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COUPLED STRUCTURAL-ACOUSTIC SYSTEMS.

APPLICATION TO STRINGED INSTRUMENTS

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Foreword

This draft has been written during a sabbatical stay at CCRMA, Stanford University, September-November 2003.

The idea of examining more closely the air-structure interaction problems in stringed instruments came first from the observation that these phenomena are not extensively discussed in the available musical acoustics literature. Most often, the presentations are restricted to very simple models: the cavity of the guitar, for example, is often described as a simple Helmholtz resonator, which is actually only true below 200 Hz. The origin and consequences of radiation damping, as well as its accurate mathematical description, are also very difficult to find in textbooks.

The selected method of presentation consists in starting first by analyzing very simple one-dimensional vibroacoustic coupled systems and by progressively generalizing the results to more complex 2-D and 3-D systems. It is hoped that this method will help the reader to gain a good understanding of the underlying physical phenomena.

During the process of writing this draft, the observation that the air-structure coupling in musical instruments can be frequently considered as *weak* has been exploited, so that a number of interesting approximations can be made. A definition of *weak* coupling has been proposed and simplified solutions for the coupled eigenfrequencies and decay times were calculated.

Recent work of the guitar made by the author with other colleagues also brought up some questions with respect to the appropriate definitions and equivalences between sound power, total energy and string's decay times. This explains the motivation for examining systematically these quantities, again starting from very simple cases and extending the results to complex systems.

In addition, one goal of computing the sound power radiated by an instrument was to determine whether or not an active control of this quantity could be developed in the future in order to obtain, through electromechanical feedback, some radiation properties which could not be achieved by means of structural modifications of the instrument only. This part of the present work should rather be considered as a preliminary study whose objectives here are limited to properly writing the fundamental equations of the problem and to list the appropriate mathematical tools that could be used for solving it. In this context, some developments are made with the help of the state-space method. This investigation is complemented by a bibliographic study on the concept of *radiation filter* which has been developed in the recent years in the context of active control of noise.

This document should be considered as a first draft which still contains a number of unanswered problems. Some questions are extensively developed while some others are only briefly summarized. Some points of interest are listed as the end of each section, and could be viewed as starting points for future studies. The objective for the near future is to progressively fill the gap so as to obtain a more structured document.

Finally, one should say that the problem solved in the appendix has not very much to do with the problem of sound-structure interaction. It is simply there because it has been the subject of interesting discussions with my friend and colleague Julius Smith. However, this development might not be found very easily in the literature.

Any comments from colleagues and students for future improvements of this text and suggestions for related studies at CCRMA and ENSTA are most welcome.

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1 Introduction

In stringed instruments, the structural-acoustic modes of the complete system are coupled, because of sound radiation. This means, in practice, that the sound pressure radiated by the structure reacts on it, which can be viewed as a feedback process. The phenomena become even more complex if we consider the presence of the cavity since, in this case, we have a multiple degree of freedom (dof) radiating structural system coupled to a multiple dof acoustic system. As a first consequence of this structural-acoustic coupling, the eigenfrequencies and damping factors are modified, compared to the "in vacuo" case. Probably even more significant is the fact that each partial of the vibrating strings will excite simultaneously many modes of the structure. In particular, if the excitation force is located at nodal points of some modes, these modes can be excited through the coupling effects.

In this document, we start with a thorough description of the coupling phenomena in the case of a 1-D longitudinally vibrating bar coupled to a semi-infinite tube, which is a simple representation of the coupling between a multiple dof structure and a radiating wave. Using the Laplace transform, it is shown how such a system can be conveniently described by a filter containing a feedback matrix whose coefficients are directly connected to the acoustic coupling. The presentation is then extended to the case where the length of the tube remains finite, with a terminating impedance whose aim is to simulate a partially radiating acoustic cavity. Finally, the previous results are generalized to 2D structures, where the phenomena are similar to those of stringed instruments. In the last section, the results obtained by using a filter representation are compared to the particular case of the guitar, where the modeling of the instrument is made by means of numerical approximations which are directly applied to the equations of the complete boundary-value problem.

The presentation continues with an energetic approach of the fluid-structure interaction phenomena. We start by computing instantaneous and mean power quantities for a single dof system. Highlight is put on the radiated acoustic power and on the radiation efficiency of the system. This latter quantity is defined here as the ratio between radiated acoustical power and input power. Links with other definitions are established. The question of whether the sound power can be derived from measurements of decay times in free vibrations is also addressed. The expressions for radiated power and efficiency are generalized to a coupled system with multiple dof. These expressions include the definition of a radiation matrix.

Finally, some preliminary mathematical derivations are made in order to obtain a state-space model for a radiating structure. The ultimate purpose here is to put the basis for the control of sound power radiated by stringed instruments.

In what follows, results on the guitar are based on previous work by Derveaux [1] and Chaigne *et al.* [2]. Considerations on acoustic power and efficiencies are inspired by previous work by Snyder and Tanaka [3], Snyder *et al.* [4], Chen [5], Chen and Ginsberg [6], Rumerman [7], Elliott and Johnson [8] and Gibbs *et al.* [9]. Active control of sound radiated by stringed instruments has been investigated by Baumann [10], and Griffin *et al.* [11] [12]. More generally, the basic results on control and state-space modeling of radiating structures can be found in textbooks by Juang and Phan [13], Meirovitch [14], Fuller *et al.* [15] and Preumont [16]. Examples of practical realizations can be found in [17]. Finally, recent results on structural-acoustics systems with applications to control are presented in [18], [19], [20] and [21].

2 Multiple dof structural system coupled to a radiating acoustic wave

2.1 Presentation of the model. Notations

We consider here the simple case of a longitudinally vibrating bar of length L coupled at one end to a 1-D infinite tube filled with air. For example purpose, it will be assumed throughout this section that the bar has a constant cross-sectional area S and that it is clamped at one end ($x = 0$) and free at the other ($x = L$). However, we will show at the end that the present method can be generalized to any boundary conditions and to structures of complex geometry.

Let us denote ρ_s the density of the bar, E its Young's modulus, $c_L = \sqrt{E/\rho_s}$ the longitudinal wave speed and $\xi(x,t)$ the longitudinal displacement at one point M of coordinate x in the bar ($0 \leq x \leq L$).
Similarly, ρ denotes the air density, c the speed of sound and $p(x,t)$ the sound pressure in the tube ($L < x < \infty$).

In the absence of excitation force (free vibrations), the equations of the problem are the following:

$$\boxed{\left\{\begin{array}{lcll} \rho_s S \dfrac{\partial^2 \xi}{\partial t^2} & = & ES\dfrac{\partial^2 \xi}{\partial x^2} - Sp(L,t)\delta(x-L) & \text{for} \quad 0 \leq x \leq L \\ p(L,t) & = & \rho c \dot{\xi}(L,t) & \\ \xi(0,t) & = & 0 & \\ p(x,t) & = & \rho c \dot{\xi}(L, t - \dfrac{x-L}{c}) & \text{for} \quad L < x < \infty \end{array}\right.} \tag{2.1}$$

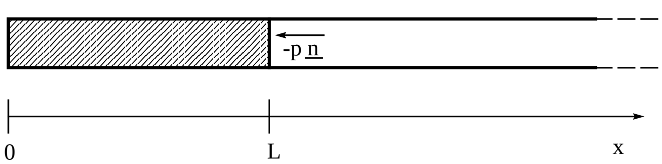

Figure 1: Longitudinally vibrating bar coupled to a semi-infinite tube.

In order to find the solution $\xi(x,t)$, an appropriate method consists in expanding this variable in terms of its *in vacuo* or *normal* modes, taking benefit of the orthogonality properties of this set of functions. Let us insist on the fact that this method does **not** imply that the bar itself vibrates in vacuo, which would be completely in contradiction with the heart of our problem where we want to emphasize the effect of air loading on the structure. Let us now take a break and review the main properties of the normal modes, using our generic example of the 1-D bar.

2.2 Brief review: normal modes of the longitudinally vibrating bar

N.B. This paragraph does not pretend to present the fundations of normal modes theory in detail. It should rather be viewed as help for facilitating the comprehension of the following parts of the paper. For more information, the reader should refer to basic textbooks in vibrations.

The *in vacuo* equations for the bar are simply obtained by letting ρ vanishing to zero in Eq. (2.1). This yields, for a clamped-free bar:

$$\left\{ \begin{array}{rcl} \rho_s S \dfrac{\partial^2 \xi}{\partial t^2} & = & ES \dfrac{\partial^2 \xi}{\partial x^2} \\ \\ \xi(0,t) & = & 0 \quad ; \quad \dfrac{\partial \xi}{\partial x}(L,t) = 0 \end{array} \right. \tag{2.2}$$

The normal modes (also called *eigenmodes*) of the clamped-free bar are obtained by looking for solutions of the form:

$$\xi(x,t) = \phi(x) \cos \omega t \tag{2.3}$$

Inserting Eq. (2.3) in Eq. (2.2) yields:

$$\phi^{''}(x) + k^2 \phi(x) = 0 \tag{2.4}$$

Where the symbol $''$ means second derivative vs. space, and where $k = \omega / c_L$ is the wavenumber. This equation yields the general solution:

$$\phi(x) = A \cos kx + B \sin kx \tag{2.5}$$

Introducing now the boundary conditions in Eq. (2.5) gives $A = 0$ and $k = k_n = (2n-1)\frac{\pi}{2L}$. The other constant B remains arbitrary. It can be shown that its value has no influence on the result: in what follows, we can decide to fix it to $B = 1$. Finally, we get the series of acceptable solutions (for $1 \leq n < \infty$):

$$\phi_n(x) = \sin{(2n-1)} \frac{\pi x}{2L} = \sin k_n x \quad \text{with} \quad k_n = \frac{\omega_n}{c_L} \tag{2.6}$$

It can be demonstrated that the eigenfunctions are orthogonal functions with respect to the mass (or to the stiffness) or the system. This means, in practice, that if we have to calculate, for example, integrals of the form:

$$\int_0^L \rho_s S \phi_n(x) \phi_m(x) \; dx \tag{2.7}$$

then this integral with always be zero for $m \neq n$. Notice that this property remains valid for heterogeneous systems ($\rho_s(x)$) and for variable sections ($S(x)$).

Because of the linearity properties of Eq. (2.2), the most general form of the solution can be written:

$$\xi(x,t) = \sum_n \phi_n(x) q_n(t) \tag{2.8}$$

Where the functions of time $q_n(t)$ are often called the *modal participation factors*.

Replacing $\xi(x,t)$ in Eq. (2.2) by Eq. (2.8), multiplying both sides by $\phi_m(x)$ and integrating over the length L of the bar yields, taking the mass- and stiffness-orthogonality properties of the normal modes into account, the following set of **independent** differential equations for the modal participation factors:

$$\ddot{q}_n + \omega_n^2 q_n = 0 \tag{2.9}$$

Each of these equations obviously yields an harmonic solution with frequency ω_n. These frequencies are often designated as the *eigenfrequencies* of the structure.

To summarize all this, one should remember that, if the bar (or any structure of finite dimensions) vibrates in vacuo, then the modal expansion of the displacement is such that **each** eigenfrequency ω_n is associated to **one unique** modal shape $\phi_n(x)$. In other words, if the structure is driven at a frequency equal to one of its eigenfrequency, then the spatial profile of the structure will be exactly described by the corresponding modal shape. This particular property is not satisfied in fluid-structure interaction problems as it will be shown below.

2.3 Free vibrations. Normal modes expansion of the solution

We go back now to our coupled problem defined in Eqs. (2.1) and we look for solutions of the form given in Eq. (2.8). It means that we impose the spatial functions $\phi_n(x)$ because of their powerful orthogonality properties. However, it can be anticipated that the functions of time $q_n(t)$ will not be the same as in the *in vacuo* case.

Now we apply the same method as previously to the first equation in Eqs. (2.1), i. e. we multiply both sides by any eigenfunction and integrate over the length of the bar, which yields:

$$\int_0^L \rho_s S \left(\sum_m \phi_m(x)\ddot{q}_m(t) \right) \phi_n(x)\ dx - \int_0^L ES \left(\sum_m \phi_m^{''}(x) q_m(t) \right) \phi_n(x)\ dx =$$
$$\int_0^L S\rho c \left(\sum_m \phi_m(L)\dot{q}_m(t) \right) \phi_n(x)\delta(x-L)\ dx \qquad (2.10)$$

This last equation can be rewritten in a more simple form, by defining the modal mass

$$m_n = \int_0^L \rho_s S \phi_n^2(x) dx \qquad (2.11)$$

Because of the mass-orthogonality property of the eigenfunctions, the terms of the series in the first integral on the left-hand side of Eq. (2.10) are zero for $m \neq n$ and thus the integral reduces to $m_n \ddot{q}_n(t)$.

Using Eq. (2.4), the second integral can be rewritten:

$$-\int_0^L ES \frac{\omega_m^2}{c_L^2} \left(\sum_m \phi_m(x) q_m(t) \right) \phi_n(x)\ dx \qquad (2.12)$$

which, due to stiffness-orthogonality properties of the eigenfunctions, reduces to $-\omega_n^2 m_n q_n(t)$. Finally, due to the properties of the delta function, the third integral can be rewritten:

$$-R_a \phi_n(L) \sum_m \phi_m(L) \dot{q}_m(t) \qquad (2.13)$$

where $R_a = \rho c S$ is an “acoustic radiation resistance”.

Remark: in the case of the clamped-free bar, we would get $\phi_n(L) = (-1)^{n+1}$ and, similarly, $\phi_m(L) = (-1)^{m+1}$. However, in what follows, we can decide to keep the formulation of Eq. (2.13) so that the equations have a more general signification.

Summary of the previous developments

- For a 1-D bar of length L coupled to a semi-infinite tube, the displacement is given by:

  $$\xi(x,t) = \sum_n \phi_n(x) q_n(t) \qquad (2.14)$$

  where the functions of times $q_n(t)$ obey to the set of coupled equations:

  $$m_n \ddot{q}_n(t) + m_n \omega_n^2 q_n(t) = -R_a \phi_n(L) \sum_m \phi_m(L) \dot{q}_m(t) \qquad (2.15)$$

- For a clamped-free bar, we have $\phi_n(x) = \sin(2n-1)\frac{\pi x}{2L}$.
- Once the displacement of the bar $\xi(x,t)$ is known, we can calculate the velocity $\dot{\xi}(x,t)$ at each point of the bar and, in particular, at point $x = L$ and thus derive the sound pressure radiated inside the tube through Eq. (2.1). We will now examine the consequences of the coupling on systems of small dimensions in order to better understand its physical meaning.

2.4 Examples of 1, 2 and 3 dof systems

Single dof system

Suppose that, for various reasons, we might be allowed to reduce the previous system to one single mode. In this case, Eq. (2.15) reduces to:

$$\ddot{q}_1(t) + \frac{R_a\phi_1^2(L)}{m_1}\dot{q}_1(t) + \omega_1^2 q_1(t) = 0 \tag{2.16}$$

We recognize here the well-known equation for a damped oscillator where the dimensionless damping factor ζ_1 is given by:

$$2\zeta_1\omega_1 = \frac{R_a\phi_1^2(L)}{m_1} \tag{2.17}$$

Thus, for a single dof system, the effect of the acoustic coupling is to add a radiation damping to the structure. This damping represents the amount of acoustic energy radiated by the vibrating bar.

2 dof sytem

We consider here only the two lowest modes of the structure. In this case, Eq. (2.15) reduces to:

$$\left\{\begin{array}{lcl} \ddot{q}_1(t) + \omega_1^2 q_1(t) & = & -\frac{R_a\phi_1(L)}{m_1}[\phi_1(L)\dot{q}_1(t) + \phi_2(L)\dot{q}_2(t)] \\ \\ \ddot{q}_2(t) + \omega_2^2 q_2(t) & = & -\frac{R_a\phi_2(L)}{m_2}[\phi_1(L)\dot{q}_1(t) + \phi_2(L)\dot{q}_2(t)] \end{array}\right. \tag{2.18}$$

This system can be rewritten as follows (omitting, for simplicity, the explicit time dependence of the variables):

$$\left\{\begin{array}{lcl} \ddot{q}_1 + 2\zeta_1\omega_1\dot{q}_1 + \omega_1^2 q_1 & = & -\frac{R_a\phi_1(L)\phi_2(L)}{m_1}\dot{q}_2 = C_{12}\dot{q}_2 \\ \\ \ddot{q}_2 + 2\zeta_2\omega_2\dot{q}_2 + \omega_2^2 q_2 & = & -\frac{R_a\phi_2(L)\phi_1(L)}{m_2}\dot{q}_1 = C_{21}\dot{q}_1 \end{array}\right. \tag{2.19}$$

Remark: Notice that we have here $-m_1C_{12} = -m_2C_{21} = R_a\phi_2(L)\phi_1(L)$. This quantity will be denoted γ in Section 4 devoted to sound power.

Several conclusions can be drawn from this last result:

- The acoustic radiation introduces damping terms $2\zeta_i\omega_i\dot{q}_i$ in each equation.
- The time functions are coupled by the radiation. The coupling coefficients C_{12} and C_{21} are not equal, in general. In the present case where the damping terms are supposed to be entirely due to radiation, notice the property $C_{12}C_{21} = 4\zeta_1\zeta_2\omega_1\omega_2$*.

Taking the Laplace transform of the system in Eq. (2.19) leads to the following characteristic equation:

$$(s^2 + 2\zeta_1\omega_1 s + \omega_1^2)(s^2 + 2\zeta_2\omega_2 s + \omega_2^2) - 4\zeta_1\zeta_2\omega_1\omega_2 s^2 = 0 \tag{2.20}$$

which can be written equivalently:

$$s^4 + 2s^3(\zeta_1\omega_1 + \zeta_2\omega_2) + s^2(\omega_1^2 + \omega_2^2) + 2s\omega_1\omega_2(\zeta_1\omega_2 + \zeta_2\omega_1) + \omega_1^2\omega_2^2 = 0 \tag{2.21}$$

Eq. (2.21) is not easy to solve analytically, in general. However, it shows that the structural-acoustic modifies the eigenfrequencies and the damping factors of the system, compared to the *in vacuo* case.

Remark: For $\zeta_1\omega_1 \ll 1$ and $\zeta_2\omega_2 \ll 1$, which are reasonable assumptions for stringed instruments, one can find first-order approximations for the modifications of both the eigenfrequencies and decay times, due to air-structure coupling. Denoting $s = \sigma + j\omega$ and replacing it in the equations (2.19) yields the new damping factors:

$$\left\{ \begin{array}{l} \sigma_1 \approx -\left[\zeta_1\omega_1 - \frac{\omega_2}{2\omega_1}C_{12}\right] \\ \\ \sigma_2 \approx -\left[\zeta_2\omega_2 - \frac{\omega_1}{2\omega_2}C_{21}\right] \end{array} \right. \tag{2.22}$$

Similarly, the new eigenfrequencies become:

$$\left\{ \begin{array}{l} \omega_1'^2 \approx \omega_1^2 - \left[2\zeta_1^2\omega_1^2 - (\zeta_1 + \zeta_2)\omega_2 C_{12} + \frac{\omega_1}{2\omega_2}C_{21}C_{12}\right] \\ \\ \omega_2'^2 \approx \omega_2^2 - \left[2\zeta_2^2\omega_2^2 - (\zeta_1 + \zeta_2)\omega_1 C_{21} + \frac{\omega_2}{2\omega_1}C_{21}C_{12}\right] \end{array} \right. \tag{2.23}$$

In a musical instrument, the eigenfrequencies of the plate are only slightly perturbed by the radiation. The main perturbation effect is due to the coupling with the cavity (see Section 3).

*This property does not hold if a structural damping is considered in addition to radiation, see Section 4.

3 dof system

A truncation of the fluid-structure system to 3 modes leads to:

$$\begin{cases} \ddot{q}_1 + 2\zeta_1\omega_1\dot{q}_1 + \omega_1^2 q_1 &= -\dfrac{R_a\phi_1(L)}{m_1}[\phi_2(L)\dot{q}_2 + \phi_3(L)\dot{q}_3] = C_{12}\dot{q}_2 + C_{13}\dot{q}_3 \\ \ddot{q}_2 + 2\zeta_2\omega_2\dot{q}_2 + \omega_2^2 q_2 &= -\dfrac{R_a\phi_2(L)}{m_2}[\phi_1(L)\dot{q}_1 + \phi_2(L)\dot{q}_2] = C_{21}\dot{q}_1 + C_{23}\dot{q}_3 \\ \ddot{q}_3 + 2\zeta_3\omega_3\dot{q}_3 + \omega_3^2 q_3 &= -\dfrac{R_a\phi_3(L)}{m_3}[\phi_1(L)\dot{q}_1 + \phi_2(L)\dot{q}_2] = C_{31}\dot{q}_1 + C_{32}\dot{q}_2 \end{cases} \tag{2.24}$$

Here again, we see that the 3 modes are coupled together by means of the C_{nm} coefficients.

Generalization

Finally, the generalized equation for the radiation structure can be written:

$$\ddot{q}_n + 2\zeta_n\omega_n\dot{q}_n + \omega_n^2 q_n = \sum_{m\neq n} C_{nm}\dot{q}_m \tag{2.25}$$

where

$$2\zeta_n\omega_n = \frac{R_a\phi_n^2(L)}{m_n} \quad \text{and} \quad C_{nm} = -\frac{R_a\phi_n(L)\phi_m(L)}{m_n} \tag{2.26}$$

2.5 Forced vibrations. Filter representation

Up to now, the analysis of the system was limited to free vibrations. We want to show here that the structural-acoustic coupling has additional effects in the case of excitation by an external force, which is much closer to the behavior of stringed instruments where the body is excited by the vibrations of the strings. This will lead us to suggest a general structure for a filter which is supposed to account for the transfer function between excitation force and bar displacement.

Eq. (2.1) is modified through introduction of a force term $F(x_o, t)$ at excitation point x_o:

$$\rho_s S\frac{\partial^2\xi}{\partial t^2} = ES\frac{\partial^2\xi}{\partial x^2} - Sp(L,t)\delta(x-L) + F(x_o,t)\delta(x-x_o) \qquad \text{for} \quad 0 \le x \le L \quad \text{and} \quad 0 \le x_o \le L \tag{2.27}$$

Using the same modal developments as in the previous section, we find:

$$\int_0^L \rho_s S\left(\sum_m \phi_m(x)\ddot{q}_m(t)\right)\phi_n(x)\,dx - \int_0^L ES\left(\sum_m \phi_m''(x)q_m(t)\right)\phi_n(x)\,dx =$$

$$\int_0^L R_a\left(\sum_m \phi_m(L)\dot{q}_m(t)\right)\phi_n(x)\delta(x-L)\,dx + \int_0^L F(x_o,t)\delta(x-x_o)\phi_n(x)\,dx \tag{2.28}$$

where each differential equation is now written:

$$\ddot{q}_n + 2\zeta_n\omega_n\dot{q}_n + \omega_n^2 q_n = \sum_{m\neq n} C_{nm}\dot{q}_m + F(x_o,t)\frac{\phi_n(x_o)}{m_n} \tag{2.29}$$

Or, in the Laplace formalism:

$$\tilde{q}_n(s) = H_n(s)\tilde{F}(x_o,s) + \sum_{m\neq n} K_{nm}(s)\tilde{q}_m(s) \tag{2.30}$$

with

$$H_n(s) = \frac{\phi_n(x_o)}{m_n(s^2 + 2\zeta_n\omega_n s + \omega_n^2)} \quad \text{and} \quad K_{nm}(s) = \frac{sC_{nm}}{s^2 + 2\zeta_n\omega_n s + \omega_n^2} \tag{2.31}$$

Finally, the displacement is written:

$$\tilde{\xi}(x,s) = \sum_n \tilde{q}_n(s)\phi_n(x) = \tilde{F}(x_o,s)\sum_n \phi_n(x)H_n(s) + \sum_n \phi_n(x)\sum_{m\neq n} K_{nm}(s)\tilde{q}_m(s) \tag{2.32}$$

2.5.1 Matrix formulation

Eq. (2.30) can be rewritten (removing the "∼" from the Laplace transforms, for convenience):

$$\begin{bmatrix} 1 & -K_{12} & \dots & -K_{1n} \\ -K_{21} & 1 & \dots & -K_{2n} \\ \dots & \dots & \dots & \dots \\ -K_{n1} & -K_{n2} & \dots & 1 \end{bmatrix} \begin{bmatrix} q_1 \\ q_2 \\ \dots \\ q_n \end{bmatrix} = F \begin{bmatrix} H_1 \\ H_2 \\ \dots \\ H_n \end{bmatrix} \tag{2.33}$$

which can be formulated in a more compact form:

$$\mathbf{K}\underline{Q} = F\underline{H} \tag{2.34}$$

In Eq. (2.34) and below, the matrices are in **bold**, and the vectors are underlined.
The displacement of the bar is written:

$$\xi = \underline{Q}^t.\underline{\phi} = \underline{Q}.\underline{\phi}^t = (\mathbf{K}^{-1}\underline{H})^t.\underline{\phi}F = (\mathbf{K}^{-1}\underline{H}).\underline{\phi}^t F \tag{2.35}$$

In some applications, it might be interesting to write Eq. (2.30) so as to obtain the characteristic equation immediately. In this case, Eq. (2.33) becomes:

$$\begin{bmatrix} s^2 + 2\zeta_1\omega_1 s + \omega_1^2 & -sC_{12} & \dots & -sC_{1n} \\ -sC_{21} & s^2 + 2\zeta_2\omega_2 s + \omega_2^2 & \dots & -sC_{2n} \\ \dots & \dots & \dots & \dots \\ -sC_{n1} & -sC_{n2} & \dots & s^2 + 2\zeta_n\omega_n s + \omega_n^2 \end{bmatrix} \begin{bmatrix} q_1 \\ q_2 \\ \dots \\ q_n \end{bmatrix} = F \begin{bmatrix} \beta_1 \\ \beta_2 \\ \dots \\ \beta_n \end{bmatrix} \tag{2.36}$$

where $\beta_n = \frac{\phi_n(x_o)}{m_n}$. The equivalent matrix notation is:

$$\mathbf{C}\underline{Q} = F\underline{\beta} \tag{2.37}$$

The displacement of the bar is written:

$$\xi = (\mathbf{C}^{-1}\underline{\beta})^t.\underline{\phi}F = (\mathbf{C}^{-1}\underline{\beta}).\underline{\phi}^t F \tag{2.38}$$

2.5.2 Inversion of the matrix C

Finding explicit solutions for the q_i (2.36) supposes to invert the matrix **C**. An example is given below for a subsystem of order 2, where we write, for convenience $D_n = s^2 + 2\zeta_n\omega_n s + \omega_n^2$. We obtain easily:

$$q_1 = \frac{\beta_1 D_2 + sC_{12}\beta_2}{D_1 D_2 - s^2 C_{12}C_{21}}F \quad \text{and} \quad q_2 = \frac{\beta_2 D_1 + sC_{21}\beta_1}{D_1 D_2 - s^2 C_{12}C_{21}}F \tag{2.39}$$

Or, alternatively, using the $(\underline{H}, \mathbf{K})$ formulation:

$$q_1 = \frac{H_1 + K_{12}H_2}{1 - K_{12}K_{21}}F = L_1 F \quad \text{and} \quad q_2 = \frac{H_2 + K_{21}H_1}{1 - K_{12}K_{21}}F = L_2 F \tag{2.40}$$

from which the displacement can be derived.

Remark 1: The coupling appears in Eq. (2.36) through the fact that q_1 (resp. q_2) is not zero, even if β_1 (resp. β_2) vanishes. In other words, one can observe oscillations at ω_n in the response of a structure coupled with air (structure displacement, sound pressure,...) even if this structure is excited on a node of the in-vacuo mode corresponding to this frequency.

Remark 2: In practice, inverting **K** corresponds to replacing the "in vacuo" transfer function H_n between F and q_n modified by the feedback matrix **K** by the new set of uncoupled transfer functions L_n. We realize, among other things, that the new eigenfrequencies and damping factors modified by the coupling are now given by the roots of the denominator $D_1 D_2 - s^2 C_{12}C_{21}$ or, more generally, by the roots of the determinant of **C**.

2.5.3 Orthogonalization of matrix C

Orthogonalizing **C** means that we are able to find an appropriate vector basis in which the relation between the displacements and the forces reduces to a diagonal form. In other words, this corresponds to decoupling the set of differential equations that govern the time evolution of the generalized displacements. In this section, much attention is paid to the case of weak coupling, in order to derive approximate equations that are valid for radiating instruments.

We start with a 2 dof system before generalization. The matrix **C** is:

$$\begin{bmatrix} D_1 & -sC_{12} \\ -sC_{21} & D_2 \end{bmatrix} \quad \text{with} \quad D_i = s^2 + 2\zeta_i\omega_i s + \omega_i^2 \tag{2.41}$$

The corresponding diagonal matrix **Λ** is written:

$$\begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} \tag{2.42}$$

where the λ_i are solutions of the equation:

$$\begin{vmatrix} D_1 - \lambda & -sC12 \\ -sC_{21} & D_2 - \lambda \end{vmatrix} = (D_2 - \lambda)(D_1 - \lambda) - s^2 C_{12}C_{21} = 0 \tag{2.43}$$

Denoting $\underline{e}_i$ the corresponding eigenvectors and $\mathbf{T} = [\underline{e}_1 \;\; \underline{e}_2]$, then we have the following matrix equations:

$$\mathbf{T\Lambda} = \mathbf{CT} \quad \Leftrightarrow \quad \mathbf{\Lambda} = \mathbf{T}^{-1}\mathbf{CT} \tag{2.44}$$

In the general case, the λ_i are given by:

$$\lambda_{1,2} = \frac{1}{2}\left[D_1 + D_2 \pm \sqrt{(D_1 - D_2)^2 + 4s^2 C_{12}C_{21}}\right] \tag{2.45}$$

At this stage, it is thus interesting to define the nondimensional coupling parameter

$$\varepsilon = \frac{C_{12}C_{21}}{D_2 - D_1} = \frac{C_{12}C_{21}}{\omega_2^2 - \omega_1^2 + 2s(\zeta_2\omega_2 - \zeta_1\omega_1)} \tag{2.46}$$

so that, for $\varepsilon \ll 1$, the eigenvalues of **C** can be written to a first-order approximation:

$$\lambda_1 = D_1 - \varepsilon s^2 \quad ; \quad \lambda_2 = D_2 + \varepsilon s^2 \tag{2.47}$$

2.5.4 Weak coupling approximations

Discussion on ε: Let us assume that the damping terms are small compared to the *in vacuo* eigenfrequencies ω_i. In this case, Eq. (2.46) shows that the coupling is *weak* if the C_{ij} are small, as it is the case for a light fluid, for example, together with the condition that the *in vacuo* eigenfrequencies ω_i and ω_j are not too close to each other. If this latter condition is not fulfilled, we can have a *strong* coupling due to radiation even in the case of a light fluid.

In what follows, the assumption of *weak* coupling, based on the criterion $\varepsilon \ll 1$, will be made. In this case, the previously defined matrices become:

$$\mathbf{T} = \begin{bmatrix} 1 & -\frac{\varepsilon s}{C_{21}} \\ \frac{\varepsilon s}{C_{12}} & 1 \end{bmatrix} \quad \Rightarrow \quad \mathbf{T}^{-1} = \frac{C_{12}C_{21}}{C_{12}C_{21} + \varepsilon^2 s^2} \begin{bmatrix} 1 & \frac{\varepsilon s}{C_{21}} \\ -\frac{\varepsilon s}{C_{12}} & 1 \end{bmatrix} \tag{2.48}$$

Defining further the vectors $\underline{\gamma} = \mathbf{T}^{-1}\underline{\beta}$ and $\underline{R} = \mathbf{T}^{-1}\underline{Q}$, Eqs. (2.37) and (2.38) can be rewritten:

$$\mathbf{\Lambda}\underline{R} = F\underline{\gamma} \quad \Rightarrow \quad \xi = \mathbf{T}\mathbf{\Lambda}^{-1}\underline{\gamma}\,\underline{\phi}^t F \tag{2.49}$$

Here, we have:

$$\mathbf{\Lambda}^{-1} = \frac{1}{D_1 D_2 - s^2 C_{12}C_{21}} \begin{bmatrix} D_2 + \varepsilon s^2 & 0 \\ 0 & D_1 - \varepsilon s^2 \end{bmatrix} \tag{2.50}$$

$$\underline{\gamma} = \frac{C_{12}C_{21}}{C_{12}C_{21} + \varepsilon^2 s^2} \begin{bmatrix} \beta_1 + \frac{\varepsilon s}{C_{21}}\beta_2 \\ \beta_2 - \frac{\varepsilon s}{C_{12}}\beta_1 \end{bmatrix} \quad \text{and} \quad \underline{R} = \frac{C_{12}C_{21}}{C_{12}C_{21} + \varepsilon^2 s^2} \begin{bmatrix} q_1 + \frac{\varepsilon s}{C_{21}} q_2 \\ q_2 - \frac{\varepsilon s}{C_{12}} q_1 \end{bmatrix} \tag{2.51}$$

To a first-order approximation, it can be shown that:

$$\mathbf{\Lambda}^{-1} \approx \mathbf{\Lambda}_0^{-1} + \varepsilon \mathbf{\Lambda}_c^{-1} \tag{2.52}$$

where $\mathbf{\Lambda}_0^{-1}$ is the in vacuo diagonal matrix given by

$$\mathbf{\Lambda}_0^{-1} = \begin{bmatrix} \frac{1}{D_1} & 0 \\ 0 & \frac{1}{D_2} \end{bmatrix} \tag{2.53}$$

and $\mathbf{\Lambda}_c^{-1}$ is the *coupling* diagonal matrix given by

$$\mathbf{\Lambda}_c^{-1} = \begin{bmatrix} \left(\frac{s}{D_1}\right)^2 & 0 \\ 0 & -\left(\frac{1}{D_2}\right)^2 \end{bmatrix} \tag{2.54}$$

This result is of particular interest for computing the perturbation due to air-loading of all significant variables of the system.

Compared to Eq. (2.38), Eq. (2.49) has several advantages:

- The matrices $\mathbf{T}$, $\mathbf{T}^{-1}$ and $\mathbf{\Lambda}$ are more easily calculated than $\mathbf{C}^{-1}$ in the general case.
- The determination of the eigenvalues of $\mathbf{C}$ naturally yields the definition of a coupling parameter ε, which allows interesting approximations in the case of weak coupling.
- From a physical point of view, it shows how to find the appropriate basis in order to decouple the system.

3 dof system and generalization

In the case of the 3 dof system (2.24), the characteristic equation is:

$$\begin{aligned}&(D_1-\lambda)(D_2-\lambda)(D_3-\lambda)\\&-s^2\left(C_{32}C_{23}(D_1-\lambda)+C_{31}C_{13}(D_2-\lambda)+C_{12}C_{21}(D_3-\lambda)\right)\\&-s^3\left(C_{12}C_{23}C_{31}+C_{13}C_{32}C_{21}\right)=0\end{aligned} \tag{2.55}$$

With the assumption of small radiation resistance, it is justified to neglect the terms in s^3 compared to the others. On can then rewrite (2.55):

$$\begin{aligned}&(D_1-\lambda)(D_2-\lambda)(D_3-\lambda)\,\times\\&\left[1-s^2\left(\frac{C_{12}C_{21}}{(D_1-\lambda)(D_2-\lambda)}+\frac{C_{13}C_{31}}{(D_1-\lambda)(D_3-\lambda)}+\frac{C_{32}C_{23}}{(D_3-\lambda)(D_2-\lambda)}\right)\right]=0\end{aligned} \tag{2.56}$$

It can be easily checked on this latter expression that first-order approximations of the eigenvalues are given by:

$$\lambda_i=D_i+\varepsilon_i \quad \text{with} \quad \varepsilon_i=-s^2\sum_j\frac{C_{ij}C_{ji}}{D_j-D_i} \quad \text{and} \quad 1\le j\le 3 \quad \text{and} \quad j\ne i \tag{2.57}$$

Equation (2.57) can be generalized to n coupled modes. In this case, the eigenvalues become:

$$\lambda_i=D_i+\varepsilon_i \quad \text{with} \quad \varepsilon_i=-s^2\sum_j\frac{C_{ij}C_{ji}}{D_j-D_i} \quad \text{and} \quad 1\le j\le n \quad \text{and} \quad j\ne i \tag{2.58}$$

2.5.5 2 dof system - approximated expressions for displacement and mode shapes

In view of the previous results, we are now able to find a first-order approximate expression for the bar displacement ξ. Like in the previous sections, we start by the simple example of a 2 dof system. Let us write first the *in vacuo* displacement:

$$\xi_0=\phi_{10}q_{10}+\phi_{20}q_{20} \tag{2.59}$$

N.B.: the notation ϕ_{i0}, which denotes the eigenmode in vacuo, should not be confused with $\phi_i(x_o)$ which denotes the value of this eigenmode at one particular point of the structure.

For the structure vibrating in air, recall that we imposed to project the solution onto the in vacuo modal shapes, in order to take advantage of their orthogonality properties. In this case, the displacement becomes:

$$\xi=\phi_{10}q_1+\phi_{20}q_2 \tag{2.60}$$

From Eq. (2.38) or (better !) from Eq. (2.49), one can derive a first-order approximation for the bar displacement:

$$\xi = \phi_{10}\left(q_{10} + \frac{sC_{12}}{D_1}q_{20}\right) + \phi_{20}\left(q_{20} + q_{10}\frac{sC_{21}}{D_2}\right) \tag{2.61}$$

Operating deflexion shapes: Eq. (2.61) can be rewritten

$$\xi = q_{10}\left(\phi_{10} + \frac{sC_{21}}{D_2}\phi_{20}\right) + q_{20}\left(\phi_{20} + \phi_{10}\frac{sC_{12}}{D_1}\right) \quad \text{with} \quad q_{i0} = \frac{\phi_{i0}}{m_i D_i}F \tag{2.62}$$

In experiments on structures, it is often made use of sinusoidal excitation. Imagine that we apply a sudden harmonic force $F(t) = H(t)\sin\omega t$ at time $t = 0$ on the structure†, with excitation location and frequency such that q_{20} is negligible compared to q_{10}. In this case, it is easy to realize that the spatial pattern of the structure is given by:

$$\phi_1 = \phi_{10} + \phi_{20}\frac{sC_{21}}{D_2} \tag{2.63}$$

Because of the time dependence of the second term (through the Lapace variable s), it can be seen that the spatial shape evolves with time. Here, we can use the Laplace limit theorem which states that the value of $\phi(t)$ as time tends to infinity is given by the product of $s\phi(s)$ as s tends to zero.

Since $\frac{s^2 C_{12}}{D_2}$ tends to zero as s tends to zero. this means that, in the time domain, the second term in the right-hand side of Eq.(2.63) will vanish after some time. Calculating the inverse Laplace transform shows that this decay time is of the order of magnitude of the decay time for the structural mode 2. After this transient regime, the spatial shape is equal to the spatial shape in vacuo ϕ_{10}.

In the more general case, Eq. (2.62) shows that F excites both q_{10} and q_{20}. After a transient regime, the bar displacement then finally converges to:

$$\xi(\omega, x) = \left[\frac{\phi_{10}\beta_1}{D_1(j\omega)} + \frac{\phi_{20}\beta_2}{D_2(j\omega)}\right] \tag{2.64}$$

The quantity between [ ] is called the *operating deflexion shape or ODS* of the structure at frequency ω. Since it is very difficult, in practice, to excite one time function q_{i0} only, this ODS is the kind of shapes which are currently observed when experimenting on vibrating structures and, in particular, on musical instruments.

2.6 State space formulation

The transfer function formulation is convenient if the system is initially at rest and for time-invariant systems. It might be useful, for other applications, to express the results in terms of

† $H(t)$ is the Heaviside function

state space variables. This formulation will be extensively used in the next sections devoted to the control of sound radiation where time-varying radiation filters will be defined.

In the present application, the mechanical state of the system is given by the position and the velocity of the dof. Since the eigenmodes ϕ_n are given, as intrinsic part of the structure, all the useful information for the state of the system is contained in the modal participation factors q_n and in their first derivatives (vs. time) $\dot{q}_n$.

This incites us to rewrite the equations for a 2-dof coupled system as follows:

$$\frac{d}{dt}\begin{bmatrix} X_1 \\ X_2 \\ X_3 \\ X_4 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -\omega_1^2 & 0 & -2\zeta_1\omega_1 & C_{12} \\ 0 & -\omega_2^2 & C_{21} & -2\zeta_2\omega_2 \end{bmatrix} \begin{bmatrix} X_1 \\ X_2 \\ X_3 \\ X_4 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ \beta_1 \\ \beta_2 \end{bmatrix} F \tag{2.65}$$

where

$$X_1 = q_1 \quad ; \quad X_2 = q_2 \quad ; \quad X_3 = \dot{q}_1 \quad ; \quad X_4 = \dot{q}_2 \tag{2.66}$$

Eq. (2.65) can be formulated equivalently:

$$\underline{\dot{X}} = \mathbf{A}\underline{X} + \underline{B}F \tag{2.67}$$

where F is the **input** and X is the **state vector**. The output Y depends on the investigated mechanical problem. If we decide, for example, to investigate the displacement, then we can write for the **output**:

$$Y = \begin{bmatrix} \phi_1 & \phi_2 & 0 & 0 \end{bmatrix}^t \underline{X} = \mathbf{\Gamma}\underline{X} \tag{2.68}$$

Equations (2.67) and (2.68) are general expressions for a linear system expressed in terms of state-space variables.

Remark: The representation presented in Eq.(2.65) is not unique. Selecting, for example

$$X_1 = q_1 \quad ; \quad X_2 = \dot{q}_1 \quad ; \quad X_3 = q_2 \quad ; \quad X_4 = \dot{q}_2 \tag{2.69}$$

obviously leads to different values for $\mathbf{A}$ and $\underline{B}$.

Remark: Denoting respectively $\mathbf{M}$, $\mathbf{R}$ and $\mathbf{K}$, the mass, resistance and stiffness matrices of the 2 dof system, then it can be seen that the matrix $\mathbf{A}$ can be rewritten more generally, using submatrices:

$$\mathbf{A} = \begin{bmatrix} \mathbf{0} & \mathbf{I} \\ -\mathbf{M}^{-1}\mathbf{K} & -\mathbf{M}^{-1}\mathbf{R} \end{bmatrix} \tag{2.70}$$

where $\mathbf{I}$ is the identity matrix.

3 Single dof structural system coupled to a multiple dof acoustic system

3.1 Presentation of the model

In stringed instruments, the soundboard is coupled to a cavity. This results in the coupling of the structural modes with the acoustic modes. Before tackling the general problem of multiple dof structural system coupled to a dissipative multiple dof acoustic system, it is of interest to examine a simplified structure (a simple dof system) coupled to a 1-D tube of cross-sectional area S and finite length L loaded at its end $x = L$ by a dissipative load. Here, we decide to express this end loading in terms of impedance Z_L, which is defined here as the ratio between pressure and acoustic velocity. In order to simplify the presentation as much as possible, we restrict ourselves here to the particular case where Z_L is real.

The selected structure is a mechanical oscillator of Mass M, stiffness $K = M\omega_o^2$ and dashpot $R = 2M\zeta_o\omega_o$ driven by a force $T(x = 0, t)$ at position $x = 0$. The motion of the mass M is assumed small, so that the acoustic velocity $v(x = 0, t)$ is equal to the mechanical velocity $\dot{\xi}(x = 0, t)$. We assume lossless wave propagation in the tube itself. Thus, the set of equations for the model is written:

$$\boxed{\begin{cases} \dfrac{1}{c^2}\dfrac{\partial^2 p}{\partial t^2} = \dfrac{\partial^2 p}{\partial x^2} \quad \text{for} \quad 0 < x < L \\[2ex] \rho\dfrac{\partial v}{\partial t} = -\dfrac{\partial p}{\partial x} \\[2ex] p(L,t) = Z_L v(L,t) \\[2ex] M\left(\ddot{\xi} + 2\zeta_o\omega_o\dot{\xi} + \omega_o^2\xi\right) = -Sp(x = 0, t) + T(t) \\[2ex] v(0,t) = \dot{\xi}(t) \end{cases}} \tag{3.1}$$

3.2 Mass displacement

As it has been done in the previous section, we can find a solution for the displacement and derive then other quantities such as velocity, force and pressure using Eq. (3.1).
One method consists of determining $p(0, t)$ as a function of mass displacement $\xi(t)$ and injecting

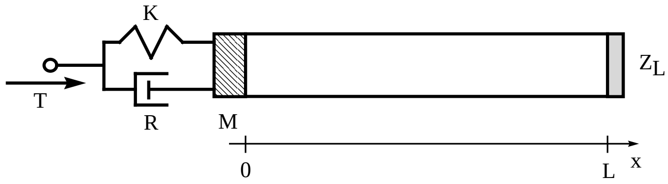

Figure 2: Single dof mechanical oscillator coupled to a finite loaded tube.

this expression in the equation that governs the motion of the oscillator.
Using again the Laplace formalism, we find the acoustic variables in the tube:

$$\begin{cases} p(x,s) = \exp(-\frac{sx}{c})\ F(s)\ +\ \exp(\frac{sx}{c})\ G(s) \\ \\ v(x,s) = \frac{1}{\rho c}[\exp(-\frac{sx}{c})\ F(s)\ -\ \exp(\frac{sx}{c})\ G(s)] \end{cases} \tag{3.2}$$

Where $F(s)$ and $G(s)$ are two functions to be determined. The boundary condition at $x = L$ yields:

$$G(s) = F(s)\frac{Z_L - \rho c}{Z_L + \rho c}\exp(-\frac{2Ls}{c}) \tag{3.3}$$

The continuity of the displacement at position $x = 0$ yields:

$$v(0,s) = \frac{1}{\rho c}F(s)\left[1 - \frac{Z_L - \rho c}{Z_L + \rho c}\exp(-\frac{2Ls}{c})\right] = s\xi(s) \tag{3.4}$$

Finally, the equation governing the oscillator motion becomes:

$$\left[s^2 + 2\zeta_o\omega_o s + \omega_o^2\right]\xi(s) = \frac{T(s)}{M} - s\frac{R_a}{M}\frac{z_L + \tanh(\dfrac{sL}{c})}{1 + z_L\tanh(\dfrac{sL}{c})}\xi(s) \tag{3.5}$$

with

$$z_L = \frac{Z_L}{\rho c} \quad \text{and} \quad R_a = \rho c S \tag{3.6}$$

Eq. (3.5) shows that, due to the loading of the oscillator by the finite tube, the damping term $2\zeta_o\omega_o$ becomes:

$$2\zeta_o\omega_o + 2\zeta_a\omega_o z(s) \tag{3.7}$$

where

$$\frac{R_a}{M} = 2\zeta_a\omega_o \quad \text{and} \quad z(s) = \frac{z_L + \tanh(\dfrac{sL}{c})}{1 + z_L\tanh(\dfrac{sL}{c})} \tag{3.8}$$

so that we can write:

$$\xi(s) = \frac{T(s)}{M}\frac{1}{s^2 + 2\omega_o\left[\zeta_o + \zeta_a z(s)\right]s + \omega_o^2} \tag{3.9}$$

F(s) is derived from Eq. (3.4):

$$F(s) = \frac{\rho cs\xi(s)}{\left[1 - \frac{z_L - 1}{z_L + 1}\exp(-\frac{2sL}{c})\right]} = \frac{\rho cs\xi(s)\exp(\frac{sL}{c})(z_L + 1)}{2\left[z_L \sinh(\frac{sL}{c}) + \cosh(\frac{sL}{c})\right]} \tag{3.10}$$

Eqs. (3.2) and (3.3) yield the pressure in the tube:

$$p(x, s) = \rho cs\xi(s)\frac{z_L\cosh(\frac{s(x-L)}{c}) - \sinh(\frac{s(x-L)}{c})}{\cosh(\frac{sL}{c}) + z_L\sinh(\frac{sL}{c})} \tag{3.11}$$

The pressure $p(x = 0, s)$ acting on the oscillator, in particular, is written:

$$p(x = 0, s) = \rho cs\xi(s)\frac{z_L\cosh(\frac{sL}{c}) + \sinh(\frac{sL)}{c})}{\cosh(\frac{sL}{c}) + z_L\sinh(\frac{sL}{c})} = \rho cs\xi(s)z(s) \tag{3.12}$$

Discussion on z(s)

Several interesting cases can be examined here.

- If the load at the end of the tube in $x = L$ is such that $Z_L = \rho c$, then $G(s) = 0$ and $F(s) = \rho cs\xi(s)$. This means that the tube is loaded by its characteristic impedance and that there is no returning wave. In this case, we have $z(s) = 1$ and the only effect of the tube is to increase the damping of the oscillator, which becomes equal to $\xi_o + \xi_a$. This increase of damping is entirely due to radiation and is, of course, similar to the one observed for the 1 dof approximation of the vibrating bar studied in the previous section.

- If the tube is closed at $x = L$, then z_L tend to ∞ and $z(s) = 1/\tanh(\frac{sL}{c})$. If, in addition, the length L of the tube is sufficiently small so that we can make the approximation $\tanh(\frac{sL}{c}) \approx \frac{sL}{c}$, then the displacement can be written:

  $$\xi(s) = \frac{T(s)}{M}\frac{1}{s^2 + 2\omega_o\zeta_o s + \omega_o^2 + \frac{2\omega_o\zeta_a c}{L}} \tag{3.13}$$

  This amounts to saying that the tube acts as an added stiffness $K_a = \frac{2\omega_o M\zeta_a c}{L}$ whose effect is to increase the eigenfrequency of the mechanical oscillator.

- If the tube is open at $x = L$ and if we neglect the radiation at this open end, then $z_L = 0$ and $z(s) = \tanh(\frac{sL}{c})$. For a small tube, or more generally for $\frac{sL}{c} \ll 1$, we obtain:

$$\xi(s) = \frac{T(s)}{M} \frac{1}{s^2 + 2\omega_o\zeta_o s + \omega_o^2 + \dfrac{2\omega_o\zeta_a s^2 L}{c}} \tag{3.14}$$

which means that the tube acts as an added mass $M_a = \dfrac{2M\omega_o\zeta_a L}{c}$ whose effect is to lower the eigenfrequency of the oscillator.

3.3 Modal representation

Here, we want to expand the sound pressure $p(x, s)$ in terms of the normal modes of the tube. For simplicity, we assume that there is no damping in the oscillator ($\zeta_o = 0$). We examine, here, before generalization, the simple case where the tube is closed at $x = L$. In what follows, we make use of some interesting properties of the transcendental functions:

$$\sinh(\frac{\pi s}{\omega_1}) = \frac{\pi s}{\omega_1} \prod_{i=1}^{\infty} \left(\frac{1}{i\omega_1}\right)^2 s^2 + i^2\omega_1^2 \quad \text{with} \quad \omega_1 = \frac{\pi c}{L} \tag{3.15}$$

$$\cosh(\frac{\pi s}{\omega_1}) = \prod_{i=1}^{\infty} \left(\frac{1}{(i - \frac{1}{2})\omega_1}\right)^2 s^2 + (i - \frac{1}{2})^2\omega_1^2 \tag{3.16}$$

We can see, in particular, on these formulae that the generic term of the product tends to 1 as i tends to ∞. This means, in practice, that it is justified to truncate this product to a finite number N. The expansion of $\sinh(\frac{\pi L}{\omega_1})$, for example, is thus equivalent here to a product of terms whose poles are given by $\omega_i = i\omega_1$. These terms correspond to the eigenfrequencies of the tube closed at both ends. The displacement of the mass is written here:

$$\xi(s) = \frac{T(s)}{M} \frac{\sinh(\dfrac{\pi s}{\omega_1})}{(s^2 + \omega_o^2)\sinh(\dfrac{\pi s}{\omega_1}) + 2\zeta_a\omega_o s \cosh(\dfrac{\pi s}{L})} \tag{3.17}$$

And the sound pressure in the tube becomes:

$$p(x, s) = \frac{\rho c s T(s)}{M} \frac{\cosh(\dfrac{s(x - L)}{c})}{(s^2 + \omega_o^2)\sinh(\dfrac{\pi s}{\omega_1}) + 2\zeta_a\omega_o s \cosh(\dfrac{\pi s}{\omega_1})} \tag{3.18}$$

We check on these expressions that, for a very small ζ_a, the eigenfrequencies of the system consists of both the oscillator frequency and the eigenfrequencies of the tube closed at both ends. Like in the previous problem of the bar coupled to the infinite tube, it is possible to find approximate solutions after defining an appropriate nondimensional term ε which quantifies the degree of coupling between the structure and the cavity.

4 Decay times and sound power

For a structural system with a single degree of freedom, it is well-known that the dissipated mechanical power can be derived from measurements of the decay time, in the case of free vibration. Similarly, the acoustical efficiency of the system can be derived from the ratio between the decay times "in vacuo" and "in air", respectively. Il will be shown that this result cannot be directly generalized to the case of the coupling between a continuous structural system with multiple degrees of freedom and a radiating acoustic wave in free space, since, in this latter case, all modes are coupled by the damping due to radiation. The problem becomes even more complex if the structure is coupled to a partially radiating cavity, as it is the case for stringed instruments.

4.1 Single degree of freedom (1dof) structural system

Consider the standard 1dof structural system given by the differential equation involving excitation force $F(t)$ and mass velocity $v(t)$:

$$F = M\frac{dv}{dt} + Rv + K\int vdt \tag{4.1}$$

which can be written alternatively, using mass displacement $\xi(t)$ and the usual reduced parameters:

$$F = M\left[\frac{d^2\xi}{dt^2} + 2\zeta_o\omega_o\frac{d\xi}{dt} + \omega_o^2\xi\right] \tag{4.2}$$

The instantaneous mechanical power $p_{mo}(t)$ put into the system is given by the scalar product between F and v, which yields:

$$p_{mo} = \frac{d}{dt}\left[\frac{1}{2}Mv^2 + \frac{1}{2}K\xi^2\right] + Rv^2 \tag{4.3}$$

The three terms on the right-hand side of Eq. (4.3) represent the kinetic energy of the mass M, the elastic energy of the spring K and the energy dissipated in the mechanical resistance R, respectively.

In a number of applications, we are mostly interested in the time-average value of $p_{mo}(t)$ rather than on the details of its time evolution. In acoustics, for example, the human ear is sensitive to the sound level which is correlated to the average value of the instantaneous acoustical power, after integration over a duration of nearly 50 ms. Therefore, after defining an integration duration T, whose appropriate selection will be discussed later in this section, we can define the average mechanical power $\mathcal{P}_{mo}(T)$:

$$\mathcal{P}_{mo}(T) = \frac{1}{2T}\left[Mv^2(T) + K\xi^2(T) - Mv^2(0) - K\xi^2(0)\right] + \frac{1}{T}\int_0^T Rv^2(t)dt \tag{4.4}$$

In a conservative system, the sum of kinetic and elastic energy remains constant over time, so that the term between [ ] in the previous equation is equal to zero. As a consequence,the average input power becomes:

$$\mathcal{P}_{mo}(T) = \frac{1}{T}\int_0^T Rv^2(t)dt = \mathcal{P}_s(T) \tag{4.5}$$

where $\mathcal{P}_s(T)$ represents the mean *structural* power dissipated in the mechanical resistance R.

4.1.1 Particular case: harmonic motion

A particular case of importance is the steady-state harmonic motion of the mechanical oscillator with angular frequency ω. Given an excitation force $F(t) = F_M \cos \omega t$, then, due to the assumed linearity of the system, the mass velocity is written $v(t) = V_M \cos(\omega_t + \phi)$. Therefore, $\mathcal{P}_{mo}(T)$ is given by:

$$\mathcal{P}_{mo}(T) = \frac{1}{T}\int_0^T F_M V_M \cos \omega t \cos(\omega t + \phi)dt \tag{4.6}$$

Denoting $\tau = \frac{2\pi}{\omega}$ the period of the motion, we can write $T = n\tau + \tau_o$, where n is a positive integer. In this case, the mean power can be rewritten:

$$\mathcal{P}_m(T) = \frac{1}{2}F_M V_M \cos \phi + \frac{F_M V_M}{4(2\pi n + \tau_o \omega)}\left[\sin \phi - \sin(2\omega\tau_o + \phi)\right] \tag{4.7}$$

Comment: This equation shows that the mean (or average) power $\mathcal{P}_{mo}(T)$ is nearly equal to $\frac{1}{2}F_M V_M \cos \phi$ only if the average duration T contains a sufficiently large number n of periods. If T is equal to $n\tau$, then the previous result is strict. In what follows, it will be assumed that this condition is fulfilled so that the dependence vs integration time T of the mean power terms will be suppressed.

With a given force, it can be easily shown that the velocity amplitude V_M and phase angle ϕ are given by:

$$V_M = \frac{F_M}{M}\frac{\omega}{\sqrt{(\omega^2 - \omega_o^2)^2 + 4\zeta_o^2\omega^2\omega_o^2}} \tag{4.8}$$

$$\cos \phi = \frac{2\zeta_o\omega_o\omega}{\sqrt{(\omega^2 - \omega_o^2)^2 + 4\zeta_o^2\omega^2\omega_o^2}} \tag{4.9}$$

and the mean power becomes:

$$\mathcal{P}_{mo} = \frac{F_M^2}{2R}\frac{4\zeta_o^2\omega_o^2\omega^2}{(\omega^2 - \omega_o^2)^2 + 4\zeta_o^2\omega^2\omega_o^2} \tag{4.10}$$

Eq. (4.10) shows that the maximum of the dissipated power, and thus the maximum of the input power, is obtained for $\omega = \omega_o$, i. e. when the excitation frequency is equal to the eigenfrequency of the oscillator. In this case, we obtain:

$$\text{Max}\{\mathcal{P}_{mo}\} = \frac{F_M^2}{2R} \tag{4.11}$$

Since F_M is known, in general, Eq. (4.11) can be used for estimating the mechanical resistance R.

Remark: Recall that Eq. (4.10) is only valid for a permanent regime, which does not correspond to many experimental (or numerical) situations where, for obvious causality reasons, the force is applied at a given instant of time, taken as origin. In this (more realistic case), the force signal should be written instead $F(t) = F_M H(t) \sin \omega t$ where $H(t)$ is the Heaviside function.
In this case, the Laplace transform of the velocity is written:

$$V(s) = \frac{F_M}{M} \frac{\omega^2}{(s^2 + \omega^2)(s^2 + 2\zeta_0 \omega_o s + \omega_o^2)} \tag{4.12}$$

Finally, the velocity can be written:

$$v(t) = \frac{F_M \omega}{M\sqrt{D(\omega)}} + A(\omega) \exp(-\zeta_o \omega_o t) \sin(\omega_o \sqrt{1 - \zeta_o^2} t + \psi(\omega)) \tag{4.13}$$

where $D(\omega) = (\omega^2 - \omega_o^2)^2 + 4\zeta_o^2 \omega^2 \omega_o^2$. $A(\omega)$ and $\psi(\omega)$ also are functions of the excitation frequency whose exact expressions do not add significant matter to the present discussion. The first term in the expression of $v(t)$ corresponds to the steady-state regime. The important features of the second term are the following:

- It is non negligible as long as the time is "small" compared to the decay time $(\zeta_o \omega_o)^{-1}$. For some lightly damped structural modes of musical instruments, this decay time can be of the order of 0.1 ms or more. In this case, this second term cannot be neglected during the first 0.5 to 1.0 s of the sound, if one wants to estimate the mean sound power correctly.
- When multiplying $v(t)$ by $F(t)$ and integrating over time, terms with frequencies $\omega + \omega_o \sqrt{1 - \zeta_0^2}$ and $\left|\omega - \omega_o \sqrt{1 - \zeta_0^2}\right|$ appear in the expression of $\mathcal{P}_{mo}$ (see Fig. 3). As a consequence, the mean power fluctuates at low frequency, which is another cause of difficulty for estimating its value properly.

4.2 Single dof structural-acoustic system

We investigate again here the simplified situation of a mechanical oscillator loaded by a semi-infinite tube. It has been shown in the previous section that the oscillator motion is governed by the equation:

$$F = M\frac{dv}{dt} + Rv + \int v dt + R_a v \qquad \text{with} \qquad R_a = \rho c S \tag{4.14}$$

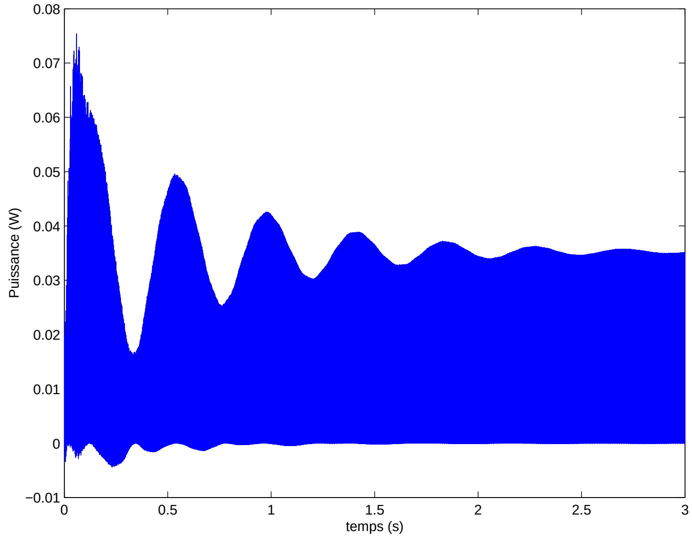

Figure 3: Time evolution of the mean power $\mathcal{P}_{mo}$.

The instantaneous input power is given by:

$$p_m(t) = \frac{d}{dt}\left[\frac{1}{2}Mv^2 + \frac{1}{2}K\xi^2\right] + (R + R_a)v^2 \tag{4.15}$$

The average input power becomes:

$$\mathcal{P}_m(T) = \frac{1}{T}\int_0^T Rv^2(t)dt + \frac{1}{T}\int_0^T R_a v^2(t)dt = \mathcal{P}_s(T) + \mathcal{P}_a(T) \tag{4.16}$$

where $\mathcal{P}_a(T)$ is the *acoustical* mean power radiated in the tube.
The *acoustical efficiency* of the system is given by:

$$\eta = \frac{\mathcal{P}_a(T)}{\mathcal{P}_m(T)} = \frac{\mathcal{P}_a(T)}{\mathcal{P}_s(T) + \mathcal{P}_a(T)} = \frac{R_a}{R + R_a} \tag{4.17}$$

Some remarks can be formulated with regards to Eq. (4.17).

- The expression of the acoustical efficiency is here independent of the integration time T.
- Though Eq. (4.17) appears simple in form, the experimental (or numerical) determination of the efficiency is generally not straightforward. As mentioned above, R can be estimated through calculation of the mechanical power *in vacuo* $\mathcal{P}_{mo}(T)$, keeping in mind the possible sources of errors (beats, transient term, value of T) presented in the previous paragraph.
- In the simple academic example discussed here, the acoustical resistance R_a, and thus the acoustical power, are directly obtained analytically. This does not correspond to the usual case where the acoustical power is estimated through measurements of the acoustic intensity in the fluid.

4.2.1 Link between estimation of sound power and free vibrations decay times

For a single dof system, an alternate interesting method for estimating the sound power consists of estimating the decay times of free vibrations through experiments or numerical calculations. In the next paragraph, the possible extension of this method to structures with multiple degrees of freedom coupled to a fluid will be discussed.

Let us take the example of the previously described oscillator loaded by the semi-infinite tube. We assume, here, that the excitation force $F(t) = 0$, whereas the mass is moved from equilibrium by a quantity $\xi(t = 0) = \xi_o$ and released without initial velocity at the origin of time. The equation of motion is written:

$$\frac{d^2\xi}{dt^2} + 2\zeta\omega_o\frac{d\xi}{dt} + \omega_o^2\xi = 0 \quad \text{with} \quad \zeta = \frac{R + R_a}{2M\omega_o} \tag{4.18}$$

The Laplace transform of the displacement is given by:

$$\xi(s) = \xi_o\frac{s + 2\zeta\omega_o}{s^2 + 2\zeta\omega_o s + \omega_o^2} \tag{4.19}$$

from which the time evolution of the mass displacement is obtained (assuming $\zeta < 1$):

$$\xi(t) = \exp(-\zeta\omega_o t)\left[\cos(\omega_o\sqrt{1-\zeta^2}t) + \frac{\zeta}{\sqrt{1-\zeta^2}}\sin(\omega_o\sqrt{1-\zeta^2}t)\right] \tag{4.20}$$

Eq. (4.20) shows that the *decay factor* α (equal to the inverse of the decay time) is equal to $\zeta\omega_o = \frac{R + R_a}{2M}$. The same mathematical derivations applied to the oscillator in vacuo yields $\alpha_o = \zeta_o\omega_o = \frac{R}{2M}$. In conclusion, this shows that, for the particular 1dof system studied here, the acoustical efficiency can be estimated in the time domain by the expression:

$$\eta = \frac{\alpha - \alpha_o}{\alpha} \tag{4.21}$$

4.3 Multiple dof structural system coupled to a radiating acoustic wave

4.3.1 2 dof system

We rewrite here the coupled equations for a 2dof structure coupled to a radiating wave presented in Section 2, with slight differences: we add here damping terms in the structure r_1 and r_2 so that we can make a comparison between the power dissipated *in vacuo* and *in air*, respectively. The damping terms r_{a1} and r_{a2} are due to radiation. Modal masses and stiffnesses are written explicitely, so that the comparison with the case of a single oscillator becomes easier. We get:

$$\begin{cases} m_1\ddot{q}_1 + (r_1 + r_{a1})\dot{q}_1 + k_1 q_1 + \gamma\dot{q}_2 = \phi_1(x_o)F \\ \\ m_2\ddot{q}_2 + (r_2 + r_{a2})\dot{q}_2 + k_1 q_2 + \gamma\dot{q}_1 = \phi_2(x_o)F \end{cases} \tag{4.22}$$

where $\gamma = -C_{12}m_1 = -C_{21}m_2$. In what follows, we use the notations:

$$\begin{cases} 2\zeta_1\omega_1 = \dfrac{r_1 + r_{a1}}{m_1} \quad ; \quad 2\zeta_{10}\omega_1 = \dfrac{r_1}{m_1} \quad ; \quad \omega_1^2 = \dfrac{k_1}{m_1} \\ \\ 2\zeta_2\omega_2 = \dfrac{r_2 + r_{a2}}{m_1} \quad ; \quad 2\zeta_{20}\omega_2 = \dfrac{r_2}{m_2} \quad ; \quad \omega_2^2 = \dfrac{k_2}{m_2} \end{cases} \tag{4.23}$$

so as to make a distinction between the damping factors in air and in vacuo.

The force F is supposed to be applied at point $x = x_o$ so that the mechanical input power is written:

$$p_m(t) = F\frac{d\xi}{dt}(x_o, t) = F\left[\phi_1(x_o)\dot{q}_1 + \phi_2(x_o)\dot{q}_2\right] \tag{4.24}$$

Using equations (4.22), we are now able to write $p_m(t)$ explicitely:

$$\begin{aligned} p_m(t) = {} & m_1\ddot{q}_1\dot{q}_1 + (r_1 + r_{a1})\dot{q}_1^2 + k_1 q_1\dot{q}_1 \\ & + 2\gamma\dot{q}_1\dot{q}_2 \\ & + m_2\ddot{q}_2\dot{q}_2 + (r_2 + r_{a2})\dot{q}_2^2 + k_2 q_2\dot{q}_2 \end{aligned} \tag{4.25}$$

Integrating $p_m(t)$ over a duration T and removing the conservative energy terms yields the mean input power:

$$\mathcal{P}_m(T) = \frac{1}{T}\int_0^T (r_1 + r_{a1})\dot{q}_1^2 + (r_2 + r_{a2})\dot{q}_2^2 + 2\gamma\dot{q}_2\dot{q}_1 \;\; dt \tag{4.26}$$

It turns out now that the input power can be seen as the sum of three terms:

- The mean power $\mathcal{P}_s(T) = \frac{1}{T}\int_0^T r_1\dot{q}_1^2 + r_2\dot{q}_2^2 \; dt$ dissipated in the structure,
- The mean acoustical power $\mathcal{P}_a(T) = \frac{1}{T}\int_0^T r_{a1}\dot{q}_1^2 + r_{a2}\dot{q}_2^2 \; dt$ radiated in the air,
- The mean *coupling* power $\mathcal{P}_c(T) = \frac{2}{T}\int_0^T \gamma\dot{q}_2\dot{q}_1 \; dt$ due to the exchange of energy between the two oscillators *via* the fluid.

Let us denote ω the steady-state excitation frequency. The transient regime is neglected, and the integration time T is supposed to be taken equal to a integer multiple of the period $T = n\tau = n\frac{2\pi}{\omega}$. In this case, the expressions of the mean input power in vacuo and in air are written:

$$\begin{aligned} \mathcal{P}_{mo} &= \frac{1}{2}\left[r_1|\dot{q}_{10}|^2 + r_2|\dot{q}_{20}|^2\right] \\ \mathcal{P}_{m} &= \frac{1}{2}\left[(r_1 + r_{a1})|\dot{q}_1|^2 + (r_2 + r_{a2})|\dot{q}_2|^2 + 2\gamma|\dot{q}_2||\dot{q}_1|\right] \end{aligned} \tag{4.27}$$

The two terms of $\mathcal{P}_{mo}$ are given by the same expression than in Eq. (4.10), so that we can write:

$$\mathcal{P}_{mo} = F_M^2\left[\frac{\phi_1^2(x_o)}{2r_1}\frac{4\zeta_{10}^2\omega_1^2\omega^2}{(\omega^2-\omega_1^2)^2 + 4\zeta_{10}^2\omega^2\omega_1^2} + \frac{\phi_2^2(x_o)}{2r_2}\frac{4\zeta_{20}^2\omega_2^2\omega^2}{(\omega^2-\omega_2^2)^2 + 4\zeta_{20}^2\omega^2\omega_2^2}\right] \tag{4.28}$$

In a free vibration regime, the total displacement is made of the linear combination of two exponentially decaying sinusoids with decay factors $\alpha_{10} = \zeta_{10}\omega_1$ and $\alpha_{20} = \zeta_{20}\omega_2$, for eigenfrequencies ω_1 and ω_2 respectively. In order to obtain each decay time experimentally, we need to excite the system successively with different values of ω. The most natural choice would be to select ω_1 and ω_2, however these in vacuo frequencies are in fact not known in practice since the experiments are generally made in the air.

$\mathcal{P}_m$ can be written in matrix form:

Denoting $\underline{\dot{Q}} = \begin{bmatrix}\dot{q}_1 \\ \dot{q}_2\end{bmatrix}$, $\mathbf{R}_s = \begin{bmatrix} r_1 & 0 \\ 0 & r_2\end{bmatrix}$ and $\mathbf{R}_a = \begin{bmatrix} r_{a1} & \gamma \\ \gamma & r_{a2}\end{bmatrix}$, we can write:

$$\mathcal{P}_m = \underline{\dot{Q}}^H\left[\mathbf{R}_s + \mathbf{R}_a\right]\underline{\dot{Q}} \tag{4.29}$$

where $\underline{\dot{Q}}^H$ is the Hermitian conjugate (conjugate transpose) of $\underline{\dot{Q}}$.

4.3.2 3 dof system and generalization

Starting from Eq. (2.24), for example, one can easily show that the mean power for a 3 dof system is written:

$$\mathcal{P}_m(T) = \frac{1}{T}\int_0^T\left[\sum_{i=1}^{3}(r_i + r_{ai})\dot{q}_i^2 + \sum_{i=1}^{3}\sum_{j\neq i=1}^{3}\gamma_{ij}\dot{q}_i\dot{q}_j\right]dt \quad \text{with} \quad \gamma_{ij} = -m_iC_{ij} \tag{4.30}$$

For the 3 dof system as well as for any system of higher number n of dof, and for an harmonic excitation at frequency ω, this mean power can be written in the same form as Eq. (4.29), where the *resistance matrix* is written:

$$\begin{bmatrix} r_1 + r_{a1} & \dots & \gamma_{1i} & \dots & \gamma_{1j} & \dots & \gamma_{1n} \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ \gamma_{i1} & \dots & r_i + r_{ai} & \dots & \gamma_{ij} & \dots & \gamma_{in} \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ \gamma_{j1} & \dots & \gamma_{ji} & \dots & r_j + r_{aj} & \dots & \gamma_{jn} \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ \gamma_{n1} & \dots & \gamma_{ni} & \dots & \gamma_{nj} & \dots & r_n + r_{an} \end{bmatrix} \tag{4.31}$$

Remark: Like previously, the resistance matrix can be viewed as the sum of a *structural* resistance matrix $\mathbf{R_s}$ and an *acoustical* resistance matrix $\mathbf{R_a}$. This leads to the expression of mean acoustical power:

$$\mathcal{P}_a = \underline{\dot{Q}}^H \mathbf{R}_a \underline{\dot{Q}} \tag{4.32}$$

and of the acoustical efficiency:

$$\eta_m = \frac{\underline{\dot{Q}}^H \left[\mathbf{R}_a\right] \underline{\dot{Q}}}{\underline{\dot{Q}}^H \left[\mathbf{R}_s + \mathbf{R}_a\right] \underline{\dot{Q}}} \tag{4.33}$$

which generalizes Eq. (4.17). Notice that all developments were made here with the assumption that the structural resistance matrix $\mathbf{R_s}$ is diagonal, which corresponds to an usual reasonable assumption for lightly damped structures. However, strong structural damping can also be the source of intermodal coupling. In this case, $\mathbf{R_s}$ is not diagonal, though the general results expressed in Eqs (4.29) and (4.33) remain valid.

4.3.3 Radiation filter

Because $\mathbf{R_a}$ is real, symmetric, definite positive, we can write the decomposition:

$$\mathbf{R_a} = \mathbf{P^t \Omega P} \tag{4.34}$$

where $\mathbf{\Omega}$ is a diagonal matrix. As a consequence, the acoustic power becomes, removing, for simplicity, the integration time T:

$$P_a = \underline{b}^H \mathbf{\Omega} \underline{b} \quad \text{where} \quad \underline{b} = \mathbf{P} \underline{\dot{Q}} \tag{4.35}$$

Which can be written explicitely:

$$P_a = \sum_n \Omega_n |b_n|^2 \tag{4.36}$$

Equation (4.36) means, in practice that, defining the appropriate basis, the acoustic power can be expressed as a sum of quadratic terms, thus removing the cross-products between the q_i seen in the previous subsections.

Another interesting consequence of the properties of $\mathbf{R_a}$ is that the sound power can be alternatively decomposed using the Cholesky method. This leads to the expression:

$$P_a = \underline{\dot{Q}}^H \mathbf{R}_a \underline{\dot{Q}} = \underline{\dot{Q}}^H \mathbf{G}^H \mathbf{G} \underline{\dot{Q}} = \underline{z}^H \underline{z} = \sum_n |z_n|^2 \tag{4.37}$$

where, by comparison with Eq. (4.36), the vector $\underline{z}(\omega)$ can be viewed as the output of a set of *radiation filters* whose transfer functions $\mathbf{G}(\omega)$ are given by:

$$\mathbf{G}(\omega) = \sqrt{\mathbf{\Omega}(\omega)}\mathbf{P}(\omega) \tag{4.38}$$

and whose input is the vector $\underline{\dot{Q}}$, so that we have:

$$\underline{z} = \mathbf{G}\underline{\dot{Q}} \tag{4.39}$$

4.3.4 Impulsively excited structure - Total radiated energy

For an impulsively excited structure, the total radiated energy is given by:

$$E_T = \int_0^\infty \underline{\dot{Q}}^H \mathbf{R}_a \underline{\dot{Q}} d\omega = \int_0^\infty \underline{z}^H(\omega)\underline{z}(\omega) d\omega \tag{4.40}$$

which, by applying Parseval's theorem, is equivalent to [Baumann, 1991]:

$$E_T = \int_0^\infty \underline{z}^t(t)\underline{z}(t) dt \tag{4.41}$$

4.3.5 State space formulation and control of the radiated energy

The interest of formulating the structural acoustic coupling in terms of state space will now appear more clearly. Denoting $\underline{r}$ the internal state of filter $\mathbf{G}$ with input $\mathbf{X}$ (or $\underline{\dot{Q}}$) and output $\underline{z}$, we can find for this filter a state space realization of the form:

$$\begin{cases} \dot{\underline{r}} = \mathbf{A}_G \underline{r} + \mathbf{B}_G \underline{X} \\ \underline{z} = \mathbf{C}_G \underline{r} + \mathbf{D}_G \underline{X} \end{cases} \tag{4.42}$$

Combining these equations with the equations of motion of the structure yields finally:

$$\begin{bmatrix} \dot{\underline{X}} \\ \dot{\underline{r}} \end{bmatrix} = \begin{bmatrix} \mathbf{A} & \mathbf{0} \\ \mathbf{B}_G & \mathbf{A}_G \end{bmatrix} \begin{bmatrix} \underline{X} \\ \underline{r} \end{bmatrix} + \begin{bmatrix} \underline{B} \\ 0 \end{bmatrix} F \tag{4.43}$$

with the output equation:

$$\underline{z} = \begin{bmatrix} \mathbf{D}_G & \mathbf{C}_G \end{bmatrix} \begin{bmatrix} \underline{X} \\ \underline{r} \end{bmatrix} \tag{4.44}$$

If, for example, the purpose is to *maximize* the total energy radiated by the instrument, then the cost function to be minimized could be:

$$\mathcal{C}_f = 1 - \frac{\displaystyle\int_0^\infty \underline{z}^t(t)\underline{z}(t)dt}{\text{Max}\{E_T\}} \tag{4.45}$$

4.4 Single dof structural system coupled to a multiple dof acoustic system

Using the notations presented in Section 3, the instantaneous input mechanical power is written here:

$$p_m(t) = T.\dot{\xi} = M\left(\ddot{\xi}\dot{\xi} + 2\zeta_o\omega_o\dot{\xi}^2 + \omega_o^2\xi\dot{\xi}\right) + Sp(x=0,t)\dot{\xi} \tag{4.46}$$

from which the average input power becomes:

$$\mathcal{P}_m(T) = \frac{1}{T}\int_0^T R\dot{\xi}^2 dt + \frac{1}{T}\int_0^T SI(x=0,t)dt = \mathcal{P}_s(T) + \mathcal{P}_a(T) \tag{4.47}$$

If the tube is terminated by a lossless impedance at $x = L$, then the pressure and velocity are in quadrature inside the tube and the acoustic intensity is equal to zero. In this case, there is no acoustic dissipation in the system.

Like it has been done previously, one can assume that the excitation is sinusoidal with angular frequency ω and that the integration time T is equal to an integer number of periods. In this case, the dependence vs T of the mean power terms can be suppressed. For a dissipative terminating impedance Z_L, and using Eq. 3.12 in the steady state regime where $s = j\omega$ yields:

$$\mathcal{P}_a = \frac{\rho c S}{2}\mathcal{R}e\left\{z(\omega)\right\}|\dot{\xi}|^2 \tag{4.48}$$

5 Extension to 2-D systems

5.1 Simply supported radiating isotropic plate

5.1.1 Equations of motion

We attempt now to generalize the fluid-structure interaction problem to the case of a 2-D rectangular baffled simply supported isotropic thin plate radiating in air. The equations of the problem are the following:

$$\left\{\begin{array}{l} D\nabla^4\xi(x,y,t) + \rho_s h\dfrac{\partial^2\xi}{\partial t^2}(x,y,t) = p(x,y,0-,t) - p(x,y,0+,t) \\ \qquad\qquad\qquad\qquad\text{for } 0 < x < L_x \text{ and } 0 < y < L_y \\ \xi(0,y,t) = \xi(L_x,y,t) = \xi(x,0,t) = \xi(x,L_y,t) = 0 \\ \xi''(0,y,t) = \xi''(L_x,y,t) = \xi''(x,y,t) = \xi''(x,L_y,t) = 0 \\ v_z(x,y,0+,t) = -v_z(x,y,0-,t) = \dot{\xi}(x,y,t) \text{ for } 0 < x < L_x \text{ and } 0 < y < L_y \\ v(x,y,0) = 0 \quad \text{for} \quad x \ni]0,L_x[\quad \text{or} \quad y \ni]0,L_y[\\ \rho\dfrac{\partial \underline{v}}{\partial t} + \underline{\nabla}p = 0 \\ \nabla^2 p - \dfrac{1}{c^2}\dfrac{\partial^2 p}{\partial t^2} = 0 \end{array}\right. \tag{5.1}$$

where h is the thickness of the plate, $D = EI = \dfrac{Eh^3}{12(1-\nu^2)}$ is the rigidity factor, E is the Young's modulus, ν is the Poisson's ratio and ρ_s is the plate's density. The transverse displacement of the plate is denoted $\xi(x,y,t)$. The acoustic variables are the pressure $p(x,y,z,t)$ and the velocity $\underline{v}(x,y,z,t)$. ρ is the air density and c is the speed of sound in air.

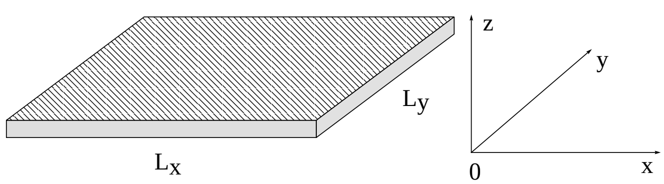

Figure 4: Geometry of the thin isotropic baffled radiating plate.

5.1.2 Eigenmodes in vacuo

Solving the plate equations in vacuo, with simply supported boundary conditions yield the eigenmodes;

$$\phi_{mn}(x,y) = \phi_m(x)\phi_n(y) = \sin\frac{\pi m x}{L_x}\sin\frac{\pi n y}{L_y} \tag{5.2}$$

allowing the displacement to be expanded as follows:

$$\xi(x,y,t) = \sum_{m,n}\phi_{mn}(x,y)q_{mn}(t) = \phi_m(x)\phi_n(y)q_{mn}(t) \tag{5.3}$$

Defining the vectors:

$$\begin{cases} \underline{Q}^t = [q_{01}\ q_{10}\ q_{11}\ \ldots\ q_{MN}] \\ \text{and} \\ \underline{\Phi}^t = [\phi_{01}\ \phi_{10}\ \phi_{11}\ \ldots\ \phi_{MN}] \end{cases} \tag{5.4}$$

than Eq. (5.3) can be written in the form:

$$\xi(x,y,t) = \underline{Q}^t\underline{\Phi} \tag{5.5}$$

which is similar to the elementary 1-D system studied in the first section.

5.1.3 Calculation of radiated sound field using wavenumber Fourier transform

One main interest in calculating the radiated sound field by using the wavenumber Fourier transform is that it allows easy calculation of the pressure on the plate surface itself, which is clearly what we want here. For a 2-D spatial function $f(x,y)$, this transformation is defined by:

$$\begin{cases} F(k_x,k_y) = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} f(x,y)e^{j(k_x x + k_y y)}dxdy \\ \\ f(x,y) = \frac{1}{(2\pi)^2}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} F(k_x,k_y)e^{-j(k_x x + k_y y)}dk_x dk_y \end{cases} \tag{5.6}$$

This allows, for example, to transform the wave equation for an harmonic pressure sound pressure in space as follows:

$$\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}(\nabla^2 + k^2)p(x,y,z)e^{j(k_x x + k_y y)}dxdy = 0 \tag{5.7}$$

This leads to the equation:

$$\left(k^2 - k_x^2 - k_y^2 + \frac{\partial^2}{\partial z^2}\right)P(k_x,k_y,z) = 0 \tag{5.8}$$

whose solution is given by:

$$P(k_x, k_y, z) = Ae^{-jk_z z} \quad \text{with} \quad k_z = \sqrt{k^2 - k_x^2 - k_y^2} \tag{5.9}$$

The arbitrary constant A is determined by using the Euler equation, which yields:

$$P(k_x, k_y, z) = \frac{\omega\rho\dot{\Xi}(k_x, k_y)}{k_z} e^{-jk_z z} \tag{5.10}$$

where $\dot{\Xi}(k_x, k_y)$ is the wavenumber transform of the transverse plate velocity $\dot{\xi}(x, y)$. The complex sound pressure in space is obtained by using the inverse transform:

$$p(x, y, z) = \frac{\omega\rho}{(2\pi)^2} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \frac{\dot{\Xi}(k_x, k_y)}{k_z} e^{-j(k_x x + k_y y + k_z z)} dk_x dk_y \tag{5.11}$$

This equation can be solved by using the method of stationary phase or the Fast Fourier Transform algorithm.

5.1.4 Radiated sound power and radiation impedance matrix

In the harmonic case, the total sound power radiated by the plate is given by:

$$\mathcal{P}_a = \frac{1}{2}\mathcal{R}e\left[\int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} p(x, y, z = 0)\ \dot{\xi}^*(x, y)\ dxdy\right] \tag{5.12}$$

where (*) denotes the complex conjugates. Using the Parseval's theorem and Eq. (5.10) yields:

$$\begin{aligned} \mathcal{P}_a &= \frac{1}{8\pi^2}\mathcal{R}e\left[\int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} P(k_x, k_y)\dot{\Xi}^*(k_x, k_y)\ dk_x dk_y\right] \\ &= \frac{\omega\rho}{8\pi^2}\mathcal{R}e\left[\int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \frac{|\dot{\Xi}(k_x, k_y)|^2}{k_z} dk_x dk_y\right] \end{aligned} \tag{5.13}$$

Notice that this last result is only valid for the wavenumbers such as $\sqrt{k_x^2 + k_y^2} \leq k$. The sound power can be alternatively written in terms of the wavenumber transform of the acoustic pressure, which yields:

$$\mathcal{P}_a = \frac{1}{8\omega\rho\pi^2} \iint_{k_x^2 + k_y^2 \leq k^2} |P(k_x, k_y)|^2\ k_z\ dk_x dk_y \tag{5.14}$$

Using the form expressed in Eq. (5.5) for the plate velocity and denoting $\underline{\Phi}(k_x, k_y)$ the wavenumber transform of $\underline{\Phi}(x, y)$ yields the modulus squared of the velocity field:

$$|\dot{\Xi}(k_x, k_y)|^2 = |\underline{\dot{Q}}^t \underline{\Phi}(k_x, k_y)|^2 = \underline{\dot{Q}}^H \underline{\Phi}^*(k_x, k_y)\underline{\Phi}^t(k_x, k_y)\underline{\dot{Q}} \tag{5.15}$$

Finally, substituting this result into Eq. (5.13) shows that the acoustic power can be written as:

$$\mathcal{P}_a = \underline{\dot{Q}}^H \mathbf{R}_a \underline{\dot{Q}} \tag{5.16}$$

with the radiation impedance matrix $\mathbf{R}_a$ defined as:

$$\mathbf{R}_a = \frac{\omega\rho}{8\pi^2}\mathcal{R}e\left\{\iint \frac{\underline{\Phi}^*(k_x,k_y)\underline{\Phi}^t(k_x,k_y)}{\sqrt{k^2-k_x^2-k_y^2}}dk_x dk_y\right\} \tag{5.17}$$

which generalizes the results obtained in Eq. (4.32) to 2-D systems .

In practice, each term $(\mathbf{R}_a)_{ij}$ of the matrix $\mathbf{R}_a$ quantifies the *mutual* radiation resistance which results from the interference between the fields due to the modes (m,n) and (m',n'), respectively. If $(m,n)=(m',n')$, then we obtain the *self* radiation resistances which are the diagonal terms of the matrix $\mathbf{R}_a$. These terms are written explicitely:

$$(\mathbf{R}_a)_{ij} = (\mathbf{R}_a)_{mn,m'n'} = \frac{\omega\rho}{8\pi^2}\mathcal{R}e\left\{\iint \frac{\Phi_m^*(k_x)\ \Phi_n^*(k_y)\ \Phi_{m'}(k_x)\ \Phi_{n'}(k_y)}{\sqrt{k^2-k_x^2-k_y^2}}dk_x dk_y\right\} \tag{5.18}$$

For a baffled simply supported plate, the radiation resistances become:

$$\begin{aligned}(\mathbf{R}_a)_{mn,m'n'} &= \frac{mm'nn'\omega\rho\pi^2}{8L_x^2L_y^2} \\ &\times \mathcal{R}e\left\{\iint \frac{f_{mm'}(k_xL_x)\ f_{nn'}(k_yL_y)dk_xdk_y}{[k_x^2-(m\pi/L_x)^2)][k_x^2-(m'\pi/L_x)^2][k_y^2-(n\pi/L_y)^2)][k_y^2-(n'\pi/L_y)^2]}\right\}\end{aligned} \tag{5.19}$$

where the functions of the form $f_{mm'}(k_xL_x)$ are given by:

$$f_{mm'}(k_xL_x) = \left\{\begin{array}{lll} 2(1-\cos k_xL_x) & \text{for } m \text{ even,} & m' \text{ even} \\ 2(1+\cos k_xL_x) & \text{for } m \text{ odd,} & m' \text{ odd} \\ 2j\sin k_xL_x & \text{for } m \text{ odd,} & m' \text{ even} \\ -2j\sin k_xL_x & \text{for } m \text{ even,} & m' \text{ odd} \end{array}\right\} \tag{5.20}$$

6 Example of the guitar

Interpreting the fluid-structure coupling phenomena in the guitar

In the present section, general results developed in the previous sections are applied to the case of a guitar [2]. The leading idea is to enhance the energetic approach in order to prepare the appropriate theoretical framework for the control of sound power. Links are made with the experiments, so as to relate measurements of decay times and sound intensity to energetic quantities.

6.1 Calculation of string's decay factors

6.1.1 Isolated string with internal damping

Let us consider an homogeneous viscoelastic string, ideally clamped at both ends. The equations of the problem are the following:

$$\boxed{\begin{cases} w(y=0) = w(y=l_s,t) = 0 \\ \\ \mu \dfrac{\partial^2 w}{\partial t^2} = T\left(1+\eta\dfrac{\partial}{\partial t}\right)\dfrac{\partial^2 w}{\partial y^2} - r\dfrac{\partial w}{\partial t} + f(t)\delta(y-y_o) \\ \end{cases}} \tag{6.1}$$

where $w(y,t)$ is the transverse displacement, μ is the mass per unit length, T is the tension, η is a viscoelastic coefficient and r is an additional damping constant. The purpose of these damping terms is to account for the fact that the higher frequencies are damped more rapidly than the lower ones, as it is currently observed on real strings. The force term $f(x_o,t) = f(t)\delta(y-y_o)$ represents the action of the finger on the string.

To solve these equations, we use a modal approach where the normal modes are those obtained for a clamped lossless string:

$$w(y,t) = \sum_n \phi_n(y)\ q_n(t) = \sum_n \sin k_n y\ q_n(t) \quad \text{with} \quad k_n = \frac{n\pi}{l_s} \tag{6.2}$$

Integrating the wave equation over the length of the string and using the mass- and stiffness orthogonality properties of the normal modes, using the same method as the one used for the bar in Sec. (2), yields the set of differential equations:

$$\ddot{q}_n + \left(\frac{r}{\mu} + \eta\omega_n^2\right)\dot{q}_n + \omega_n^2 q_n = f(t)\frac{\phi_n(y_o)}{m_n} \tag{6.3}$$

where $\omega_n = k_n c = k_n \sqrt{\frac{T}{\mu}}$ and where the modal masses are given by:

$$m_n = \int_0^{l_s} \mu \phi_n^2(y)\, dy \tag{6.4}$$

. For an homogeneous string, we have $m_n = \mu l_s/2$. Assuming that the damping coefficients are sufficiently small, then it can be shown (using, for example, the Laplace transform) that the time participation factor of the n_{th}- mode is of the form:

$$q_n(t) = e^{-\alpha_n t} \left(A_n \cos \omega_n \sqrt{1-\alpha_n^2} t + B_n \sin \omega_n \sqrt{1-\alpha_n^2} t \right) \tag{6.5}$$

where the damping factors α_n are given by:

$$\alpha_n = \frac{r_n}{2m_n} = \frac{1}{2} \left[\frac{r}{\mu} + \eta \omega_n^2 \right] = \zeta_n \omega_n \tag{6.6}$$

and where $\omega_n \sqrt{1-\alpha_n^2} \approx \omega_n$.

6.1.2 String with internal damping loaded by the soundboard in vacuo

We examine first the case of a free lossless string fixed at point $y = 0$ and coupled to a mechanical load, such as a soundboard, at the other end $y = l_s$. The equations of the problem are formulated in the Laplace domain where s is the Laplace variable and $w = w(y, s)$. In what follows, the motion of the moving end is described in terms of the *admittance* $Y_l(s)$, defined as the ratio between velocity $\dot{w} = sw(y, s)$ and force $\mathcal{F}(y, s)$ at point $y = l_s$. This yields:

$$\begin{cases} w(y=0,s) = 0 \\ \mu s^2 w = T \dfrac{\partial^2 w}{\partial y^2} \\ sw(y=l_s,s) = Y_l(s)\ \mathcal{F}(y=l_s,s) = -TY(s)\ \dfrac{\partial w}{\partial y}(y=l_s,s) \end{cases} \tag{6.7}$$

Let us denote $F(s)$ the wave propagating towards the end (in the positive direction) and $G(s)$ the reflected wave from the moving end in the negative direction. The general solution of the wave equation in the Laplace domain is written:

$$w(y,s) = \exp\left(-s\frac{y}{c}\right) F(s) + \exp\left(s\frac{y}{c}\right) G(s) \tag{6.8}$$

and the boundary condition at end $y = l_s$ yields the reflection coefficient $R(s)$ between outgoing and incoming wave:

$$R(s) = \frac{G(s)}{F(s)} = \frac{Z_cY_l - e^{-\frac{sl_s}{c}}}{Z_cY_l + e^{\frac{sl_s}{c}}} \tag{6.9}$$

where $Z_c = \frac{T}{c} = \sqrt{T\mu}$ is the *characteristic impedance* of the string.

We now examine the consequence of Eq. (6.9) for the particular case of an incoming harmonic wave with pulsation ω. For simplicity, we also modify the origin of the y-axis with a translation equal to $-l_s$ so that the reflection now takes place at the origin, which allows us to replace the exponential terms by 1 in Eq. (6.9). This does not modify, of course, the calculation of the decay times which are independent of the position of the load along the string. Denoting now $Y_l(\omega) = G(\omega) + jB(\omega)$, then Eq. (6.9) becomes [22]:

$$R(\omega) = |R|e^{j\beta} = \frac{Z_cG - 1 + jB}{Z_cG + 1 + jB} = e^{-a+j\beta} \tag{6.10}$$

where

$$|R|^2 = 1 - \frac{4Z_cG}{(Z_cG+1)^2 + B^2Z_c^2} \quad \text{and} \quad \tan\beta = \frac{2BZ_c}{Z_c^2(G^2+B^2) - 1} \tag{6.11}$$

Since the moving end is assumed to be dissipative, $G(\omega)$ is positive and thus Eq. (6.11) shows that $|R(\omega)| < 1$, which means that the returning wave is attenuated compared to the incoming one. Assuming that this attenuation is weak ($Z_cG \ll 1$), as it is the case for a string attached to a soundboard, than we can use the approximation:

$$|R| \approx 1 - \frac{2Z_cG}{(Z_cG+1)^2 + B^2Z_c^2} = 1 - a \tag{6.12}$$

For a string of finite length l_s, the attenuation takes place at each period of time $T = \frac{2l_s}{c}$. Therefore the damping factor per period of the wave along the string (in s^{-1}) is:

$$\alpha_{nl} = \frac{a}{T} = \frac{c}{l_s}\frac{Z_cG}{(Z_cG+1)^2 + B^2Z_c^2} \tag{6.13}$$

The consequence of the phase shift β is to modify the eigenfrequencies of the string. In guitars, this shift is generally very small so that we can assume $\beta \approx 0$. With the abovementioned assumption $Z_cG \ll 1$, the damping factor for the n_{th} eigenfrequency of the string becomes:

$$\alpha_{nl} \approx \frac{c}{l_s}Z_cG(\omega_n) = \frac{T}{l_s}\mathcal{R}e\left\{Y(\omega_n)\right\} \tag{6.14}$$

We combine now the results obtained for the string and the moving end, respectively. During each period of the oscillation, the wave is continuously damped along the string with factor α_n

and, at each reflexion at the moving end, with an additional factor α_{nl} so that the total damping factor for the string loaded by the plate *in vacuo* is given by:

$$\alpha_n(\omega_n) \approx \frac{r_n}{2m_n} + \frac{T}{l_s}\mathcal{R}e\{Y(\omega_n)\} = \frac{1}{2m_n}\left[r_n + Z_c^2\mathcal{R}e\{Y(\omega_n)\}\right] \quad (6.15)$$

In terms of impedance, we have $Z_l(\omega) = \frac{1}{Y_l} = R(\omega) + jX(\omega)$, so that the damping factors can be written alternatively:

$$\alpha_n(\omega_n) \approx \frac{r_n}{2m_n} + \frac{T}{l_s}\frac{R}{R^2 + X^2} \quad (6.16)$$

Eq. (6.15) shows that the damping always increases with the loading by the plate, compared to the isolated string, since both terms at the right-hand side are positive. For a string stretched at a fixed point on a soundboard, it can be seen that the damping factor increases with the tension T. This is due to the fact that the characteristic impedance increases with T, thus facilitating the transfer of energy between string and soundboard. Similarly, the damping factor increases as the length of the string decreases. This can be understood by observing that the amount of energy transferred to the load in a given time interval increases as the travelling time between two consecutive reflexions becomes smaller.

In the low frequencies, the frequencies for which the admittance (also called mobility) $Y(\omega)$ reaches a local maximum roughly correspond to the eigenfrequencies of the soundboard. If there is coincidence at those frequencies with the eigenfrequencies of the string, then these frequencies will die more rapidly.

On a mathematical point of view, the maxima for α_n correspond to the frequencies $\omega_n = \omega_i$ for which $X(\omega_i) = 0$. Notice that measuring ω_i experimentally is an hard task, because of the difficulties in conducting experiments in vacuo. For these frequencies, we have:

$$\alpha_{nMAX} \approx \frac{r_n}{2m_n} + \frac{T}{Rl_s} \quad (6.17)$$

6.1.3 String with internal damping loaded by the soundboard coupled to air and cavity

The exact calculation of the decay factors in this case requires accurate modeling of the air-structure interaction. In practice, radiation and cavity modes modify both the real and imaginary part of the impedance Z_l. Let us denote $R_a(\omega)$ and $X_a(\omega)$ the additional resistance and reactance due to air loading of the soundboard. R_a is positive whereas X_a can be either positive or negative, depending on whether the inertial effects or the stiffening effects due to the air and cavity are predominant at the attachment point of the string. The considerations developed in the previous paragraph now lead to:

$$\alpha_n(\omega_n) \approx \frac{r_n}{2m_n} + \frac{T}{l_s}\frac{R+Ra}{(R+R_a)^2+(X+X_a)^2} \tag{6.18}$$

In this case, the maxima of the damping factors are obtained at the frequencies $\omega_n = \omega_j$ for which we have $[X + X_a](\omega_j) = 0$, which yields:

$$\alpha_{nMAX}(\omega_j) \approx \frac{r_n}{2m_n} + \frac{T}{(R+Ra)l_s} \tag{6.19}$$

We obtain here an apparent paradox in the sense that the highest damping factors are smaller than in the vacuo case, despite the fact that energy is now dissipated in both the plate and air. In fact, one should not only consider the particular case of the highest damping factors, but rather the total balance of power in the system. The condition that govern the increase of damping compared to the in vacuo case is given by:

$$\frac{R+Ra}{(R+R_a)^2+(X+X_a)^2} - \frac{R}{R^2+X^2} > 0 \tag{6.20}$$

Considering, for example, the frequencies ω_j for which $X + X_a = 0$, which are easily measured, then we see that the difference expressed in Eq. (6.20) is positive under the condition:

$$X_a^2 > RR_a \tag{6.21}$$

Finally, returning now to Eq. (6.18), one can see that the damping factors for the string mounted on the complete instrument in air can be decomposed into the sum of three contributions related to the string (α_{nc}), the structure (α_{ns}) and the air (α_{na}), respectively:

$$\begin{aligned}\alpha_n(\omega_n) &= \alpha_{nc} + \alpha_{ns} + \alpha_{na} \\ &\approx \frac{r_n}{2m_n} + \frac{T}{l_s}\frac{R}{(R+R_a)^2+(X+X_a)^2} + \frac{T}{l_s}\frac{Ra}{(R+R_a)^2+(X+X_a)^2}\end{aligned} \tag{6.22}$$

Notice, in particular, on this equation that the structural contribution to the damping α_{ns} is modified compared to the structural damping in vacuo in Eq. (6.16).

6.2 Sound power and radiation efficiency of the instrument

6.2.1 Power dissipated in the string

Using the same method as the one presented in Section 4, we start by calculating the instantaneous power $p_{ms}(t)$ imparted to the string. It is assumed that the input is localized at position $y = y_o$ and that the force is not propagating. We treat the case of a viscoelastic string. Using Eq. (6.1), we obtain:

$$\begin{aligned} p_{ms}(t) &= \int_0^{l_s} f(t)\dot{w}(y,t)\delta(y-y_o)\, dy \\ &= \int_0^{l_s} \mu \frac{\partial^2 w}{\partial t^2}\dot{w}\, dy - \int_0^{l_s} T\left(1+\eta\frac{\partial}{\partial t}\right)\frac{\partial^2 w}{\partial y^2}\dot{w}\, dy + \int_0^{l_s} r\frac{\partial w}{\partial t}\dot{w}\, dy \end{aligned} \tag{6.23}$$

Using integration by parts, this expression can be rewritten as:

$$\begin{aligned} p_{ms}(t) &= \frac{\partial}{\partial t}\left\{\int_0^{l_s} \mu\left(\frac{\partial w}{\partial t}\right)^2 dy + \int_0^{l_s} T\left(\frac{\partial w}{\partial y}\right)^2 dy\right\} + \int_0^{l_s}\left[r\dot{w}^2 + \eta T\left(\frac{\partial \dot{w}}{\partial y}\right)^2\right] dy \\ &= \frac{\partial E_s}{\partial t} + \int_0^{l_s}\left[r\dot{w}^2 + \eta T\left(\frac{\partial \dot{w}}{\partial y}\right)^2\right] dy \end{aligned} \tag{6.24}$$

The quantity E_s is a constant, since it corresponds to the total energy for a conservative system. Therefore the instantaneous power for the dissipative string reduces to:

$$p_{ms}(t) = \int_0^{l_s}\left[r\dot{w}^2 + \eta T\left(\frac{\partial \dot{w}}{\partial y}\right)^2\right] dy \tag{6.25}$$

where we can identify the contributions of *fluid* and *viscoelastic* losses, respectively.

In the case of a steady-state harmonic motion with frequency ω, we can write the velocity of the string, using the result obtained in Sec. (6.1):

$$\begin{aligned} \dot{w}(y,\omega) &= \sum_n \dot{w}_n(y,\omega) = \sum_n \phi_n(y)\dot{q}_n(\omega) \\ &= \sum_n \sin k_n y \frac{F_M \sin k_n y_o e^{j\omega t}}{m_n\left[j\omega + \dfrac{\omega_n^2}{j\omega} + 2\zeta_n\omega_n\right]} = \sum_n \sin k_n y \frac{F_M \sin k_n y_o}{m_n D_n(\omega)} e^{j\omega t} \end{aligned} \tag{6.26}$$

where F_M is the magnitude of the force applied at the excitation point y_o. Replacing this expression in Eq. (6.25) and averaging in time over a period $\tau = 2\pi/\omega$ yields the expression of the mean power:

$$\mathcal{P}_{ms}(\omega) = \frac{1}{\tau}\int_0^{\tau}\left\{\int_0^{l_s}\left[r\dot{w}^2 + \eta T\left(\frac{\partial \dot{w}}{\partial y}\right)^2\right]dy\right\}dt \tag{6.27}$$

Integrating first this expression in space over the length of the string yields:

$$\mathcal{P}_{ms}(\omega) = \frac{l_s}{2\tau}\int_0^{\tau}\left\{r\sum_n \dot{q}_n^2 + \eta T\sum_n k_n^2\dot{q}_n^2\right\}dt \tag{6.28}$$

then, the integration versus time yields:

$$\mathcal{P}_{ms}(\omega) = \frac{\mu l_s}{4}\sum_n\left(\frac{r}{\mu} + \eta\omega_n^2\right)|\dot{q}_n|^2 = \frac{\mu l_s F_M^2}{2}\sum_n \frac{\zeta_n\omega_n \sin^2 k_n y_o}{m_n^2|D_n(\omega)|^2} \tag{6.29}$$

Recalling that $\alpha_n = \zeta_n\omega_n$, then Eq. (6.29) gives the relation between the power dissipated in the string and the damping factors, for a given excitation frequency ω. If ω is equal to one particular eigenfrequency ω_n of the string, and continuing to assume further that $\alpha_n \ll \omega_n$, then the approximate expression for the dissipated power is:

$$\mathcal{P}_{ms}(\omega_n) \approx \frac{F_M^2}{4m_n}\frac{\sin^2 k_n y_o}{\zeta_n\omega_n} = \frac{F_M^2}{2r_n}\sin^2 k_n y_o \tag{6.30}$$

Introducing the quantity:

$$A_n = \frac{F_M \sin k_n y_o}{r_n\omega_n} \tag{6.31}$$

thus we can express the dissipated power at this frequency as follows:

$$\mathcal{P}_{ms}(\omega_n) \approx \frac{1}{2}r_n\omega_n^2 A_n^2 = \alpha_n k_n^2 A_n^2 \tag{6.32}$$

Eq. (6.32) shows that $\mathcal{P}_{ms}(\omega_n)$ is proportional to α_n, and that the coefficient of proportionality is completely determined by both the length and boundary conditions of the string (through k_n) and by the excitation (through A_n).

6.2.2 Energetic considerations

For the guitar, the normal use of the instrument is rather to establish free vibrations than driving the string with a constant frequency. In this case, integrating Eq. (6.24) over time shows that the time history of the instantaneous energy $E(t)$ of the string after the pluck is given by:

$$E(t) = E(0) - \int_0^t \left\{ \int_0^{l_s} \left[r\dot{w}^2 + \eta T \left(\frac{\partial \dot{w}}{\partial y} \right)^2 \right] dy \right\} dt \tag{6.33}$$

which, in view of the previous results, can be written equivalently:

$$E(t) = E(0) - \frac{l_s}{2} \int_0^t \sum_n \left(r + \eta T k_n^2 \right) \dot{q}_n^2(t) dt \tag{6.34}$$

The time functions $q_n(t)$ can be written as follows:

$$q_n(t) = A_n e^{-\alpha_n t} \sin \omega_n t \tag{6.35}$$

where the A_n are determined here by the initial conditions of the pluck. This leads to:

$$\begin{aligned} E(t) &= E(0) \\ &- \frac{1}{2} \int_0^t \sum_n \left(r + \eta T k_n^2 \right) A_n^2 e^{-2\alpha_n t} \left[\omega_n^2 \cos^2 \omega_n t + \alpha_n^2 \sin^2 \omega_n t - \alpha_n \omega_n \sin 2\omega_n t \right] dt \end{aligned} \tag{6.36}$$

Using the approximation $\alpha_n \ll \omega_n$ yields, to a first-order approximation:

$$E(t) \approx E(0) - \left[\sum_n \frac{\alpha_n \omega_n^2 \mu l_s}{2} A_n^2 \right] t \tag{6.37}$$

which shows that the total energy of the string decreases *linearly* with time. This result is confirmed by direct simulations of the string equation in the time domain (see Fig. 5).

Finally, the link between time history of the energy and the dissipated power is obtained by taking the time derivative of $-E(t)$, which yields:

$$-\frac{dE}{dt} = \sum_n \frac{\alpha_n \omega_n^2 \mu}{2} A_n^2 = \sum_n \frac{r_n \omega_n^2}{2} A_n^2 = \sum_n \mathcal{P}_{ms}(\omega_n) \tag{6.38}$$

Comparing this result with the expression of power in Eq. (6.32) shows that, to a first-order approximation, the slope of the energy curve vs time is equal to the sum of the dissipated power terms at the eigenfrequencies of the string.

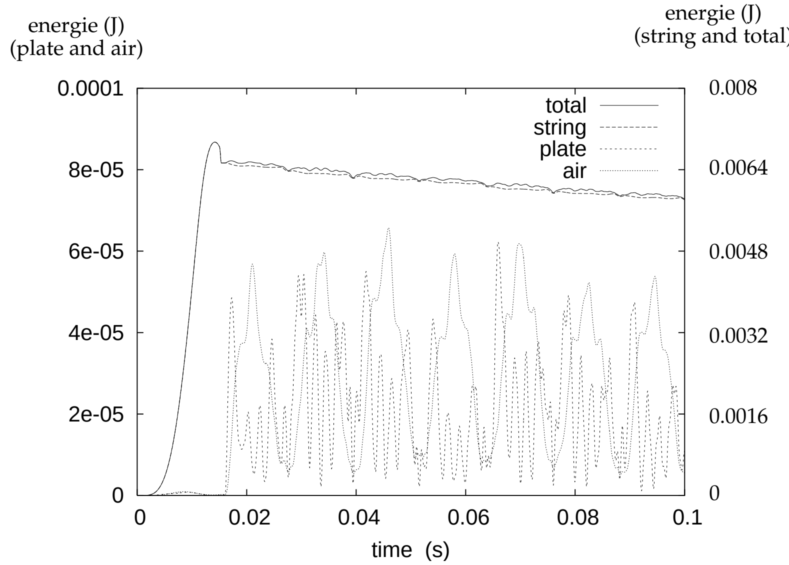

Figure 5: Time history of the total energy of the string

6.2.3 Structural power dissipated in the soundboard in vacuo

If the end of the string at $y = l_s$ is allowed to move, then the instantaneous power becomes:

$$p_m(t) = p_{ms}(t) - T\dot{w}(l_s,t)\frac{\partial w}{\partial y}(l_s,t) \tag{6.39}$$

where $p_{ms}(t)$ denotes the instantaneous power in the string calculated in previous paragraph. The expression of the mean power becomes:

$$\mathcal{P}_m(\omega) = \mathcal{P}_{ms}(\omega) + \frac{1}{2}\mathcal{R}e\left\{Y(\omega)\right\}|T\frac{\partial w}{\partial y}|^2(l_s,\omega) = \mathcal{P}_{ms}(\omega) + \mathcal{P}_{ml}(\omega) \tag{6.40}$$

where $Y(\omega)$ still denotes the admittance at the end $y = l_s$ of the string.

Assuming that the perturbation of the string's eigenfrequencies due to the load are negligible, then we can assume that the slope of the string at the end is given by:

$$\frac{\partial w}{\partial y}(\omega, l_s) = \sum_n k_n cos k_n l_s q_n(\omega) \approx \sum_n \frac{n\pi}{l_s}(-1)^n q_n(\omega) \tag{6.41}$$

Therefore:

$$|T\frac{\partial w}{\partial y}|^2(l_s,\omega) = \frac{T^2\pi^2}{l_s^2}\left[\sum_n n^2 q_n^2(\omega) + 2\sum_{n\neq m} nm(-1)^{n+m} q_n(\omega)q_m^*(\omega)\right] \tag{6.42}$$

As a consequence, we can write the power dissipated at the loaded end:

$$\mathcal{P}_{ml}(\omega) = \underline{\dot{Q}}^H \mathbf{R_l} \underline{\dot{Q}} \tag{6.43}$$

where the elements of the resistance matrix $\mathbf{R_l}$ are given by:

$$(R_l)_{mn}(\omega) = \frac{T^2\pi^2}{l_s^2} \mathcal{R}e\left\{Y(\omega)\right\} \frac{(-1)^{n+m}}{2\omega^2} \tag{6.44}$$

which, in view of the definition of the damping factor due to the load α_l given in Eq.(6.14), becomes:

$$(R_l)_{mn}(\omega) = \frac{T\pi^2}{l_s} \frac{(-1)^{n+m}}{2\omega^2} \alpha_l(\omega) \tag{6.45}$$

As mentioned in the previous sections, the frequencies of interest for a plucked instrument are the eigenfrequencies of the string ω_n, and thus, in this case one should simply replace ω by ω_n in Eqs. (6.40)-(6.45).

6.2.4 Radiated sound power and acoustical efficiency of the instrument

For a stringed instrument, the acoustical efficiency can be defined as:

$$\sigma = \frac{\mathcal{P}_a}{\mathcal{P}_{tot}} = \frac{\mathcal{P}_a}{P_a + \mathcal{P}_s + \mathcal{P}_l} \tag{6.46}$$

Where $\mathcal{P}_a$ is the radiated acoustical power, $\mathcal{P}_s$ is the power dissipated in the string and $\mathcal{P}_l$ is the power dissipated in the structure. All these quantities are functions of frequency.

For a given eigenfrequency of the string ω_n, the previous developments show that $\mathcal{P}_{tot}(\omega_n)$ can be estimated by $\alpha_n \omega_n^2 A_n^2 m_n$, where $2\alpha_n m_n = r_n + Z_c^2 \mathcal{R}e\left\{Y(\omega)\right\}$, and where $\mathcal{R}e\left\{Y(\omega)\right\}$ represents the admittance at the bridge for the soundborad loaded by the air and cavity. The determination of $\mathcal{P}_a(\omega_n)$ is more difficult, since it needs the determination of the sound power radiated by the instrument in an anechoic chamber, by means of intensity measurements and integration over a closed surface, for example.

In general, as shown in Eq. (6.22), it is not possible to separate the structural losses from the radiated losses, since the loading of the air modifies the response of the structure, compared to the *in vacuo* case. However, an interesting alternative can be found, by defining the efficiency of the instrument, as the ratio between the power dissipated in both *structure + air* compared to the total dissipated power:

$$\sigma' = \frac{\mathcal{P}_a + \mathcal{P}_l}{P_a + \mathcal{P}_s + \mathcal{P}_l} \tag{6.47}$$

In this case, it is straightforward to show that:

$$\sigma^{'}(\omega_n) = \frac{T}{l_s}\frac{\mathcal{R}e\left\{Y(\omega_n)\right\}}{\alpha_n(\omega_n)} \tag{6.48}$$

Eq. (6.48) shows that determining $\sigma^{'}$ is very simple. It requires measurement of string length, tension, as well as decay time and admittance at the bridge for the frequencies of interest.

Finally, following Eq. (6.38) for a given note played on the guitar, with multiple eigenfrequencies ω_n, we would get the modified efficiency by calculating the sum:

$$\sigma^{'}_{tot} = \sum_n \sigma^{'}(\omega_n) = \frac{T}{l_s}\sum_n \frac{\mathcal{R}e\left\{Y(\omega_n)\right\}}{\alpha_n(\omega_n)} \tag{6.49}$$

If one can show (through measurements or simulations) that the power dissipated in the structure (mostly in the soundboard) is significantly lower than the radiated acoustic power at a given freequency, then Eq. (6.49) yields a good estimate of the *true* acoustical efficiency. Otherwise, one should consider $\sigma^{'}$ only as an order of magnitude. In this case, accurate determination of the efficiency is only given by σ, which requires masurement of the acoustic power, in order to separate structural and acoustic contributions.

7 Summary and conclusions

In this report, a systematic theoretical approach for both string-structure and air-structure coupling has been presented. The initial objective was to summarize the main useful results for tackling the problem of sound radiation and efficiency in stringed musical instruments.
The second goal of this study was to explore as much as possible whether some of these basic equations of mechanics and acoustics could be formulated in terms of filters and state-space variables, in order to prepare eventual future work on mixed "physics-signal processing" synthesis and control of sound produced by stringed instruments. This was mostly the purpose of Sec. 2.
In case of a coupling between a thin structure and a cavity filled with air, the assumption of weak coupling is usually not justified and one has to solve the equations derived in Sec. 3 numerically.
The relationships between modal coupling, energy and sound power were developed extensively in Sec. 4. The concept of efficiency and radiation impedance matrix is generalized in Sec. 5 in the case of plates for which the calculations can be conducted analytically.
Finally, the main concepts are applied to the case of a guitar and compared to results obtained through numerical simulations in Sec. 6. Comparisons with experiments on real guitars in an anechoic space should be conducted in the near future.

8 Appendix: Perturbation of a string subjected to an in-plane load at one end

Fundamental frequency $f_1 = \frac{c}{2L}$
Longitudinal force P at one end so that the new length become $L - \Delta L = L(1 - \varepsilon)$.
Initial tension T and initial mass per unit length μ, so that the initial wave velocity is given by $c = \sqrt{\frac{T}{\mu}}$.
The initial tension T is given by $\frac{T}{S} = E\frac{L - L_o}{L_o}$ where E is the Young's modulus and S is the cross-sectional area *after* initial stretching. L_o is the length of the string *before* initial stretching.

Remark: For nylon guitar strings, the ratio $\frac{L - L_o}{L_o}$ usually lies within the interval $[0.1; 0.5]$, depending on Young's modulus of the material and cross-section.

After initial stretching, the string volume is $V = SL$. The density is ρ. With a perturbation P, the conservation of mass allows to write:

$$\rho SL = \rho^{'} S^{'} L(1 - \varepsilon) \quad \text{or} \quad \mu = \mu^{'}(1 - \varepsilon) \tag{8.1}$$

The new tension $T^{'}$ of the string is given by:

$$\frac{T}{S(L - L_o)} = \frac{T^{'}}{S^{'}(L - \varepsilon L - L_o)} \tag{8.2}$$

Finally, the new transverse wave velocity along the string is:

$$c^{'} = \sqrt{\frac{T^{'}}{\mu^{'}}} \tag{8.3}$$

And the new fundamental frequency is given by:

$$f_1^{'} = \frac{c^{'}}{2L(1 - \varepsilon)} \tag{8.4}$$

So far, the problem cannot be solved without getting an idea on the relative variation of section of the string, consecutive to the perturbation. The solution is given by the Poisson's ratio ν, which governs the relative variation of lateral dimension compared to the relative variation of length for a given material. For a string made of an isotropic material, it is a standard result that:

$$\frac{S^{'}}{S} = 1 + 2\nu\varepsilon \tag{8.5}$$

For a metallic steel string, we have $\nu = 0.3$ and for a nylon string, we have $\nu = 0.4$. The Poisson's ratio is always < 0.5.

Using Eqs. (8.1)-(8.5) together, we obtain the ratio between the wave velocities:

$$c'^2 = c^2 \left[\frac{L - L_o - \varepsilon L}{L - L_o}\right] \left[\frac{1 + 2\nu\varepsilon}{1 + \varepsilon}\right] \tag{8.6}$$

which yields, to a first-order approximation:

$$c' = c\left[1 - \frac{\varepsilon}{2}\left(\frac{L}{L - L_o} + 1 - 2\nu\right)\right] \tag{8.7}$$

Since $\nu < 0.5$, we see that the transverse wave velocity always *decrease* due to a compression at one end.

Finally, the fundamental frequency becomes:

$$f' = f\frac{c'}{c(1 - \varepsilon)} \approx 1 - \frac{\varepsilon}{2}\left(\frac{L}{L - L_o} - 1 - 2\nu\right) \tag{8.8}$$

We see that the fundamental frequency will *decrease* under the condition:

$$\frac{L}{L - L_o} = \frac{ESL}{TL_o} > 1 + 2\nu \tag{8.9}$$

which is usually the case for strings mounted on musical instruments. However, we see that for very flexible and thin strings (low E and S), it might happen that the fundamental frequency increases, which means that the effect of decreasing the length dominates the systematic decrease in transverse wave velocity.

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