

Brahms at the Piano: An Analysis of Data from the Brahms Cylinder

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BACKGROUND

On 2 December 1889, Theo Wangemann, a representative of Thomas Edison, recorded Johannes Brahms introducing himself and performing two segments of music at the piano. The recording took place at the house of the Fellingner family [1]. The works recorded included measures 13–72 of a solo arrangement of Brahms's "Ungarischen Tanz No. 1" (First Hungarian Dance) and a segment of a paraphrase of Strauss's "Libelle" [2].

Features of Hungarian music became an integral part of Brahms's musical style. He performed Hungarian gypsy music on concert tours with the violinist Remenyi earlier in his career. Originally composed between 1852 and 1864, the four-hand set of 20 dances was transcribed by Brahms for two hands in 1872 and for orchestra in 1874. A violin and piano arrangement of these pieces was later prepared by Joachim. In addition to the Hungarian dances, Brahms wrote the

"Variations on a Hungarian Song" (op. 21, no. 1) in 1857 and the "11 Zigeunerlieder" (op. 103) in 1887. The "Piano Quartet" (op. 25), written in 1861, features a rondo alla zingarese and a set of gypsy poems in translation that comprise four of the op. 112 quartets (1891).

The *Ungarische Tanze* were enormously popular during

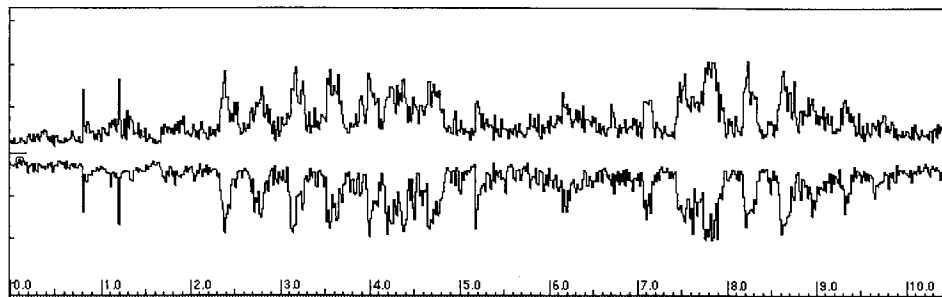
ABSTRACT

In 1889, Johannes Brahms recorded a segment of the first of his *Ungarische Tanze* (Hungarian Dances) in an arrangement for solo piano. The authors describe the analysis and reconstruction of this recording and examine the implications of this work as a contribution to the understanding of performance practices in the late nineteenth century.

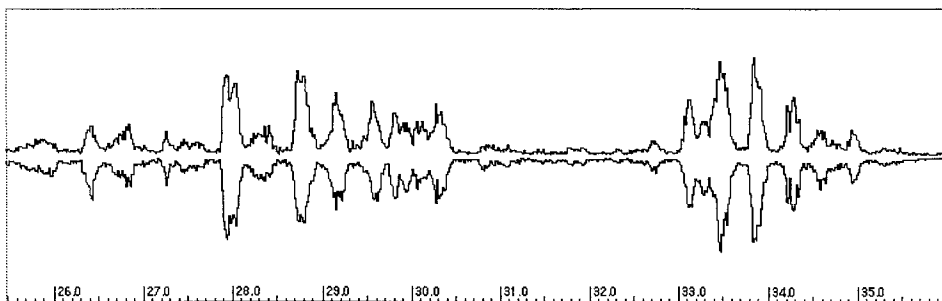
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(a)



(b)

Fig. 1. Sound pressure graphs of (a) the original and (b) reconstructed sound files of Brahms's "Ungarischen Tanz No. 1." The segments show graph measures 13–24 of the composition.

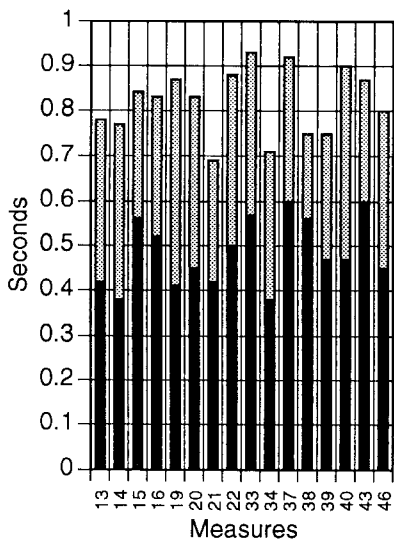


Fig. 2. Durations of dotted quarter (black) and eighth (grey) notes in measures 13–46 of “Ungarischen Tanz No. 1.”

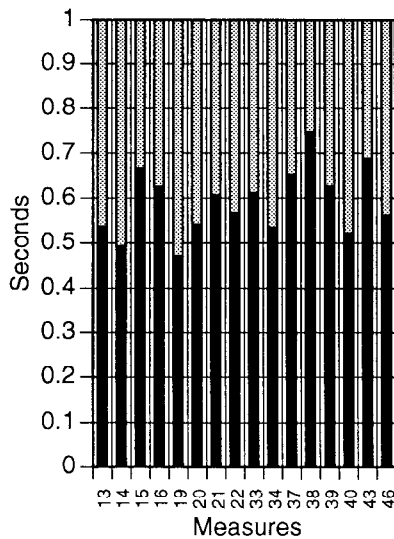
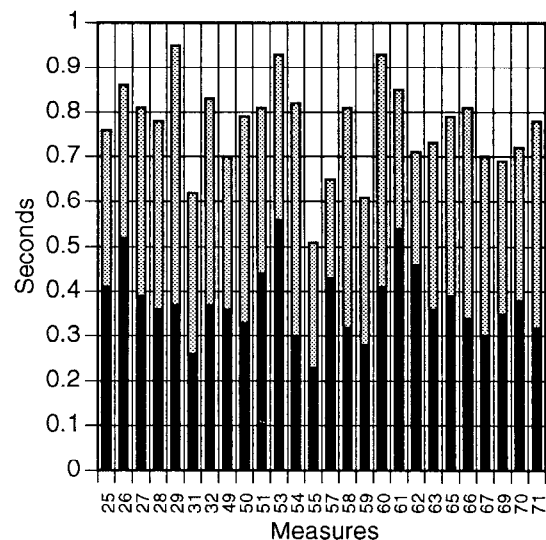


Fig. 3. Scaled durations of dotted quarter (black) and eighth (grey) notes in measures 13–46 of “Ungarischen Tanz No. 1.”

Fig. 4. Durations of first and second quarter notes in measures 25–32 and 49–71 of “Ungarischen Tanz No. 1.”



Brahms’s lifetime. Whether or not this had any bearing on Brahms’s choice of the first dance for the recording is open to conjecture.

REVIEW OF PREVIOUS WORK

In 1935, Fritz Bose attempted to copy the cylinder to long-playing (LP) disc by placing a microphone into the horn of the cylinder-playing mechanism and directly cutting a disc. The cylinder was stored at the Preussische Staatsbibliothek. It was thought to be lost from the end of World War II until 1983 but was found by Kowar, Lechleitner and Schiller at what was then the Deutsche Staatsbibliothek, Berlin, East Germany. This team attempted unsuccessfully to transfer the cylinder recording to more modern media.

The British Library National Sound Archive owns acetates of the Brahms recording from the collection of Ludwig Koch. It is assumed that these are direct copies of Bose’s attempt at an LP conversion of the recording [3].

The team from the Vienna Phonogrammarchiv released two versions of the British Library recording, one with a band-pass filtering between 90 and 5,000 Hz and another that was highly processed. The latter resulted in a recording that is virtually free of noise but lacks both temporal resolution and significant musical information about the original performance.

As a result of the restorative efforts by the Vienna Phonogrammarchiv, an extensive performance analysis of this work was published by Gerda Lechleitner [4]. Lechleitner attempted to compare data derived from the Brahms recording to other recordings of the work.

Our analysis of the Brahms cylinder began as a test of an environment for signal analysis and reconstruction using wavelet packets. Although we were primarily concerned with evaluating the efficacy of this method of signal manipulation, we were intrigued by how radically our results differed from those of Lechleitner. In addition, we were interested in generalizing the temporal data we gathered into performance attributes in order to determine how consistent Brahms was in his performance with the musical context and structural features of the work.

Fig. 5. Scaled durations of first and second quarter notes in measures 25–71 of “Ungarischen Tanz No. 1”

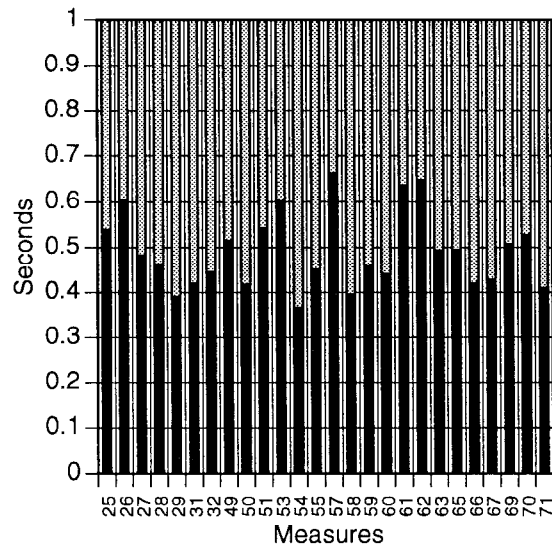


Fig. 6. Transcription of measures 49–72 of “Ungarischen Tanz No. 1.” Dotted bar lines indicate a shift of metric weight of the second beat of measure 69 until the close of the segment.



DESCRIPTION OF DENOISING AND RECONSTRUCTION TECHNIQUE

In 1946, Denis Gábor, the inventor of the hologram, proposed that conventional representations of signals as continuous functions of time could be supplanted by a representation that combined time and frequency. Considering the finite information present in actual signals, Gábor divided this information plane into “cells.” Bell-like Gaussian curves are modulated in amplitude by sinusoids (called Gábor grains) and are later supplanted by more flexible variants of generalized wave representations. These variants spawn children that amplify characteristics of the signal being represented.

Reconstruction of a signal is possible with the use of a paired highpass and lowpass filter formula called a quadrature mirror filter (QMF). Different QMFs produce mother wavelets with varying degrees of frequency resolution.

WPLab is a computer application that performs signal analysis with orthogonal trigonometric and wavelet bases [5]. The environment provides a visual interface between a signal waveform representation and the analysis of that signal, as well as a set of tools to reconstruct portions of the signal from the analysis window. WPLab reads ASCII collections of floating-point numbers and 8-bit mulaw or 16-bit linear monophonic sound files.

We have been actively investigating the applicability of this environment to music, specifically the implications of resynthesis of selectively chosen portions of a musical signal in terms of reconstruction and high-level analysis.

Briefly, the process of reconstruction includes: (1) the digitization of the musical segment, (2) the choice of QMF, (3) performance of wavelet analysis, (4) selective reconstruction of the sound file and (5) generation of a score file or MIDI file transcription of the analysis using timing data extracted from the reconstruction process.

Currently, the selection of wavelet basis is done by trial and error. We are currently developing tools to automate the process of frequency detection and windowing boundary selection. We are also exploring the use of linear, predictive coding techniques to isolate certain attributes of the signal [6].

Figure 1a is a sound pressure graph of measures 13–24 of Brahms’s performance of “Ungarischen Tanz No. 1” taken

Fig. 7. Transcription of measures 17–18, 23–24, 29–30, 35–36, 41–42, 47–48 of “Ungarischen Tanz No. 1.” Onset times for each extracted event relative to the beginning of its measure are listed above each note.





Fig. 8. (a) Perceptible and (b) prominent notes in measure 24 of “Ungarischen Tanz No. 1.”



Fig. 9. (a) Measures 71–72 as they appear in the score of “Ungarischen Tanz No. 1” and (b) transcription of measures 71–72 of the composition as Brahms performed it.

from a digital recording of an LP [7]. Figure 1b is a graph of the same segment, reconstructed using WPLab. Both waveform representations show time on the horizontal x plane and amplitude on the vertical y plane. The noticeable difference in amplitude variance between the two graphs is a result of selectively choosing frequencies to reconstruct in segments of variable window lengths. Thus, we were able to eliminate a significant amount of masking noise and focus on the piano sound.

The reconstruction method was, however, far less successful at clarifying the inner voices of the work than we had hoped. The basic premise of our analytical system is that once data is identified to be salient, it can be separated from the whole. The reconstruction process preserves the residue as well as the extracted data, so as our analysis tools become refined, we can continue to extract more detail from the residue data.

Our method of transcription was based primarily on a combination of (1) careful repeated listening and marking and (2) measuring events in a sound pressure graph representation of the score using the SoundWorks signal editing program. We generally worked on individual phrases, focusing

on smaller units. As we gained more precise information, we reevaluated our understanding of the larger musical contexts. In certain instances, we reconstructed what we were hearing in a synthetic monophonic line and overlaid it with the sound file in order to assure ourselves that we were correct in our transcription. In many cases, each of us did an independent set of measurements, which we then compared to the other’s work. In addition to the use of wavelet-based re-synthesis, we used a variety of locally applied digital filters when we felt that this might add new information or resolve a conflicting interpretation. Most recently, we experimented with averaging compressed and resynthesized versions of the already processed sound file to the reconstructed sound file in order to enhance the resolution of musical detail.

GENERAL OBSERVATIONS REGARDING THE RECORDING

Despite others’ attempts at filtering and enhancing the LP transfers, the poor quality of the cylinder recording resulted in a general consensus that the recording was not of significant musicological value. In Gregor Benko’s words, “any musical value . . . heard [in the cylinder recording as released by the International Piano Archives] can be charitably described as the product of a pathological imagination” [8].

By applying orthogonal trigonometric and wavelet-based analysis techniques, we were able to reconstruct enough meaningful musical data for us to challenge this long-held view.

In addition to Brahms’s liberal rubato and some protracted fermatti, the performance lapses into improvisation at a number of points. Our initial success in charting and transcribing these aspects suggests that further work in this direction may yield much information about performance practice.

DATA ANALYSIS

The reconstructed sound file allowed us to measure and compare various temporal aspects of the performance. Since the dance uses a small set of rhythmic units, we first subdivided the data by rhythmic types. The most recurrent of these types is the ♩ measure unit. There are 16 occurrences of this rhythm. These units occur in four consequent six-measure phrases, each divided by a terminating measure in which a half note is accompanied by arpeggiation.

The second section of the dance is characterized by two four-measure phrases of sequential upper-neighbor note motion in three ♩ note groups separated by a ♩ measure unit. This section is cadenced by a two-measure sixteenth-note sequence of descending conjunct tetrachords followed by a sixteenth-note ascending scale passage terminated by ♩ . The score repeats the last phrase with the neighbor note embellishment replaced by octave skips and a sextuplet arpeggio replacing the sixteenth-note sequence.

For reasons that are not immediately obvious, Brahms commenced his recording of the “Ungarischen Tanz” segment on the consequent of the first phrase, thus starting on a V_9 harmony. The phrase structure and harmonic rhythm of measures 1–12 is:

$$[(((2 + 2) + 2) + (2(1 + 1) + 2))] \\ i \quad vii, V \quad i \quad i \quad V_7/III \quad V_7/iv \quad iv$$

Brahms begins his performance at measure 13, which continues the six-measure phrases as follows:

$$[(((2 + 2) + 2) + (2(1 + 1) + 2))] \\ V_{(9)} \quad ii_4 \quad ii_6 \quad V \quad ii_6 \quad V \quad V_7 \quad i \\ \quad \quad \quad 3 \quad 5 \quad \quad \quad 5$$

Measures 25–48 continue the 6+6 phrase structure.

The second section of the piece is subphrased into four-measure groups, preserving the twelve-measure phrase structure as (4+4+4) rather than (6+6). Although the recording ends at this point, the score follows with a transition of one (4+4+4) phrase followed by one (6+6) phrase that leads into a recapitulation of the opening section.

A detailed analysis of the performance of each of these rhythmic types in the work follows.

♩♩ Measure Groups

While the overall durations of the three large phrase groups that incorporate dotted quarter- and eighth-note measure groups (measures 13–24, 25–36 and 37–48) do not differ radically, the internal lengths and proportions of each measure unit is remarkably flexible. Measures 13–24 have an overall duration of 10.82 seconds; measures 25–36, 9.28 seconds; and measures 37–48, 9.86 seconds. The length of measure units ranges from 0.69 seconds for measure 21 to 0.93 seconds for measure 33.

Figure 2 graphs ♩ and ♩♩ durations in milliseconds in each ♩♩ measure unit: ♩ durations are shown in black, and ♩♩ durations are shown in grey, each bar representing one measure. Although in the score the ♩♩ pattern continues in measures 25–36, Brahms alters this group considerably in his performance, subverting the ♩♩ units. Consequently, measures 25–

28 and 31–32 are not included in the graph. In addition, measures 44 and 45 were excluded from the graph because we were unable to accurately measure the onset of measure 45.

In order to compare dotted quarters between measures, we measured from the onset of dotted quarter notes to the onset of eighth notes and from the onset of eighth notes to the onset of subsequent dotted quarter notes. We then divided the overall duration of the measure by four and multiplied the dividend by three. This provided us with a measure of the optimum duration of a dotted quarter note for each measure to which we could compare the actual duration of each dotted quarter note. The average duration for all optimum dotted quarter notes is .615 seconds, whereas the average duration for all actual dotted quarter notes is .485 seconds, suggesting that in this particular performance Brahms underdotted the dotted quarter notes.

Figure 3 represents the dotted quarter and eighth notes as a percentage of the measure. Again, the ♩ durations are shown in black and ♩♩ durations are shown in grey. By scaling the proportions of the ♩♩ measures to one, we see that the general tendency towards underdotted becomes apparent. The eighth note exceeds the duration of its preceding dotted quarter note in measures 14 and 19 and approaches the duration of a quarter note in measures 20, 34 and 40. Brahms gives the dotted quarter note its full value only once, in measure 38. This rather surprising feature of the performance runs contrary to our expectations based on the persistent and intuitive tendency for performers to extend the longer duration at the expense of the shorter rhythmic value.

Fig. 10. Transcription of measures 25–36 of “Ungarischen Tanz No. 1” as performed by Brahms. Onset times relative to the duration of the entire excerpt are indicated above each note.

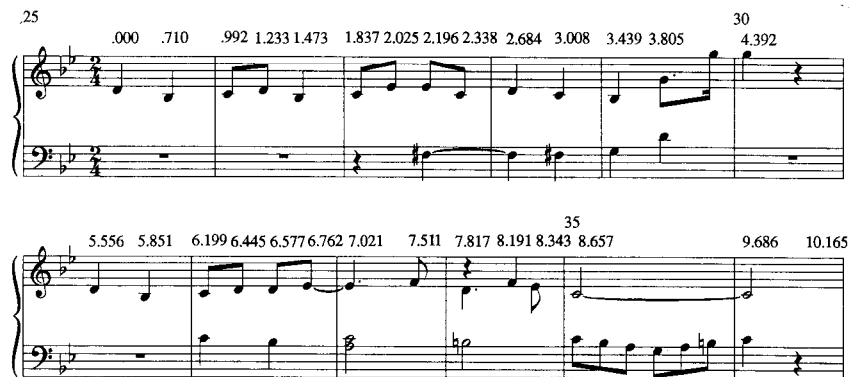


Fig. 11. Measures 25–36 as they appear in the score of “Ungarischen Tanz No. 1.”



♪♪ Rhythmic Units

The middle section of the work is comprised largely of three sixteenth-note elaborative figures characterized by an upper neighbor in the center of each figure. We were able to detect the structural line that was being elaborated. However, due in part to the proximity in frequency of the structural notes to their elaborating neighbors, we were unsuccessful in our attempts to detect and separate the elaborations. Figure 4 graphs inter-onset intervals (IOIs)—the time between the onsets of beats—of first and second beats in measures 49–71. Although musically divergent, IOIs of first and second beats in measures 25–32 are included for comparison.

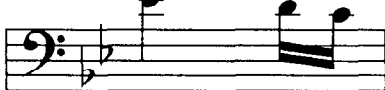
Once again, in order to compare Brahms's rhythmic performance against the notated durations in the manuscript, we added first and second beat IOIs and divided the sum by two. This provided us with an optimum IOI measurement for first and second beats in each measure. We then compared the actual IOI measurements to the optimum IOIs, subtracting the former from the latter. The difference between actual and optimum IOIs ranges from -0.12 seconds to +0.11 seconds with an average difference of 0.0075 seconds.

16



Fig. 12. Transcription of improvisation in measures 16, 20, 39 and 46 of "Ungarischen Tanz No. 1."

20



39



46



Figure 5 shows the IOIs as a percentage of the duration of the measure. With each measure scaled to one, it is easier to see that Brahms was more accurate with his placement of first and second beats in measures 49–71 than with his placement of eighth notes following dotted quarter notes in measures 13–46. The average scaled IOI of first beats in measures 25–71 is 0.4906 seconds (compared to the optimum 0.5 seconds), whereas the average scaled duration of dotted quarter notes in measures 13–46 is 0.5915 seconds (compared to the optimum 0.75 seconds).

♪♪ Rhythmic Units

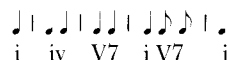
The amphibrach ♪♪♪ permeates both Hungarian speech and dance patterns. Also, exaggeration of the internal quarter note is a feature of numerous dances in Eastern Europe. The most exaggerated temporal fluctuations occur in ♪♪♪ measures that are terminators of the four-bar phrase groups in measures 49–68.

Although the second and third occurrences of these measures are distorted (possibly by media deformity), the first and last instances of this type place a long caesura on the inner quarter note. These measures become significantly extended, with overall durations of 1.033 seconds for measure 52 and 1.146 seconds for measure 68, in contrast to the average duration of 806 milliseconds per measure.

The concluding four-measure phrase that Brahms recorded (measures 69–72) poses interesting problems in transcription. In order to facilitate a closing cadence, Brahms shifts the scansion of the phrase from the feminine



to a complete masculine cadence



This is achieved by shifting the metric weight to the second beat of measure 69 and preserving this scansion until the close of the segment. This shift is notated by dotted bar lines in Fig. 6.

Arpeggios and Scalar Runs

Terminating each phrase unit of the dance is either a sixteenth-note arpeggiation or a sixteenth-note scalar run. In general, Brahms performs these measures far more strictly than any of the other rhythmic types in the recording. Our rather limited success at transcribing these measures is charted in Fig. 7, in which onset times for each extracted event relative to the beginning of its measure are listed above each note.

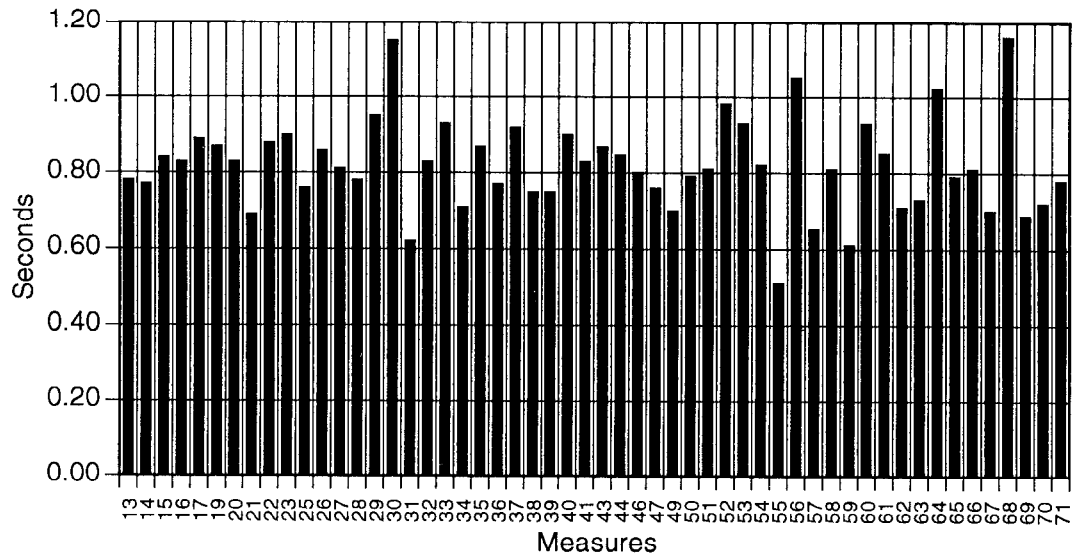
The most noticeable feature of the performance of these units is the augmentation of the second beat of measure 24. This change effectively creates an added beat such that the duple rhythm ♪♪♪♪ is played ♪♪♪♪. Although the eighth-note arpeggio shown in Fig. 8a is perceptible, the prominent notes in our reconstruction are those shown in Fig. 8b.

Detection of arpeggios and groups of notes with short durations proved problematic at this stage of our research. We hope to improve the resolution of this data in future work.

IMPROVISATION IN BRAHMS'S PERFORMANCE

Improvised segments in this recording are of two types: prominent melodic insertion within the phrase structure of

Fig. 13. IOIs between first beats of measures 13–71 of “Ungarischen Tanz No. 1.”



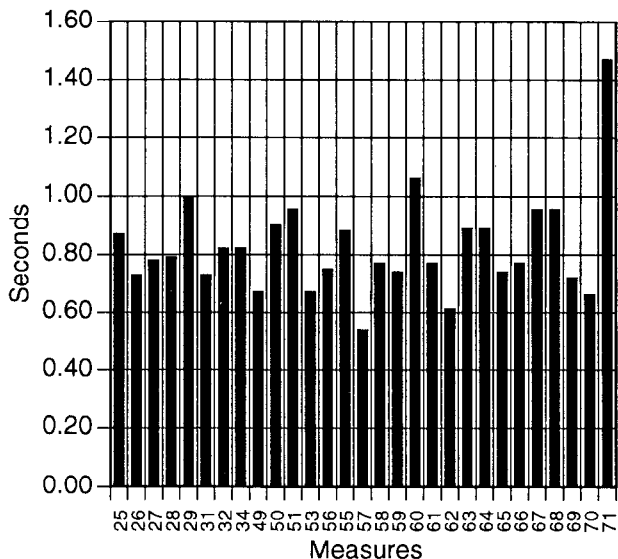
the original score, and alteration of the phrase structure in order to facilitate closure at a nonterminal point of the piece.

Of the latter, the most prominent is the augmentation of the cadence of measures 71–72, which facilitated closure with a masculine ending. Brahms performs the measure shown in Fig. 9a as the measure shown in Fig. 9b.

Of the former type of improvisation, the most prominent change is a distortion of the metric layout of the second phrase that Brahms plays—measures 25–36. The resulting line bears a rather prominent and independent melodic character. Although we have yet to isolate the accompaniment and all the internal detail of this phrase, we have arrived at the transcription of these measures shown in Fig. 10. Compare Fig. 10 to the corresponding phrase in the score, shown in Fig. 11.

The shift from ♩ rhythmic units with accompanying ♩♩ figures, as notated in the score, to ♩♩ rhythmic units in Brahms’s performance is consistent with the composer’s predilection for metric ambiguity.

Fig. 14. IOIs between second beats of measures 25–71 of “Ungarischen Tanz No. 1.”



Measure 60 in Fig. 6 and measure 17 in Fig. 7 show additional improvisation we discovered when transcribing Brahms’s performance. These added ornaments can also be found in measures 16, 20, 39 and 46, as illustrated in Fig. 12.

MEDIA DEFORMITIES AND TEMPORAL ALTERATIONS

Had the original cylinder been preserved, a physical analysis of the media could have qualified our interpretation of timings. However, the cylinder was deemed seriously damaged by the Wien Phonogrammarchiv. Thus, we attempted to search out periodic temporal distortion patterns within the sound file that could have resulted from warping or machine imbalance.

The following steps were taken to search for temporal irregularities in the sound file:

1. a systematic plotting of the onset times of various recurrent rhythmic patterns
2. comparison of the proportions derived from the onset times between similar sections of the piece
3. a search for patterns of recurrent proportions derived from the rhythmic onset times and, finally,
4. a determination of which patterns more strongly suggest musically logical rubato and which resemble recurrent or cyclic physical distortion of the wax cylinder.

The first recurrent rhythmic pattern we plotted was the downbeat of every measure. Since all but five of the measures have clearly audible downbeats, a measure of their IOIs provided us with a comprehensive outline of the temporal structure of the recording. Figure 13 shows much longer durations for measures 30, 56, 64 and 68, and much shorter durations for measures 31 and 55.

The elongation of measure 30 occurs at the end of a six-bar phrase. The previous measure, 29, is also lengthened, suggesting a *ritardando* at the end of this phrase. Measures 56, 64 and 68 are all at the end of four-bar phrases, and are also probably due to rubato or *ritardando*. The shorter durations, measures 31 and 55, immediately follow or precede a lengthened measure, suggesting a musical compensation for time gained or lost.

Having found no evidence of nonmusical temporal distortion from the IOIs of downbeats, we continued to search with IOIs between second beats. Figure 14 shows the results of these measurements.

Two outstanding second-beat IOIs, in measures 29 and 71, occur during arpeggiation in the penultimate measure of the phrase. The other longer second beat IOI, in measure 60, is at the end of a 12-bar phrase. Both cases are easily interpreted as musically motivated elongation.

A TRANSCRIPTION OF BRAHMS'S PERFORMANCE

Although we realized early in our work that a musically pleasing wavelet-based reconstruction of this recording was not currently feasible, we were interested in the idea of reconstructing the performance synthetically using MIDI and/or DSP synthesis techniques. Detected frequencies were converted back to corresponding pitch representation while detected onsets were preserved. We then synthesized the piece and overlaid the result onto the original sound file. This proved to be an effective method of substantiating our results.

CONCLUSIONS

Aside from the musicological and theoretical relevance of this performance data, we believe that our work is a first step towards the development of robust tools for analysis, reconstruction and resynthesis of historical musical recordings [9]. We have succeeded in exposing a significant amount of data that previously was either misinterpreted or deemed inaccessible. In so doing, we believe that, however modest, this work is a contribution to studies of performance practice in the nineteenth century. It is also of interest in that we provide a glimpse of a composer of enormous stature taking leave of the score in his own performance. We plan to continue our efforts with the Brahms cylinder as well as perform similar analyses of recordings of Debussy and Grieg performing their own works.

Acknowledgments

Ronald Coifman, of the Applied Mathematics Department at Yale University, has been at the forefront of research and development of wavelet-based analysis and reconstruction methods. Our research is entirely based upon his insights. Sean Gugler assisted in coding frequency-based selection methods in WPLab. David Rochberg assisted us in recoding portions of WPLab. We acknowledge the assistance of Richard Warren, curator of the Yale Collection of Historical Sound Recordings, and Werner Deutsch of the Vienna Programmarchiv. In addition to making the original suggestion to work on the Brahms cylinder, Russel Caplan was helpful at every stage of our research. We also thank Leon Plantinga and Robert Morgan for their thoughtful comments. This research was funded in part by the Wavelet Research Group at Yale University.

References and Notes

1. See Helmut Kowar, Franz Lechleitner and Dietrich Schiller, "Zur Wiederherausgabe des einzigen Tondokuments von Johannes Brahms durch das Phonogrammarchiv," *Schallarchiv* (14 December 1983). This research team suggests that it is the voice of Wangemann rather than Brahms that introduces the recording.
 2. The Strauss "Libelle" segment of the recording, long believed lost, has been found and transcribed by Helmut Kowar.
 3. Kowar et al. [1] note that the reissue on PHA EP 5 corrected the speed so that $A4 = 440$. Thus, the actual 1889 pitch cannot be determined from this recording.
 4. Gerda Lechleitner, "Der Brahms Zylinder: Kuriositat oder musikalisches Vermachtnis," in *Bruckner Symposium 5* (Linz: Anton Bruckner Institut/Linzer Veranstaltungsgesellschaft, 1985) p. 225.
 5. WPLab (from Wavelet Packet Lab), developed by Victor Wickerhauser and David Rochberg) is based upon the Adaptive Wavelet Analysis Library, developed by Ronald Coifman and Wickerhauser.
 6. A detailed technical description of musical applications of wavelets can be found in R. Kronland-Martinet and A. Grassman, "Application of Time-Frequency and Time-Scale Methods (Wavelet Transforms) to the Analysis, Synthesis and Transformation of Natural Sounds," in G. de Poli, A. Piccialli and C. Roads, eds., *Representations of Musical Signals* (Cambridge, MA: MIT Press, 1991) p. 45. A detailed description of our research in this area can be found in Berger, Coifman, Goldberg and Nichols, "Digital Audio Applications of Wavelets," CSMT Technical Notes No. 4117 (Princeton, NJ: CSMT, Yale Univ., 1993).
 7. The dub was done at the Yale Collection of Historical Sound Recordings by Richard Warren, curator of the Yale Archives of Recorded Sound.
 8. Gregor Benko, liner notes of Desmar/IPA 117, OP. Quoted by Wil Crutchfield in an article that includes a detailed description of Crutchfield's interpretation of the recording in *Opus* (August 1986).
 9. A digital recording of the original transfers, as well as a recording that overlays a synthesized version of our transcriptions upon the original recording, is available by special arrangement from the Center for Studies in Music Technology. For additional information contact the authors at <jberger@alice.music.yale.edu>.
- The reconstructed and processed Brahms cylinder is scheduled for commercial release on a compact disc from MusicMasters that includes other piano performances from the acoustic era by major composers and leading interpreters.

Glossary

masculine closing cadence—an ending in which the cadential tonic is sounded on a strong metric position.

mother wavelet—a wavelet transform uses this mother, or analyzing, wavelet to decompose a signal into elementary parts.

wavelet packets—a family of orthonormal bases for $L_2(\mathbb{R})$ consisting of a library of modulated wave forms, out of which wavelet bases, the walsh function, and rapidly oscillating wave packet bases are obtained.