Waveguide Filter Tutorial

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Abstract

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Contents

1 Introduction

Digital Waveguide Filters (DWF) have proven useful for building computational models of acoustic systems which are both physically meaningful and efficient for digital synthesis. The physical interpretation opens the way to capturing valued aspects of real instruments which have been difficult to obtain by more abstract synthesis techniques. Waveguide filters were derived for the purpose of building reverberators using lossless building blocks [6], but any linear acoustic system can be approximated using waveguide networks. For example, the bore of a wind instrument can be modeled very inexpensively as a digital waveguide [7]. Similarly, a violin string can be modeled as a digital waveguide with a nonlinear coupling to the bow [7]. When the basic model is physically meaningful, it is often obvious how to introduce nonlinearities correctly, thus leading to realistic behaviors far beyond the reach of purely analytical methods.

A basic feature of DWF building blocks is the exact physical interpretation of the contained digital signals as traveling *pressure waves* or *velocity waves.* A byproduct of this formulation is the availability of *signal power* defined *instantaneously* with respect to both *space* and *time.* This instantaneous handle on signal power yields a simple picture of the effects of round-off error on the growth or decay of the signal energy within the DWF system [8]. Another nice property of waveguide filters is that they can be reduced in special cases to standard lattice/ladder digital filters which have been extensively developed in recent years [4]. One immediate benefit of this connection is a body of techniques for realizing *any* digital filter transfer function as a DWF. Waveguide filters are also very closely related to *Wave Digital Filters (WDF)* which have been developed primarily by Fettweis [2]. Waveguide filters can be viewed as a generalized framework incorporating aspects of lattice and ladder digital filters, wave digital filters, one-dimensional waveguide acoustics, and classical network theory [1].

A *waveguide* for our purposes is any medium in which wave motion can be characterized by the one-dimensional *wave equation* [5]. In the lossless case, all solutions can be expressed in terms of left-going and right-going *traveling waves* in the medium. The traveling waves propagate unchanged as long as the *wave impedance* of the medium is constant. The wave impedance is the square root of the of the "massiness" times the "stiffness" of the medium; that is, it is the geometric mean of the two sources of resistance to motion: the inertial resistance of the medium due to its mass, and the spring-force on the displaced medium due to its elasticity. For example, the wave impedance R of a vibrating string is $R = \sqrt{T\rho} = \rho c$, where ρ is string density (mass per unit length) and T is the tension of the string.

When the wave impedance changes, *signal scattering* occurs, i.e., a traveling wave impinging on an impedance discontinuity will partially reflect and partially transmit at the junction in such a way that energy is conserved. Real-world examples of waveguides include the bore of a clarinet, the vocal tract in speech, microwave antennas, electric transmission lines, and optical fibers.

2 Reduction to Standard Forms

Digital waveguide filters (DWF) are obtained (conceptually) by sampling the unidirectional traveling waves which occur in a system of ideal, lossless waveguides. Sampling is across time and space. Thus, variables in a DWF structure are equal *exactly* (at the sampling times and positions, to within numerical precision) to variables propagating in the physical medium in an interconnection of uniform transmission-lines.

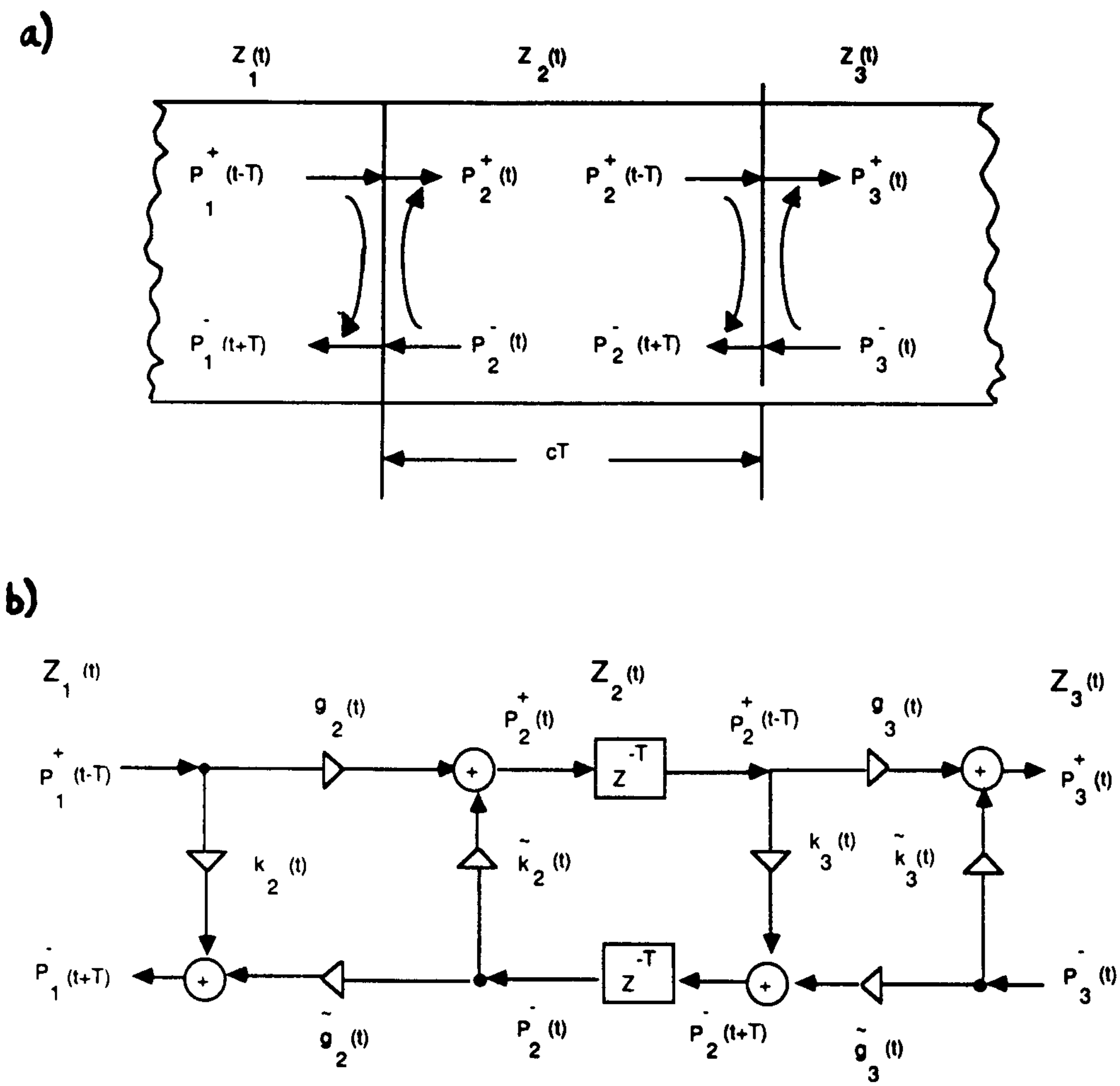

Figure 1: Waveguide digital filter structure.

A cascade chain of DWF sections, terminated by a pure reflection on the right, is shown in Fig. 1. Each box enclosing the symbol $k_i(t)$ denotes a scattering junction characterized by that reflection coefficient. While we have mentioned only the Kelly-Lochbaum and one-multiply junction, any type of lossless scattering junction will do [4]. The DWF employs delays between each scattering junction along both the top and bottom signal paths, unlike conventional ladder and lattice filters. As a result, it has a direct physical interpretation as a sampled acoustic tube.

The delays preceding the two inputs to a junction can be "pushed" into the junction so that they emerge on the outputs and combine with the delays there. (Show this using the Kelly-Lochbaum scattering junction.) By performing this operation on every other section in the DWF chain, the filter structure of Fig. 2 is obtained. This structure has some advantages worth considering: (1) it consolidates delays to length $2T$ as do conventional lattice/ladder structures, (2) it does not require a termination by an infinite wave impedance, allowing it to be extended to networks of arbitrary topology (e.g., multiport branching, intersection, and looping), and (3) there is no long delay-free signal path along the upper rail as in conventional lattice/ladder structures—a pipeline segment is only two sections long. This structure appears to have better overall characteristics than any other digital filter structure for many applications. Advantage (2) makes it especially valuable for modeling physical systems.

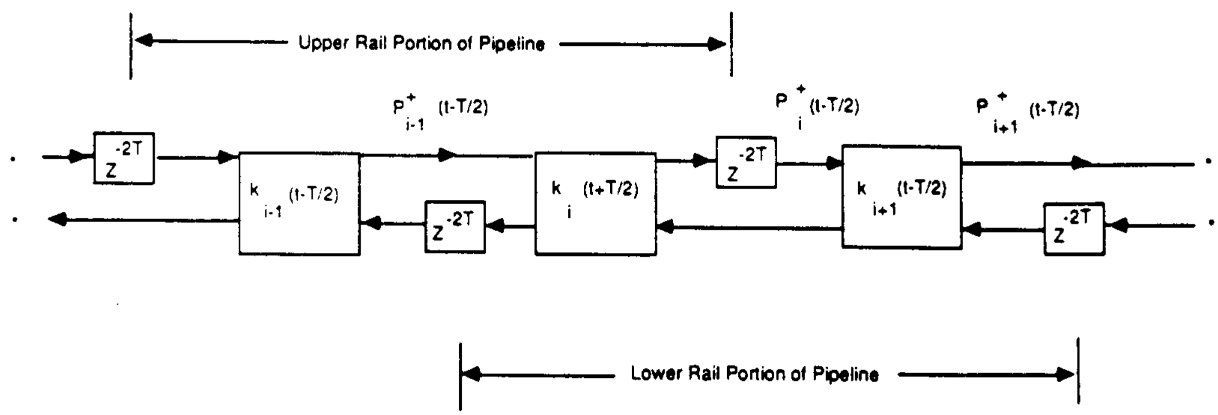

Figure 2: Pipelineable, physically extendible, consolidated-delay, waveguide filter.

Given a reflecting termination on the right, the half-rate DWF chain of Fig. 2 can be reduced further to the conventional ladder/lattice structure of Fig. 3. Every delay on the upper rail is pushed to the right until they have all been worked around to the bottom rail. In the end, each bottom-rail delay becomes $2T$ seconds instead of T seconds. Such an operation is possible because of the termination at the right by an infinite (or zero) wave impedance. In the time-varying case, pushing a delay through a multiply results in a corresponding time advance of the multiplier coefficient. The time arguments of the reflection coefficients in the figure indicate the amount of the time shift for each section. Note that because of the reflecting termination, conventional lattice filters cannot be extended to the right in any physically meaningful way. Also, creating network topologies more complex than a simple series (or acyclic tree) of waveguide sections is not immediately possible because of the delay-free path along the top rail. In particular, the output cannot be fed back to the input.

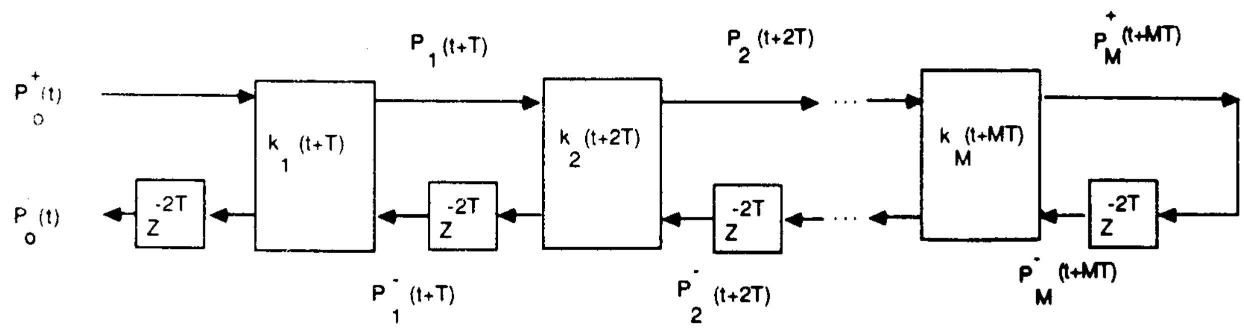

Figure 3: Conventional ladder/lattice filter structure.

3 Power-Normalized Waveguide Filters

Above, we adopted the convention that the time variation of the wave impedance did not alter the traveling force waves $f_i^{\pm}$. In this case, the power represented by a traveling force wave is modulated by the changing wave impedance as it propagates. The actual power becomes inversely proportional to wave impedance:

$$\mathcal{I}_i(t,x) = \mathcal{I}_i^+(t,x) + \mathcal{I}_i^-(t,x) = \frac{[f_i^+(t,x)]^2 - [f_i^-(t,x)]^2}{R_i(t)}$$

In some applications (e.g. [6]), it may be desirable to compensate for the power modulation so that changes in the wave impedances of the waveguides do not affect the power of the signals propagating within.

In [8], three methods are discussed for making signal power *invariant* with respect to time-varying branch impedances: (1) The *normalized waveguide* scheme compensates for power modulation by scaling the signals leaving the delays so as to give them the same power coming out as they had going in. It requires two additional scaling multipliers per waveguide junction. (2) The *normalized wave* approach [4] propagates *rms-normalized waves* in the waveguide. In this case, each delay-line contains $\tilde{f}_i^+(t,x) = f_i^+(t,x)/\sqrt{R_i(t)}$ and $\tilde{f}_i^-(t,x) = f_i^-(t,x)/\sqrt{R_i(t)}$. In this case, the power stored in the delays does not change when the wave impedance changes. This is the basis of the *normalized ladder filter* (NLF) [3, 4]. Unfortunately, four multiplications are obtained at each scattering junction. (3) The *transformer-normalized waveguide* approach to normalization changes the wave impedance at the output of the delay back to what it was at the time it entered the delay using a "transformer."

A *transformer* joins two waveguide sections of differing wave impedance in such a way that signal power is preserved and no scattering occurs. From Ohm's Law for traveling waves, and from the definition of power waves, we see that to bridge an impedance discontinuity with no power change and no scattering requires the relations

$$\frac{[f_i^+]^2}{R_i(t)} = \frac{[f_{i-1}^+]^2}{R_{i-1}(t)} \qquad \frac{[f_i^-]^2}{R_i(t)} = \frac{[f_{i-1}^-]^2}{R_{i-1}(t)}$$

Therefore, the junction equations for a *transformer* [1] can be chosen as

$$f_i^+ = g_i(t) f_{i-1}^+ \qquad f_{i-1}^- = g_i^{-1}(t) f_i^- \tag{1}$$

where

$$g_i(t) \triangleq \sqrt{\frac{R_i(t)}{R_{i-1}(t)}} = \sqrt{\frac{1+k_i(t)}{1-k_i(t)}} \tag{2}$$

The choice of a negative square root corresponds to the *gyrator* [1]. The gyrator is equivalent to a transformer in cascade with a dualizer [9]. A dualizer is a direct implementation of Ohm's Law for traveling waves (to within a scale factor): the forward path is unchanged while the reverse path is negated. On one side of the dualizer there are force waves, and on the other side there are velocity waves. Ohm's law can thus be interpreted as a gyrator in cascade with a transformer whose scale factor equals the wave admittance.

The transformer-normalized DWF junction is shown in Fig. 4a. We can now modulate a single junction, even in arbitrary network topologies, by inserting a transformer immediately to the left or right of the junction. Conceptually, the wave impedance is not changed over the delay-line portion of the waveguide section; instead, it is changed to the new time-varying value just before (or after) it meets the junction. When velocity is the wave variable, the coefficients g_i and g_i^{-1} in Fig. 4a are swapped (or inverted).

So, as in the normalized waveguide case, for the price of two extra multiplies per section, we can implement time-varying digital filters which do not modulate stored signal energy. Moreover, transformers enable the scattering junctions to be varied independently, without having to propagate time-varying impedance ratios throughout the waveguide network.

It can be shown [9] that cascade waveguide chains built using transformer-normalized waveguides are *equivalent* to those using normalized-wave junctions. Thus, the transformer-normalized DWF in Fig. 4a and the wave-normalized DWF in Fig. 4b are equivalent. One simple proof is to start with a transformer and a Kelly-Lochbaum junction, move the transformer scale factors inside the junction, combine terms, and arrive at Fig. 4b. One practical benefit of this equivalence is that the normalized ladder filter (NLF) can be implemented with only three multiplies and three additions instead of four multiplies and two additions.

4 Conclusions

Waveguide digital filters were derived by sampling ideal waveguide networks with respect to space and time. It was shown that the DWF can be transformed into well known ladder and lattice digital filter structures simply by pushing delays around to the bottom rail (in the special case of a cascade, reflectively terminated chain of waveguides). The DWF structure gives a precise implementation of physical wave phenomena in time-varying media; consequently, waveguide filters are useful as building blocks for computational models of acoustic systems.

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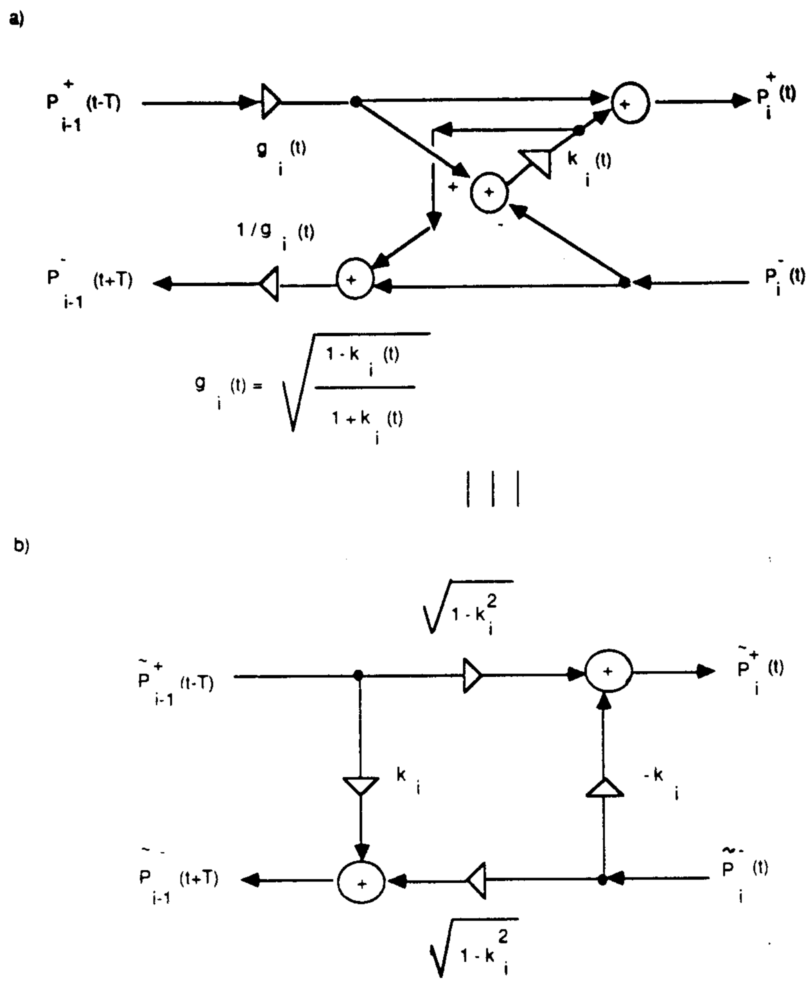

Figure 4: a) Transformer-normalized waveguide digital filter section, for transformer on left of junction. b) Normalized ladder filter section. The two are equivalent.

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