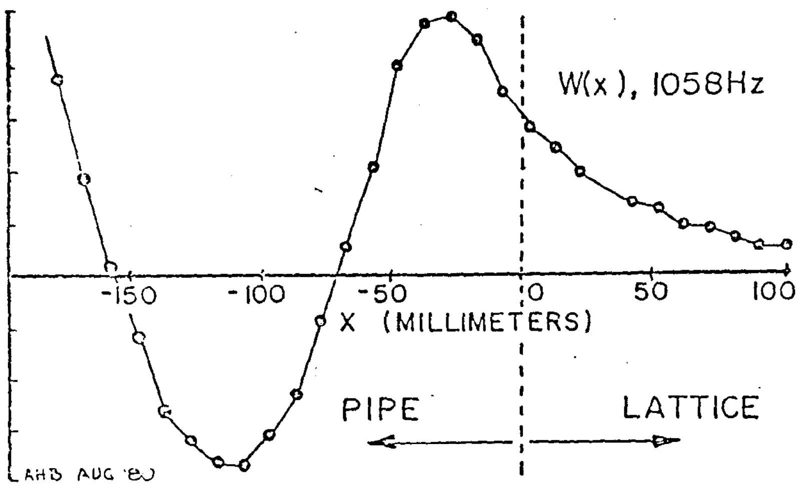

W(x)
p(x)
PIPE
LATTICE
AHB AUG '80


W(x)
p(x)
PIPE
LATTICE

(6)

(7)

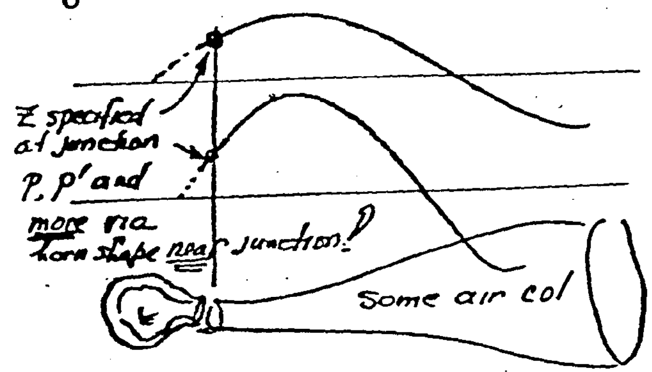




(8)

OTHER PERTURBATIONS:

(I) Tinkering with the end conditions alters the eigen values.



Simplest version is similar to lowering of energy levels in a potential well wt walls of finite height →

Note however that these end-modifications not only stretch and shrink p'' in the main air column, they also can slide it lengthwise along it —

For this reason, End perts — alter middle bore perts

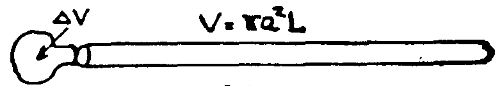
This can cause endless confusion if one forgets it.

(II) Temperature, humidity, composition gradients, viscous, thermal boundary layers; turbulence effects etc etc. COMPLEX EIGENVALUES → ALTERS PERT NEAR NODES OF PRESSURE, AND OF FLOW.

(9)

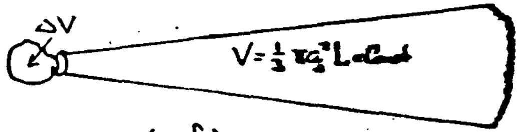
MOUTHPIECE AND REED CAVITY

(1)



$$\left(\frac{\Delta f}{f}\right) \cong -\frac{\Delta V}{V}$$ INDEP OF n

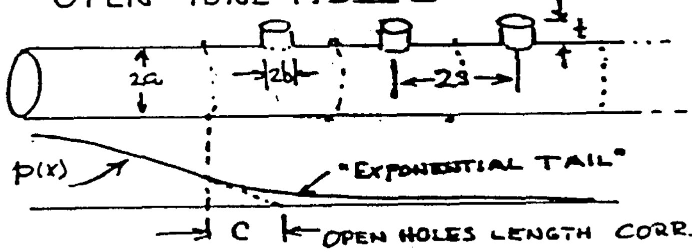
(2)



$$\left(\frac{\Delta f}{f}\right) \cong -\left(\frac{\Delta V}{V}\right)\{\mathrm{CONST}\}\cdot n^2$$

RISES RAPIDLY WITH n

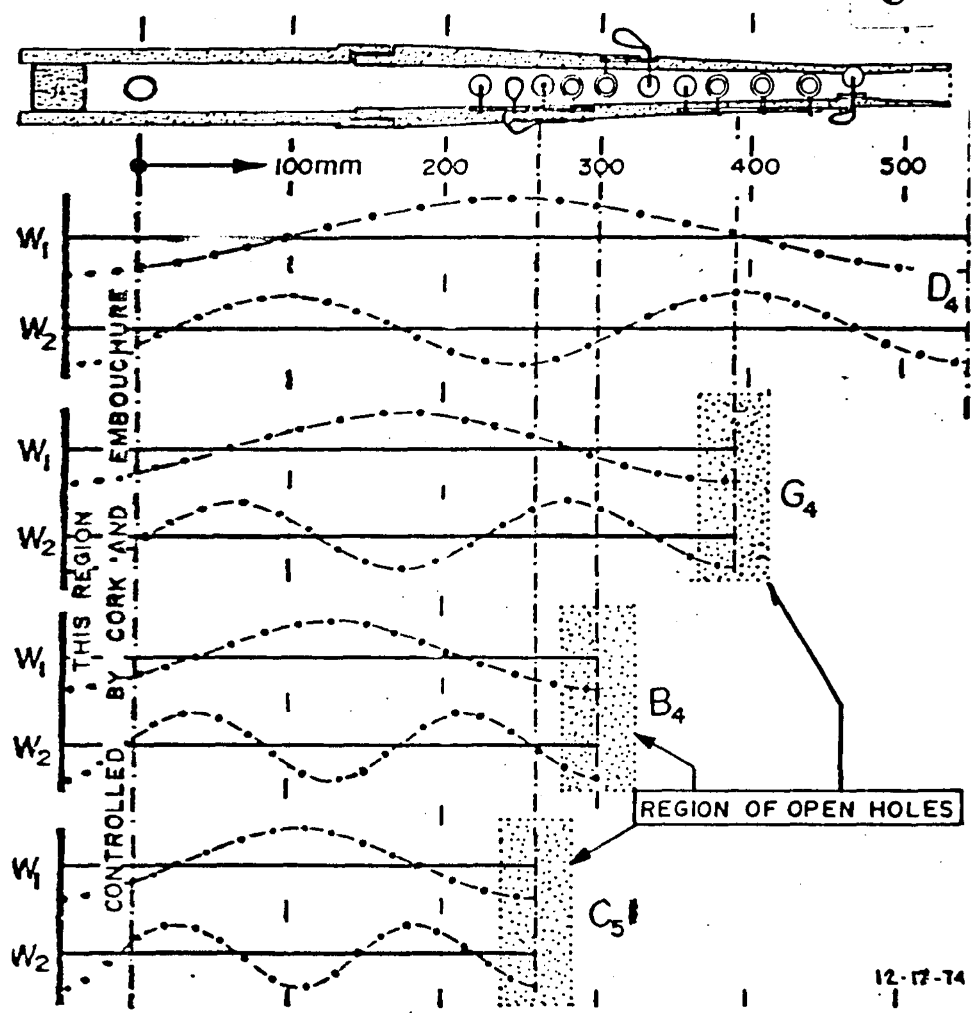
OPEN TONE HOLES



$$C \cong s\sqrt{1 + 2(t/s)(a/b)^2} \quad \text{FOR } f_n \ll f_c$$

$$f_c = \left(\frac{c}{2\pi}\right)(b/a)\sqrt{1/2st}$$

(10)



(11)